Research Article

Fuzzy Stability of Quadratic Functional Equations in General Cases

Ehsan Movahednia

Department of Mathematics, Damghan University, Damghan, Iran

Correspondence should be addressed to Ehsan Movahednia, ehsar.m@gmail.com

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The aim of this paper is to investigate fuzzy Hyers-Ulam-Rassias stability of the general case of quadratic functional equation $f(ax + by) + f(ax - by) = (a/2)f(x + y) + (a/2)f(x - y) + (2a^2 - a)f(x) + (2b^2 - a)f(y)$, where $a, b \geq 1$ and fixed integers with $a \neq 2b^2$. These functional equations are equivalent. This has been proven by Ulam, 1964.

1. Introduction and Preliminaries

The stability problem of functional equations was raised by Ulam [1] in 1964. In fact he posed the question "Assume that a function satisfies a functional equation approximately according to some convention. Is it then possible to find near this function a function satisfying the equation accurately?" In 1941 Hyers gave a significant partial solution to this problem in his paper [2].

Hyers' result was generalized by Aoki [3] for additive mappings. In 1978, Rassias and Song [4] generalized Hyers' result, a fact which rekindled interest in the field. Such type of stability is now called the Ulam-Hyers-Rassias stability of functional equations. We refer the curious readers for further information on such problems to, for example, [5–7].

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad (1.1)$$

is said to be a simple quadratic functional equation. The first person that investigated the stability of the simple quadratic equation was Skof [8]. He proved that, if f is a mapping from a normed space X into a Banach space Y satisfying

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \epsilon \quad \text{for some } \epsilon > 0, \quad (1.2)$$

then there is a unique simple quadratic function $g : X \rightarrow Y$ such that

$$\|f(x) - g(x)\| \leq \frac{\epsilon}{2}. \quad (1.3)$$

In 1984, Katsaras [9] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a linear space from various points of view [10, 11]. In particular, in 2003, Bag and Samanta [12], following Cheng and Mordeson [13], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [14]. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces.

Recently, considerable attention has been increasing to the problem of fuzzy stability of functional equations. Several fuzzy stability results concerning Cauchy, Jensen, simple quadratic, and cubic functional equations have been investigated [15–18].

Definition 1.1. Let X be a real vector space. A function $N : X \times R \rightarrow [0, 1]$ is called *fuzzy normed* on X if for all $x, y \in X$ and all $s, t \in R$

- (N₁) $N(x, t) = 0$ for $t \leq 0$,
- (N₂) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$,
- (N₃) $N(cx, t) = N(x, t/|c|)$ if $c \neq 0$,
- (N₄) $N(x + y, t + s) \geq \min\{N(x, t), N(y, s)\}$,
- (N₅) $N(x, \cdot)$ is a nondecreasing function of R and $\lim_{t \rightarrow \infty} N(x, t) = 1$,
- (N₆) for $x \neq 0$, $N(x, \cdot)$ is continuous on R ,

the pair (X, N) is called a *fuzzy normed vector space*.

Example 1.2. Let $(X, \|\cdot\|)$ be a normed linear space. One can easily verify that, for each $k > 0$,

$$N_k(x, t) = \begin{cases} \frac{t}{t + k\|x\|}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases} \quad (1.4)$$

defines a fuzzy norm on X .

Definition 1.3. Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converges if there exists an $x \in X$ such that $\lim_{n \rightarrow \infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$, and one denotes it by

$$N - \lim_{n \rightarrow \infty} N(x_n - x, t) = x. \quad (1.5)$$

Definition 1.4. Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\epsilon > 0$ and each $\delta > 0$ there exists an $n_0 \in N$ such that

$$N(x_m - x_n, \delta) > 1 - \epsilon (m, n \geq n_0). \quad (1.6)$$

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

2. Main Results

Let X be a linear space and (Z, N) a fuzzy normed space. Let (Y, N') be a fuzzy Banach space and $f : X \rightarrow Y$ a function satisfying

$$\begin{aligned} N' \left(f(ax + by) + f(ax - by) - \frac{a}{2}f(x + y) - \frac{a}{2}f(x - y) - (2a^2 - a)f(x) \right. \\ \left. - (2b^2 - a)f(y), t + s \right) \geq \min \{ N(\varphi(x), t^q), N(\varphi(y), s^q) \}, \end{aligned} \quad (2.1)$$

where $a, b \geq 1$, $a \neq 2b^2$ and for all $t, s > 0$, $q > 1/2$, such that $\varphi : X \rightarrow Z$ is a function, and

$$\varphi(ax) = \alpha\varphi(x), \quad \forall x \in X, \quad (2.2)$$

for some number α with $0 < |\alpha| < a$. Then, there exists a unique quadratic functional equation $Q : X \rightarrow Y$ such that

$$N'(Q(x) - f(x), t) \geq N \left(\varphi(x), \left(\frac{(a^2 - \alpha^p)t}{2} \right)^q \right). \quad (2.3)$$

Proof. Putting $y = 0$ and $t = s$ in (2.1), we have that

$$N' \left(2f(ax) - 2a^2f(x), 2t \right) \geq N(\varphi(x), t^q) \quad (\forall x \in X, t > 0). \quad (2.4)$$

