Research Article

A Restriction for Singularities on Collapsing Orbifolds

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Every point p in an orbifold X has a neighborhood that is homeomorphic to $G_p \backslash B_r(0)$, where G_p is a finite group acting on $B_r(0) \subset \mathbb{R}^n$, so that $G_p(0) = 0$. Assume X is a Riemannian orbifold with isolated singularities that is collapsing, that is, X admits a sequence of metrics g_i with uniformly bounded curvature, so that, for any $x \in X$, the volume of $B_1(x)$, with respect to the metric g_i , goes to 0 as $i \rightarrow \infty$. For such X , we prove that $|G_p| \leq (2\pi/0.47)^{n(n-1)}$ for all singularities $p \in X$.

1. Introduction

An n -dimensional Riemannian orbifold, X , is a metric space so that the following is true: for any $x \in X$, there exists $r = r(x) > 0$ and a Riemannian metric $\tilde{g}_x$ on $B_{2r}(0) \subset \mathbb{R}^n$, a finite group G_x (the isotropy group) acting on $(B_r(0), \tilde{g}_x)$ by isometries, so that $G_x(0) = 0$, and there is an isometry $\iota_x : B_r(x) \rightarrow G_x \backslash B_r(0)$ with $\iota_x(x) = 0$ (see [1]). We call $x \in X$ a *regular point* if $|G_x| = 1$; otherwise, x is a *singular point*. We say the curvature of X satisfies

$$|K_X| \leq \kappa^2, \quad (1.1)$$

if the sectional curvature K of every $(B_r(0), \tilde{g}_x)$ above satisfies $|K| \leq \kappa^2$. We say X is *collapsing*, if X admits a sequence of metrics, g_i , with uniformly bounded curvature, so that, for any $x \in X$,

$$\lim_{i \rightarrow \infty} \text{Vol}_{g_i}(B_1(x)) = 0. \quad (1.2)$$

As an example, consider the standard $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ action on the sphere S^2 .

The quotient orbifold $X_m = \mathbb{Z}_m \backslash S^2$ will be arbitrarily collapsed when $m \rightarrow \infty$ (see Figure 1). However, for any fixed m , X_m can be collapsed only to a certain degree; it does not support a sequence of collapsing metrics. In fact, for each one of the two singularities

Figure 1

on X_m , there is a neighborhood that is isometric to $\mathbb{Z}_m \setminus \mathbb{R}^2$, where $\mathbb{R}^2$ is equipped with some $\mathbb{Z}_m$ invariant metric. Therefore if g_i is a collapsing sequence of metrics on X_m , we get a corresponding sequence $\tilde{g}_i$ of pullback metrics on S^2 ; every $\tilde{g}_i$ is smooth. Observe $\text{Vol}(S^2, \tilde{g}_i) = m \text{Vol}(X_m, g_i)$, where m is fixed and $\lim_{i \rightarrow \infty} \text{Vol}(X_m, g_i) = 0$, thus $\lim_{i \rightarrow \infty} \text{Vol}(S^2, \tilde{g}_i) = 0$. If the diameter of (X_m, g_i) stays bounded, we immediately get a contradiction to the Gauss-Bonnet theorem; in general, we can use the result in [2] to conclude that S^2 admits an F -structure, in particular the Euler characteristic $\chi(S^2)$ vanishes—this is a contradiction since clearly $\chi(S^2) = 2$.

On the other hand, consider the double of a 2-dimensional rectangle. Clearly it admits a flat metric, thus we obtain a sequence of collapsing metrics by rescale. Notice, in this example, for each of the four singularities, the isotropy group G_x has order 2, a quite small number.

Intuitively, these examples suggest that when an orbifold is collapsing, a conelike singularity cannot be too “sharp,” that is, there should be some bound in $|G_x|$. The main result of this paper is as follows.

Theorem 1.1. *Assume X is a compact, collapsing orbifold, $p \in X$ is an isolated singularity. Then $|G_p| \leq (2\pi/0.47)^{n(n-1)}$.*

If X has an isolated singularity p , then the dimension of X must be even, and $G_p \subset SO(n)$. Theorem 1.1 fails if we drop the requirement that x is an *isolated* singularity; for example, we can take *any* orbifold X' and let $X = X' \times S^1$; by shrinking the S^1 factor, we see X is collapsing while there is no restriction on singularities of X' . The bound $|G_p| \leq (2\pi/0.47)^{n(n-1)}$ has its root in the Bieberbach theorem of crystallographic groups and Gromov’s almost flat manifold theorem.

Clearly, Theorem 1.1 is a corollary of the following.

