Research Article

Existence for Nonoscillatory Solutions of Higher-Order Nonlinear Differential Equations

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The existence of nonoscillatory solutions of the higher-order nonlinear differential equation

\[
[r(t)(x(t) + P(t)x(t - \tau))^{(n-1)}] + \sum_{i=1}^{m} Q_i(t)f_i(x(t - \sigma_i)) = 0, \quad t \geq t_0,
\]

where \(m \geq 1, n \geq 2\) are integers, \(\tau > 0, \sigma_i \geq 0, r, P, Q_i \in C([t_0, \infty), R), f_i \in C(R, R), i = 1, 2, \ldots, m\), is studied. Some new sufficient conditions for the existence of a nonoscillatory solution of above equation are obtained for general \(Q_i(t) (i = 1, 2, \ldots, m)\) which means that we allow oscillatory \(Q_i(t) (i = 1, 2, \ldots, m)\). In particular, our results improve essentially and extend some known results in the recent references.

1. Introduction

Consider the higher-order nonlinear neutral differential equation

\[
\left[r(t)(x(t) + P(t)x(t - \tau))^{(n-1)}\right] + \sum_{i=1}^{m} Q_i(t)f_i(x(t - \sigma_i)) = 0, \quad t \geq t_0.
\]  

(1.1)

With respect to (1.1), throughout, we shall assume the following:

(i) \(m \geq 1, n \geq 2\) are integers, \(\tau > 0, \sigma_i \geq 0\),

(ii) \(r, P, Q_i \in C([t_0, \infty), R), r(t) > 0, f_i \in C(R, R), i = 1, 2, \ldots, m\).

Let \(\rho = \max_{t \in [t_0, \infty)} \{\tau, \sigma_i\}\). By a solution of (1.1), we mean a function \(x(t) \in C([t_1 - \rho, \infty), R)\) for some \(t_1 \geq t_0\) which has the property that \(x(t) + P(t)x(t - \tau) \in C^{n-1}([t_1, \infty), R)\) and \(r(t)(x(t) + P(t)x(t - \tau))^{(n-1)} \in C([t_1, \infty), R)\) and satisfies (1.1) on \([t_1, \infty)\).

A nontrivial solution of (1.1) is called oscillatory if it has arbitrarily large zeros, and, otherwise, it is nonoscillatory.
The existence of nonoscillatory solutions of higher-order nonlinear neutral differential equations received much less attention, which is due mainly to the technical difficulties arising in its analysis.

In 1998, Kulenovic and Hadziomerspahic [1] investigated the existence of nonoscillatory solutions of second-order nonlinear neutral differential equation

\[
(x(t) + cx(t - \tau))^\prime' + Q_1(t)x(t - \sigma_1) - Q_2(t)x(t - \sigma_2) = 0, \quad t \geq t_0, \quad (E_0)
\]

where \(c\) is a constant.

In 2006, Zhang and Wang [2] investigated the second neutral delay differential equation with positive and negative coefficients:

\[
[r(t)(x(t) + P(t)x(t - \tau))]' + Q_1(t)f(x(t - \sigma_1)) - Q_2(t)g(x(t - \sigma_2)) = 0, \quad t \geq t_0, \quad (E)
\]

where \(\tau > 0, \sigma_i \geq 0, Q_1, Q_2 \in \mathcal{C}([t_0, \infty), R^+), f, g \in \mathcal{C}(R, R), x f(x) > 0, x g(x) > 0, (x \neq 0)\). By using Banach contraction mapping principle, they proved the following theorem which extends the results in [1].

**Theorem A** ([2, Theorem 2.3]). Assume that

\((H_1)\) \(f\) and \(g\) satisfy local Lipschitz condition and \(x f(x) > 0, x g(x) > 0, \) for \(x \neq 0;\)

\((H_2)\) \(Q_i(t) \geq 0, i = 1, 2, a Q_1(t) - Q_2(t)\) is eventually nonnegative for every \(a > 0;\)

\((H_3)\) \(\int_{t_0}^{\infty} \int_{t_0}^{t} (Q_i(t)/r(s)) ds dt < \infty, i = 1, 2\) hold

if one of the following two conditions is satisfied:

\((H_4)\) \(P(t) > 1\) eventually, and \(0 < P_2 \leq P_1 < P_2 < +\infty,\)

\((H_5)\) \(P(t) < -1\) eventually, and \(-\infty < P_2 \leq P_1 < -1,\)

where \(P_1 = \limsup_{t \to \infty} P(t), P_2 = \liminf_{t \to \infty} P(t),\) then (1.1) has a nonoscillatory solution.