Theorem 1.2. *For any $L > 0$, there is $\epsilon = \epsilon(n, L)$ so that if X is an orbifold with all singularities $q \in X$ satisfying $|G_q| < L$, $\text{Vol}(B_1(q)) < \epsilon$, then $|G_p| \leq (2\pi/0.47)^{n(n-1)}$ for any isolated singularity $p \in X$.*

Remark 1.3. The bound $|G_p| \leq (2\pi/0.47)^{n(n-1)}$ in Theorem 1.1 is not sharp. When $n = 2$, it is not hard to see that either $|G_p| = 2$ or X is a flat orbifold. Therefore by Polyá and Niggli’s classification of crystallographic groups on $\mathbb{R}^2$ [3, page 105] or [4, page 228], we actually have $|G_p| \leq 6$ for collapsing 2 orbifolds.

A *nilmanifold*, $\Gamma \setminus N$, is the quotient of the (left) action of a discrete, uniform subgroup $\Gamma \subset N$, on a simply connected nilpotent Lie group N . Left invariant vector fields (LIVFs) can be defined on $\Gamma \setminus N$. An affine diffeomorphism of $\Gamma \setminus N$ is a diffeomorphism that maps any local LIVF to some local LIVF. In general, a right invariant vector field (RIVF) cannot be defined globally in $\Gamma \setminus N$, unless this vector field is in the center of the Lie algebra of N . However, the right invariant vector fields, not the left invariant ones, are Killing fields of left invariant metrics on $\Gamma \setminus N$. An *infranil orbifold* is the quotient of a nilmanifold by the action of a finite group H of affine diffeomorphisms. If the action H is free, we get an *infranil manifold*.

In our previous work, [5], we generalized the Cheeger-Fukaya-Gromov nilpotent Killing structure [6] and the Cheeger-Gromov F-structure, [2, 7], to collapsing orbifolds. In particular, sufficiently collapsed X can be decomposed into a union of *orbits*. Each orbit $\mathcal{O}_p$ is the orbit of the action of a sheaf $\mathfrak{n}$ of nilpotent Lie algebras, which comes from local RIVFs on a nilmanifold fibration in the frame bundle FX . Therefore every $\mathcal{O}_p$ is an infranil orbifold. The proof of Theorem 1.2 is based on the relation between singularities on X and singularities within an orbit $\mathcal{O}_p$ in X , as well as the nilmanifold fibration on FX .

X is called *almost flat*, if

$$\sup |K_X|^{1/2} \cdot \text{Diam } X \leq \delta_n, \quad (1.3)$$

where $\text{Diam } X$ is the diameter of X , δ_n is a small constant that depends only on n . In [8], Gromov proved that an almost flat manifold M has a finite, normal covering space $\widetilde{M} = \Gamma \backslash N$ that is a nilmanifold. Subsequently, Ruh [9] proved that M is diffeomorphic to $\Lambda \backslash N$, where $\Lambda \supset \Gamma$ is a discrete subgroup in the affine transformation group of N . In [10], Ghanaat generalized this to an almost flat orbifold X , under the assumption that X is *good* in the sense of Thurston [1], that is, X is the *global* quotient of a simply connected manifold M . There are examples of orbifolds that are not good, see [1]. In fact, without much effort, one can remove the assumption that X is good.

Proposition 1.4. *If X is an almost flat orbifold, then X is an infranil orbifold.*

Precisely, there is a nilmanifold $\widetilde{X} = \Gamma \backslash N$, a finite group H acting on $\widetilde{X}$ by affine diffeomorphism, so that X is diffeomorphic to $H \backslash \widetilde{X}$. The order of H is bounded by $c_n \leq (2\pi/0.47)^{n(n-1)/2}$. Moreover, there is a sequence of metrics g_j so that $\text{Diam}(X, g_j) \rightarrow 0$.

The proof is almost the same as [11, 12]; the only difference is one must replace the exponential map by the *develop map* (see [5, 13]) and modify the definition of *Gromov product* in [11] accordingly.

The proof of Theorem 1.2 does not depend on Proposition 1.4. On the other hand, Proposition 1.4 implies Theorem 1.2 for almost flat orbifolds immediately, even without the assumption that the singularities are isolated.

Remark 1.5. If $p \in X$ is an isolated singularity, then, near p , X is homeomorphic to (and in the metric sense, close to) a metric cone over a space form of dimension $n - 1$. When $n = 4$, the $4 - 1 = 3$ -dimensional space forms were first classified by Threlfall and Seifert, they used the fact that $SO(4)$ is locally isomorphic to $SO(3) \times SO(3)$; [3, chapter 7] or [4] for details.

Remark 1.6. By the work of Anderson, Gao, Nakajima, Tian, Yang, and others, orbifolds with discrete singularities appear naturally as Gromov-Hausdorff limits of noncollapsing Einstein metrics with a uniform $L^{n/2}$ curvature bound; see [14] for a recent survey. In particular, for Kähler-Einstein metrics, there is a complex structure on the limit X .

2. Proof of Theorem 1.2

If X is an infranil orbifold, then it is easy to obtain the bound in Theorem 1.1. Since the proof contains some ideas for the general case, we give full details.

Lemma 2.1. *Assume X is an infranil orbifold. Then $|G_X| \leq (2\pi/0.47)^{n(n-1)/2}$.*

Proof. Assume $X = \Lambda \backslash N$, where N is a simply connected nilpotent Lie group, Λ is a discrete group of affine diffeomorphisms on N so that $X = \Lambda \backslash N$ is compact. If N is

abelian, then X is a flat orbifold, Λ is a discrete group of isometries on $N = \mathbb{R}^n$ that acts properly discontinuously. So the conclusion follows from (the proof of) Bieberbach's theorem on crystallographic groups. In fact, it is well known that the maximal rotational angle of any $\lambda \in \Lambda$ is either 0 or at least $1/2$. Thus the bound comes from a standard packing argument; notice $n(n-1)/2 = \dim SO(n)$ and the bi-invariant metric on $SO(n)$ has positive curvature.

We prove the general case by induction on dimension of X . Remember that Λ contains a normal subgroup Γ of finite index, so that Γ is a uniform, discrete subgroup of N and X is the quotient of the Λ/Γ action on the nilmanifold $\tilde{X} = \Gamma \backslash N$. Clearly G_x embeds in Λ/Γ , that is, $G_x = \{\bar{\lambda} \in \Lambda/\Gamma \mid \bar{\lambda}\tilde{x} = \tilde{x}\}$; here we choose a point $\tilde{x}$ in $\tilde{X} = \Gamma \backslash N$ that projects to $x \in \Lambda \backslash N$.

Let C be the center of N , then C is connected, of positive dimension. Since any $\lambda \in \Lambda$ is affine diffeomorphism, λ moves a C -coset in N to a C -coset. Therefore Λ/Γ acts on the nilmanifold $\tilde{X}^* = (\Gamma/(\Gamma \cap C)) \backslash (N/C)$, the quotient X^* is an infranil orbifold of lower dimension. Let $\pi : \tilde{X} \rightarrow \tilde{X}^*$ be the projection, and assume $\pi(\tilde{x}) = \tilde{x}^*$. Thus we have a homomorphism

$$h : G_x \longrightarrow G_{x^*}. \quad (2.1)$$

$\tilde{X}$ is a torus bundle over $\tilde{X}^*$, the fiber is $T = (\Gamma \cap C) \backslash C$. Assume $\bar{\lambda} \in \Lambda/\Gamma$ is in $\text{Ker } h$, the kernel of h , then $\bar{\lambda}$ fixes every T fiber in $\tilde{X}$. If, in addition, $\bar{\lambda}$ fixes *every point* in the T fiber passing through $\tilde{x}$, we claim $\bar{\lambda}$ must be identity. In fact, on N we have $\lambda(z) = a \cdot A(z)$, where $a \in N$ and A is a Lie group automorphism of N ; if $\bar{\lambda}$ fixes every point in *one* T fiber, then A is identity on the center $C \subset N$. This implies that $\bar{\lambda}$ is a *translation* on every T fiber. Since $\bar{\lambda}$ is of finite order and fixes every point in *one* T -fiber, $\bar{\lambda}$ must be identity. Therefore any element $\bar{\lambda} \in \text{Ker } h$ is decided by its restriction on the T fiber passing through x ; so $\text{Ker } h$ is isomorphic to a finite group of affine diffeomorphisms on T that fixes $\tilde{x} \in T$, thus $|\text{Ker } h|$ can be bounded by Bieberbach's theorem. Since

$$|G_x| \leq |G_{x^*}| \cdot |\text{Ker } h|, \quad (2.2)$$

the conclusion follows by induction. $\square$

In [5], the existence of nilpotent Killing structure of Cheeger-Fukaya-Gromov [6] is generalized to sufficiently collapsed orbifolds. We briefly review this construction.