In 2007, Zhou [3] studies the existence of nonoscillatory solution of the following second-order nonlinear differential equation.

\[
[r(t)(x(t) + P(t)x(t - \tau))]' + \sum_{i=1}^{m} Q_i(t)f_i(x(t - \sigma_i)) = 0, \quad t \geq t_0, \quad (E')
\]

where \(f_i \in \mathcal{C}(R, R) (i = 1, 2, \ldots, m).\) By using Krasnoselskii’s fixed point theorem, they proved the following theorem.

**Theorem B** ([3, Theorem 1]). Assume that there exist nonnegative constants \(c_1\) and \(c_2\) such that \(c_1 + c_2 < 1, -c_2 \leq P(t) \leq c_1.\) Further, assume that

\[
\int_{t_0}^{\infty} \int_{t_0}^{t} \frac{|Q_i(t)|}{r(s)} ds dt \leq \infty, \quad i = 1, 2, \ldots, m. \quad (1.2)
\]

Then (1.1) has a bounded nonoscillatory solution.
In this paper, by using Krasnoselskii’s fixed point theorem and some new techniques, we obtain some sufficient conditions for the existence of a nonoscillatory solution of (1.1) for general $Q_i(t)$ ($i = 1, 2, \ldots, m$) which means that we allow oscillatory $Q_i(t)$ ($i = 1, 2, \ldots, m$). Meanwhile, we extend the main results of [2, 3].

### 2. Main Result

The following fixed point theorem will be used to prove the main results in this section.

**Lemma 2.1** (see [3, Krasnoselskii’s fixed point theorem]). Let $X$ be a Banach space, let $\Omega$ be a bounded closed convex subset of $X$, and let $S_1, S_2$ be maps of $\Omega$ into $X$ such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If $S_1$ is a contraction and $S_2$ is completely continuous, then the equation

$$S_1x + S_2x = x$$

has a solution in $\Omega$.

**Theorem 2.2.** Assume that there exist nonnegative constants $c_1$ and $c_2$ such that $c_1 + c_2 < 1$, $-1 < -c_2 \leq P(t) \leq c_1 < 1$. Further, assume that

$$\int_{t_0}^{\infty} \int_{t_0}^{t} \frac{s^{n-2}|Q_i(t)|}{r(s)} ds dt < \infty, \quad i = 1, 2, \ldots, m.$$  

(2.2)

Then (1.1) has a bounded nonoscillatory solution.

**Proof.** By interchanging the order of integral, we note that (2.2) is equivalent to

$$\int_{t_0}^{\infty} s^{n-2} \int_{t_0}^{\infty} \frac{|Q_i(t)|}{r(s)} ds dt < \infty, \quad i = 1, 2, \ldots, m.$$  

(2.3)

By (2.3), we choose $T > t_0$ sufficiently large such that

$$\frac{1}{(n-2)!} \int_{t_0}^{T} s^{n-2} \int_{s}^{\infty} \frac{M}{r(s)} \sum_{i=1}^{m} |Q_i(u)| du ds < \frac{1-c_1-c_2}{4},$$

(2.4)

where $M = \max_{(1-c_1-c_2)/2 \leq x \leq 1} \{|f_i(x)| : 1 \leq i \leq m\}$.

Let $C([t_0, \infty), R)$ be the set of all continuous functions with the norm $\|x\| = \sup_{t \geq t_0} |x(t)| < \infty$. Then $C([t_0, \infty), R)$ is a Banach space. We define a bounded, closed, and convex subset $\Omega$ of $C([t_0, \infty), R)$ as follows:

$$\Omega = \left\{ x = x(t) \in C([t_0, \infty), R) : \frac{1-c_1-c_2}{2} \leq x(t) \leq 1, \ t \geq t_0 \right\}.$$  

(2.5)
Define two maps $S_1$ and $S_2 : \Omega \rightarrow C([t_0, \infty), R)$ as follows:

\[
(S_1 x)(t) = \begin{cases} 
\frac{3 + c_1 - 3c_2}{4} - P(t)x(t - \tau), & t \geq T, \\
(S_1 x)(T), & t_0 \leq t \leq T,
\end{cases}
\]

\[
(S_2 x)(t) = \begin{cases} 
\frac{(-1)^{n-1}}{(n-2)!} \int_t^\infty (s-t)^{n-2} \int_s^\infty \frac{1}{r(s)} \left( \sum_{i=1}^m |Q_i(u)f_i(y(u - \sigma_i))| \right) du ds, & t \geq T, \\
(S_2 x)(T), & t_0 \leq t \leq T.
\end{cases}
\] (2.6)

(i) We shall show that for any $x, y \in \Omega$, $S_1 x + S_2 y \in \Omega$.
In fact, $x, y \in \Omega$, and $t \geq T$, we get

\[
(S_1 x)(t) + (S_2 y)(t) \leq \frac{3 + c_1 - 3c_2}{4} - P(t)x(t - \tau) + \frac{1}{(n-2)!} \int_t^\infty (s-t)^{n-2} \int_s^\infty \frac{1}{r(s)} \left( \sum_{i=1}^m |Q_i(u)f_i(y(u - \sigma_i))| \right) du ds
\]
\[
\leq \frac{3 + c_1 - 3c_2}{4} + c_2 + \frac{1}{(n-2)!} \int_t^\infty \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du ds
\]
\[
\leq \frac{3 + c_1 - 3c_2}{4} + c_2 + \frac{1 - c_1 - c_2}{4} = 1.
\] (2.7)

Furthermore, we have

\[
(S_1 x)(t) + (S_2 y)(t) \geq \frac{3 + c_1 - 3c_2}{4} - P(t)x(t - \tau) - \frac{1}{(n-2)!} \int_t^\infty s^{n-2} \int_s^\infty \frac{1}{r(s)} \left( \sum_{i=1}^m |Q_i(u)f_i(y(u - \sigma_i))| \right) du ds
\]
\[
\geq \frac{3 + c_1 - 3c_2}{4} - c_1 - \frac{1}{(n-2)!} \int_t^\infty s^{n-2} \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du ds
\]
\[
\geq \frac{3 + c_1 - 3c_2}{4} - c_1 - \frac{1 - c_1 - c_2}{4} = \frac{1 - c_1 - c_2}{2}.
\] (2.8)

Hence,

\[
\frac{1 - c_1 - c_2}{2} \leq (S_1 x)(t) + (S_2 y)(t) \leq 1, \quad \text{for } t \geq t_0.
\] (2.9)

Thus, we have proved that $S_1 x + S_2 y \in \Omega$ for any $x, y \in \Omega$.

(ii) We shall show that $S_1$ is a contraction mapping on $\Omega$.
In fact, for $x, y \in \Omega$ and $t \geq T$, we have

\[
| (S_1 x)(t) - (S_1 y)(t) | \leq |P(t)||x(t - \tau) - y(t - \tau)| \leq c_0\|x - y\|,
\] (2.10)
where \( c_0 = \max\{c_1, c_2\} \). This implies that

\[
\|S_1x - S_1y\| \leq c_0\|x - y\|. \tag{2.11}
\]

Since \( 0 < c_0 < 1 \), we conclude that \( S_1 \) is a contraction mapping on \( \Omega \).

(iii) We now show that \( S_2 \) is completely continuous.