As in the manifold case, one can define the *frame bundle* FX of an orbifold X . If $B_r(x) \subset X$ is isometric to $G_x \backslash B_r(0)$, where G_x is a finite group acting on $B_r(0) \subset \mathbb{R}^n$, then locally FX is $G_x \backslash FB(0, r)$, where $FB(0, r)$ is the orthonormal frame bundle over $B_r(0)$, and G_x acts on $FB(0, r)$ by differential, that is, $\tau \in G_x$ moves a frame u to $\tau_* u$. Therefore FX is a manifold; strictly speaking, FX is not a fiber bundle. Let $\pi : FX \rightarrow X$ be the projection.

Moreover, there is a natural $SO(n)$ action on FX ; on the frames over regular points, this $SO(n)$ action is the same one as in the manifold case; however, at (the frames over) singular points, this action is *not free*. As in the work of Fukaya [15], see also [5], any Gromov-Hausdorff limit Y of a collapsing sequence FX_i is a manifold. Following [6], for sufficiently collapsing orbifolds, *locally* we have an $SO(n)$ -equivariant fibration

$$Z \longrightarrow FX \xrightarrow{f} Y, \quad (2.3)$$

where the fiber Z is a nilmanifold, Y is a smooth manifold with controlled geometry.

As in [6], we can put a *canonical* affine structure on the Z fibers, that is, a canonical way to construct a diffeomorphism from a fiber Z to the nilmanifold $\Gamma \backslash N$. In particular, there is a sheaf $\mathfrak{n}$, of a nilpotent Lie algebra of vector fields on FX . Sections of $\mathfrak{n}$ are local right invariant vector fields on the nilmanifold fibers Z . By integrating $\mathfrak{n}$, we get a *local* action of a simply connected nilpotent Lie group, $\mathfrak{N}$, on FX . Therefore we also call a Z fiber an *orbit*, and we can write $Z = \tilde{\mathcal{O}}$.

The fibration $f : FX \rightarrow Y$ is $SO(n)$ -equivariant, so any $Q \in SO(n)$ moves a Z fiber to a (perhaps another) Z fiber by affine diffeomorphism. Moreover, the $SO(n)$ action on $\mathfrak{n}$ is *locally trivial*, that is, if $A \in \mathfrak{so}(n)$ is sufficiently small, then $e^A \in SO(n)$ moves a section, $\mathfrak{n}(U)$, of $\mathfrak{n}$ on any open set $U \subset FX$, to itself (over $U \cap Ue^A$) ([6, Proposition 4.3]). In particular, the sheaf $\mathfrak{n}$ induces a sheaf, which we also denote by $\mathfrak{n}$, on the orbifold X away from the singular points. An orbit $\tilde{\mathcal{O}}_{\tilde{q}} = Z$ on X projects down to an *orbit* $\mathcal{O}_q$ on X .

Assume $\tilde{q} \in FX$ is any frame over $q \in X$. Let

$$I(q) = \{Q \in SO(n) \mid Z_{\tilde{q}}Q = Z_{\tilde{q}}\} \quad (2.4)$$

be the the isotropy group of an orbit $Z_{\tilde{q}} = \mathcal{O}_{\tilde{q}} \subset FX$. We will simply write $I(q)$ by I . Let I_0 be the identity component of I . It can be shown that, restricted on $Z = \tilde{\mathcal{O}}_{\tilde{q}}$, the action of I_0 is identical to the action of a torus, and the Lie algebra of this torus, I_0 , is in the center of $\mathfrak{n}$ (see [5, 6] for more details). Consider the nilmanifold

$$\tilde{\mathcal{O}}_{\tilde{q}} = \frac{\tilde{\mathcal{O}}_{\tilde{q}}}{I_0}. \quad (2.5)$$

Therefore $Z_{\tilde{q}} = \tilde{\mathcal{O}}_{\tilde{q}}$ is a torus bundle over $\tilde{\mathcal{O}}_{\tilde{q}}$. Notice, on $Z = \tilde{\mathcal{O}}_{\tilde{q}}$, I moves I_0 fibers to I_0 fibers, thus the orbit $\mathcal{O}_q$ is the quotient of $\tilde{\mathcal{O}}_{\tilde{q}}$ by the action of the finite group I/I_0 . Therefore $\mathcal{O}_q$ is an infranil orbifold. In particular, the singularities within $\mathcal{O}_q$ satisfy the bound in Lemma 2.1.

It is important to remark that the above structure is not trivial.