First, we will show that \( S_2 \) is continuous. Let \( x_k = x_k(t) \in \Omega \) be such that \( x_k(t) \to x(t) \) as \( k \to \infty \). Because \( \Omega \) is closed, \( x = x(t) \in \Omega \). For \( t \geq T \), we have

\[
|\langle S_2x_k \rangle(t) - \langle S_2x \rangle(t)|
\leq \frac{1}{(n-2)!} \int_t^\infty s^{n-2} \int_s^\infty \frac{1}{r(s)} \left( \sum_{i=1}^m |Q_i(u)||f_i(x_k(u - \sigma_i)) - f_i(x(u - \sigma_i))| \right) du \, ds \tag{2.12}
\]

Since \( |f_i(x_k(t - \sigma_i)) - f_i(x(t - \sigma_i))| \to 0 \) as \( k \to \infty \) for \( i = 1, 2, \ldots, m \), by applying the Lebesgue dominated convergence theorem, we conclude that \( \lim_{k \to \infty} \|\langle S_2x_k \rangle(t) - \langle S_2x \rangle(t)\| = 0 \). This means that \( S_2 \) is continuous.

Next, we show that \( S_2 \Omega \) is relatively compact. It suffices to show that the family of functions \( \{S_2x : x \in \Omega\} \) is uniformly bounded and equicontinuous on \( [t_0, \infty) \). The uniform boundedness is obvious. For the equicontinuity, according to Levitan’s result [4], we only need to show that, for any given \( \varepsilon > 0 \), \( [T, \infty) \) can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have change of amplitude less than \( \varepsilon \). By (2.3), for any \( \varepsilon > 0 \), take \( T^* \geq T \) large enough so that

\[
\frac{1}{(n-2)!} \int_{t_0}^\infty s^{n-2} \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du \, ds < \frac{\varepsilon}{2}. \tag{2.13}
\]

Then, for \( x \in \Omega \), \( t_2 \geq t_1 \geq T^* \),

\[
|\langle S_2x \rangle(t_2) - \langle S_2x \rangle(t_1)| \leq \frac{1}{(n-2)!} \int_{t_1}^{t_2} s^{n-2} \int_s^\infty \frac{1}{r(s)} \left( \sum_{i=1}^m |Q_i(u)||f_i(x(u - \sigma_i))| \right) du \, ds
+ \frac{1}{(n-2)!} \int_{t_1}^{t_2} s^{n-2} \int_s^\infty \frac{1}{r(s)} \left( \sum_{i=1}^m |Q_i(u)||f_i(x(u - \sigma_i))| \right) du \, ds \leq \frac{1}{(n-2)!} \int_{t_1}^{t_2} s^{n-2} \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du \, ds \tag{2.14}
\]

\[
+ \frac{1}{(n-2)!} \int_{t_1}^{t_2} s^{n-2} \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du \, ds < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
For $x \in \Omega$, $T \leq t_1 < t_2 \leq T^* + 1$,

$$|(S_2x)(t_2) - (S_2x)(t_1)|$$

$$\leq \frac{1}{(n-2)!} \int_{t_1}^{t_2} (s-t_1)^{n-2} \int_s^\infty \frac{1}{r(s)} \left( \sum_{i=1}^m Q_i(u)f_i(x(u-\sigma_i)) \right) du \, ds$$

$$+ \frac{1}{(n-2)!} \int_{t_1}^{t_2} (s-t_2)^{n-2} - (s-t_1)^{n-2} \int_s^\infty \frac{1}{r(s)} \left( \sum_{i=1}^m Q_i(u)f_i(x(u-\sigma_i)) \right) du \, ds$$

$$\leq \frac{1}{(n-2)!} \int_{t_1}^{t_2} \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du |s^{n-2}\int_s^\infty$$

$$+ \frac{1}{(n-3)!} (t_2-t_1) \int_{t_1}^{t_2} (s-\xi)^{n-3} \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du ds$$

$$\leq \frac{1}{(n-2)!} \int_{t_1}^{t_2} \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du ds$$

$$+ \frac{1}{(n-3)!} (t_2-t_1) \int_{t_1}^{t_2} \int_s^T \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du ds,$$

(2.15)

where $t_1 < \xi < t_2$.

Then there exists $\delta > 0$ such that

$$|(S_2x)(t_2) - (S_2x)(t_1)| < \epsilon, \quad \text{if} \ 0 < t_2 - t_1 < \delta.$$  (2.16)

For any $x \in \Omega$, $t_0 \leq t_1 < t_2 \leq T$, it is easy to see that

$$|(S_2x)(t_2) - (S_2x)(t_1)| = 0 < \epsilon.$$  (2.17)

Therefore, $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$, and hence $S_2\Omega$ is relatively compact. By Lemma 2.1, there is $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. It is easy to see that $x_0(t)$ is a nonoscillatory solution of (1.1). The proof is complete. \qed

**Theorem 2.3.** Assume that $-\infty < c_1 \leq P(t) \leq c_2 < -1$ and (2.2) holds. Then (1.1) has a bounded nonoscillatory solution.

**Proof.** We choose positive constants $M_1$, $M_2$, $\alpha$ such that $-c_1M_1 < \alpha < (-c_2 - 1)M_2$. $c = \min\{|(\alpha + M_1c_1)c_2/c_1, ((-c_2 - 1)M_2) - \alpha\}$. Choosing $T > t_0$ sufficiently large such that

$$\frac{1}{(n-2)!} \int_{t_1}^{t_2} \int_s^\infty \frac{M'}{r(s)} \sum_{i=1}^m |Q_i(u)| du ds < c,$$

(2.18)

where $M' = \max_{M_1S \leq M_2} \{|f_i(x)| : 1 \leq i \leq m\}$. 


Let $C([t_0, \infty), R)$ be the set as in the proof of Theorem 2.2. We define a bounded, closed, and convex subset $\Omega$ of $C([t_0, \infty), R)$ as follows:

$$\Omega = \{ x = x(t) \in C([t_0, \infty), R) : M_1 \leq x(t) \leq M_2, t \geq t_0 \}. \quad (2.19)$$

Define two maps $S_1$ and $S_2 : \Omega \to C([t_0, \infty), R)$ as follows:

$$(S_1x)(t) = \begin{cases} \frac{-\alpha}{P(t + \tau)} x(t + \tau), & t \geq T, \\ (S_1x)(T), & t_0 \leq t \leq T, \end{cases}$$

$$(S_2x)(t) = \begin{cases} \frac{(1)^{n-1}}{(n-2)!} \frac{1}{P(t + \tau)} \int_{t+\tau}^{\infty} (s-t-\tau)^{n-2} \int_{s}^{\infty} \frac{1}{r(s)} \left( \sum_{i=1}^{m} Q_i(u)f_i(x(u-\sigma_i)) \right) du \, ds, & t \geq T, \\ (S_2x)(T), & t_0 \leq t \leq T. \end{cases} \quad (2.20)$$

(i) We shall show that for any $x, y \in \Omega$, $S_1x + S_2y \in \Omega$.
In fact, for every $x, y \in \Omega$, and $t \geq T$, we get

$$(S_1x)(t) + (S_2y)(t) \geq \frac{-\alpha}{c_1} + \frac{c}{c_2} \geq M_1,$$

$$(S_1x)(t) + (S_2y)(t) \leq \frac{-\alpha}{c_2} - \frac{M_2}{c_2} - \frac{c}{c_2} \leq M_2. \quad (2.21)$$

Thus, we have proved that $S_1x + S_2y \in \Omega$. Since $-\infty < c_1 \leq P(t) \leq c_2 < -1$, we get that $S_1$ is a contraction mapping. We also can prove that $\{ S_2x : x \in \Omega \}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$, and hence $S_2\Omega$ is relatively compact. So by Lemma 2.1, there is $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. That is,

$$x_0(t) = \frac{-\alpha}{P(t + \tau)} x_0(t + \tau) - \frac{1}{P(t + \tau)} \int_{t+\tau}^{\infty} (s-t-\tau)^{n-2} \int_{s}^{\infty} \frac{1}{r(s)} \left( \sum_{i=1}^{m} Q_i(u)f_i(x_0(u-\sigma_i)) \right) du \, ds. \quad (2.22)$$

It is easy to see that $x_0(t)$ is a bounded nonoscillatory solution of (1.1).

The proof is complete. $\square$

**Theorem 2.4.** Assume that $1 < c_1 \leq P(t) \leq c_2 < +\infty$ and (2.2) holds. Then (1.1) has a bounded nonoscillatory solution.