Lemma 2.2. *Let L be any integer. Then there is $\epsilon = \epsilon(n, L)$, so that if X is an orbifold with $|G_x| \leq L$, $\text{Vol}(B_1(x)) \leq \epsilon$ for all $x \in X$, then every $\mathfrak{n}$ -orbit $\mathcal{O}$ on X is of positive dimension.*

Proof (sketch). For any unit vector $A \in \mathfrak{so}(n)$, the bound in $|G_x|$ implies that e^{tA} does not have fixed point in $\tilde{\mathcal{O}}_{\tilde{q}}$ unless $t = 0$ or $|t| > cL^{-1}$. However, for sufficiently collapsed orbifolds, there is a vector B in the center of $\mathfrak{n}$ so that B generates a closed loop in $\tilde{\mathcal{O}}_{\tilde{q}}$ that is shorter than cL^{-1} , therefore B cannot be in the Lie algebra of I_0 , which is in both $\mathfrak{so}(n)$ and the center of $\mathfrak{n}$. Thus the orbit $\tilde{\mathcal{O}}_{\tilde{q}}$ is not contained in a single $SO(n)$ orbit in FX , so $\mathcal{O}_q$ is of positive dimension in $X = FX/SO(n)$ (see [5] for more details). $\square$

Proof of Theorem 1.2. Assume $p \in X$ is an isolated singular point, $\tilde{p} \in \pi^{-1}(p)$ is in FX . $Z_{\tilde{p}} = \tilde{\mathcal{O}}_{\tilde{p}}$ is the fiber that projects to $\mathcal{O}_p$. Let $I(p)$, I_0 , $\tilde{\mathcal{O}}_{\tilde{q}}$ be as above. Let

$$K_p = \{Q \in SO(n) \mid \tilde{p}Q = \tilde{p}\}. \quad (2.6)$$

Thus K_p is a subgroup of I , and $|K_p| = |G_p|$. Let

$$K_p^- = \{Q \in K_p \mid (\tilde{p}'I_0)Q = (\tilde{p}'I_0) \ \forall \ \tilde{p}' \in Z_{\tilde{p}}\}. \quad (2.7)$$

Thus K_p^- is a normal subgroup of K_p .

Lemma 2.3. *If $Q \in K_p$ fixes every point in $Z_{\tilde{p}}$, then Q is the identity in $SO(n)$.*

Proof. Potentially Q may fix every point in $Z_{\tilde{p}}$ while moving some points of FX that are outside $Z_{\tilde{p}}$. We will rule out this possibility.

By assumption, p is an isolated singularity. For any Q that is not identity, the connected component of the fixed point set of Q that passes through $\tilde{p}$ must project to p under $\pi : FX \rightarrow X$, because away from $\pi^{-1}(p)$ the $SO(n)$ action is free. Therefore $\pi(Z_{\tilde{p}}) = p$ is a single point in X , and this contradicts the fact that the n -orbits on X are of positive dimension; see Lemma 2.2. $\square$

In particular, we have a *faithful* representation of K_p in the affine group of $Z_{\tilde{p}}$, that is, we can identify K_p with the *restricted action* of the group K_p on $Z_{\tilde{p}}$. Take any $Q \in K_p^-$ that is not identity in $SO(n)$. By definition Q fixes $\tilde{p}$. If Q fixes *every* point in $\tilde{p}I_0$, as in Lemma 2.1, Q is a translation on *every* I_0 fiber; because Q is of finite order and Q moves every I_0 fiber to itself, Q necessarily fixes every point in Z ; thus Q is identity. So Q rotates the tangent plane of $\tilde{p}I_0$ at $\tilde{p}$. Therefore K_p^- is isomorphic to a finite group of affine diffeomorphisms on the torus $\tilde{p}I_0$. By the Bieberbach theorem,

$$|K_p^-| \leq \left(\frac{2\pi}{0.47} \right)^{k(k-1)/2}, \quad k = \dim I_0. \quad (2.8)$$

Recall that Bieberbach's theorem implies that all finite subgroups of $SL(n, \mathbb{Z})$ have a uniform upper bound in order. We have

$$\mathcal{O}_p = \frac{H}{(Z_{\tilde{p}}/I_0)}, \quad (2.9)$$

where $H = I/I_0$ is a finite group. Let H_p be the subgroup of H that fixes p . Now we get an *embedding*

$$\frac{K_p}{K_p^-} \subset H_p. \quad (2.10)$$

By Lemma 2.1,

$$|H_p| \leq \left(\frac{2\pi}{0.47} \right)^{i(i-1)/2}, \quad i = \dim \mathcal{O}_p. \quad (2.11)$$

Thus

$$|G_p| = |K_p| = \left| \frac{K_p}{K_p^-} \right| \cdot |K_p^-| \leq \left(\frac{2\pi}{0.47} \right)^{n(n-1)}. \quad (2.12)$$

$\square$

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