**Proof.** We choose positive constants $M_3, M_4, \alpha$ such that $M_4 + c_2 M_3 < \alpha < c_1 M_4, c = \min\{\alpha - M_4 - c_2 M_3, c_1 M_4 - \alpha\}$. Choosing $T > t_0$ sufficiently large such that
\[
\frac{1}{(n-2)!} \int_0^\infty s^{n-2} \int_s^\infty \frac{M''}{r(s)} \sum_{i=1}^m |Q_i(u)| du \, ds < c, \tag{2.23}
\]

where \( M'' = \max_{M_i \leq x \leq M_4} \{|f_i(x)| : 1 \leq i \leq m\} \).

Let \( C([t_0, \infty), R) \) be the set as in the proof of Theorem 2.2. We define a bounded, closed, and convex subset \( \Omega \) of \( C([t_0, \infty), R) \) as follows:

\[
\Omega = \{ x = x(t) \in C([t_0, \infty), R) : M_3 \leq x(t) \leq M_4, t \geq t_0 \}. \tag{2.24}
\]

Define two maps \( S_1 \) and \( S_2 : \Omega \to C([t_0, \infty), R) \) as follows:

\[
(S_1 x)(t) = \begin{cases} 
\frac{\alpha}{P(t+\tau)} \frac{x(t+\tau)}{P(t+\tau)}', & t \geq T, \\
(S_1 x)(T), & t_0 \leq t \leq T,
\end{cases}
\]

\[
(S_2 x)(t) = \begin{cases} 
\frac{(-1)^{n-1}}{(n-2)!} \frac{1}{P(t+\tau)} \int_{t+\tau}^\infty (s-t-\tau)^{n-2} \int_s^\infty \frac{1}{r(s)} \left( \sum_{i=1}^m Q_i(u) f_i(x(u-\sigma_i)) \right) du \, ds, & t \geq T, \\
(S_2 x)(T), & t_0 \leq t \leq T.
\end{cases} \tag{2.25}
\]

(i) We shall show that for any \( x, y \in \Omega \), \( S_1 x + S_2 y \in \Omega \).

In fact, for every \( x, y \in \Omega \) and \( t \geq T \), we get

\[
(S_1 x)(t) + (S_2 y)(t) \geq \frac{1}{c_2} (\alpha - M_4 - c) \geq M_3, \quad (S_1 x)(t) + (S_2 y)(t) \leq \frac{\alpha}{c_1} + \frac{c}{c_1} \leq M_4. \tag{2.26}
\]

Thus, we have proved that \( S_1 x + S_2 y \in \Omega \). Since \( 1 < c_1 \leq P(t) \leq c_2 < +\infty \), we get \( S_1 \) is a contraction mapping. We also can prove that \( \{S_2 x : x \in \Omega\} \) is uniformly bounded and equicontinuous on \([t_0, \infty)\), and, hence, \( S_2 \Omega \) is relatively compact. So by Lemma 2.1, there is \( x_0 \in \Omega \) such that \( S_1 x_0 + S_2 x_0 = x_0 \). That is,

\[
x_0(t) = \frac{\alpha}{P(t+\tau)} \frac{x_0(t+\tau)}{P(t+\tau)}' + \frac{(-1)^{n-1}}{(n-2)!} \frac{1}{P(t+\tau)} \int_{t+\tau}^\infty (s-t-\tau)^{n-2} \int_s^\infty \frac{1}{r(s)} \left( \sum_{i=1}^m Q_i(u) f_i(x_0(u-\sigma_i)) \right) du \, ds. \tag{2.27}
\]

It is easy to see that \( x_0(t) \) is a bounded nonoscillatory solution of (1.1).

The proof is complete. \( \square \)

Remark 2.5. If we let \( n = 2 \) in Theorem 2.2, we get the Theorem 1 in [3]. In the case where \( n = 2, r(t) \equiv 1 \), Theorem 2.2 improves essentially Theorem 2.2 in [5].

Remark 2.6. The conditions of Theorem 2.4 relaxing the hypotheses \((H_k)\) of Theorem 3 in [2].
Remark 2.7. Theorems 2.3 and 2.4 improve essentially Theorem 3 in [2], we allow that $Q_i(t)$ $(i = 1, 2, \ldots, m)$ are oscillatory.

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