Therefore

$$N' \left(\frac{1}{a^2}f(ax) - f(x), \frac{t}{a^2} \right) \geq N(\varphi(x), t^q) \quad (\forall x \in X, t > 0), \quad (2.5)$$

and

$$N' \left(\frac{1}{a^2}f(ax) - f(x), \frac{t^p}{a^2} \right) \geq N(\varphi(x), t). \quad (2.6)$$

Now, replacing $x = ax$ in (2.6),

$$N' \left(\frac{1}{a^4}f(a^2x) - \frac{1}{a^2}f(ax), \frac{t^p}{a^4} \right) \geq N(\varphi(ax), t), \quad (2.7)$$

and then by the assumption that $\varphi(ax) = \alpha\varphi(x)$ and property (N_3) of Definition 1.1 we obtain that

$$N'\left(\frac{1}{a^4}f(a^2x) - \frac{1}{a^2}f(ax), \frac{(\alpha t)^p}{a^4}\right) \geq N(\varphi(x), t). \quad (2.8)$$

By comparing (2.6) and (2.8) and using property (N_4) we obtain that

$$N'\left(\frac{1}{a^4}f(a^2x) - f(x), \frac{t^p}{a^2} + \frac{(\alpha t)^p}{a^4}\right) \geq N(\varphi(x), t) \quad (\forall x \in X, t > 0). \quad (2.9)$$

Again, by replacing $x = ax$, in (2.9),

$$N'\left(\frac{1}{a^6}f(a^3x) - \frac{1}{a^2}f(ax), \frac{t^p}{a^4} + \frac{(\alpha t)^p}{a^6}\right) \geq N(\varphi(ax), t). \quad (2.10)$$

Thus

$$N'\left(\frac{1}{a^6}f(a^3x) - \frac{1}{a^2}f(x), \frac{(\alpha t)^p}{a^4} + \frac{\alpha^{2p}t^p}{a^6}\right) \geq N(\varphi(x), t). \quad (2.11)$$

By comparing (2.6), and (2.11) we obtain that

$$N'\left(\frac{1}{a^6}f(a^3x) - f(x), \frac{t^p}{a^2} + \frac{(\alpha t)^p}{a^4} + \frac{\alpha^{2p}t^p}{a^6}\right) \geq N(\varphi(x), t) \quad (\forall x \in X, t > 0). \quad (2.12)$$

With following this process we obtain that

$$N'\left(\frac{1}{a^{2n}}f(a^n x) - f(x), \sum_{k=1}^n \alpha^{(k-1)p} a^{-2k} t^p\right) \geq N(\varphi(x), t). \quad (2.13)$$

If $m \in N$, $n > m > 0$, then $n - m \in N$. Replacing n by $n - m$ in (2.13) gives

$$N'\left(a^{2m-2n}f(a^{n-m}x) - f(x), \sum_{k=1}^{n-m} \alpha^{(k-1)p} a^{-2k} t^p\right) \geq N(\varphi(x), t) \quad (\forall x \in X, t > 0, n \in N). \quad (2.14)$$

By replacing $x = a^m x$ in (2.14) we obtain that

$$N'\left(a^{-2n}f(a^n x) - a^{-2m}f(a^m x), a^{-2m} \sum_{k=1}^{n-m} \alpha^{(k-1)p} a^{-2k} t^p\right) \geq N(\varphi(a^m x), t). \quad (2.15)$$

Thus

$$N'\left(a^{-2n}f(a^n x) - a^{-2m}f(a^m x), a^{-2m} \sum_{k=1}^{n-m} \alpha^{(k-1)p} a^{-2k} \alpha^{mp} t^p\right) \geq N(\varphi(x), t). \quad (2.16)$$

It follows that

$$N'\left(a^{-2n}f(a^n x) - a^{-2m}f(a^m x), \sum_{k=m+1}^n \alpha^{(k-1)p} a^{-2k} t^p\right) \geq N(\varphi(x), t) \quad (\forall x \in X, t > 0). \quad (2.17)$$

Let $c > 0$, and let ϵ be given. Since $\lim_{t \rightarrow \infty} N(\varphi(x), t) = 1$, there is some $t_0 > 0$ such that

$$N(\varphi(x), t_0) \geq 1 - \epsilon. \quad (2.18)$$

Fix some $t > t_0$. The convergence of series $\sum_{k=1}^{\infty} (\alpha^p/a^2)^k \alpha^{-p} t^p$ guarantees that there exists some $n_0 \geq 0$ such that, for each $n > m > n_0$, the inequality $\sum_{k=m+1}^n (\alpha^p/a^2)^k \alpha^{-p} t^p < c$ holds. It follows that

$$\begin{aligned} & N'\left(a^{-2n}f(a^n x) - a^{-2m}f(a^m x), c\right) \\ & \geq N'\left(a^{-2n}f(a^n x) - a^{-2m}f(a^m x), \sum_{k=m+1}^n \left(\frac{\alpha^p}{a^2}\right)^k \alpha^{-p} t_0^p\right) \geq N(\varphi(x), t_0) \geq 1 - \epsilon. \end{aligned} \quad (2.19)$$

Hence $\{f(a^n x)/a^{2n}\}$ is a Cauchy sequence in fuzzy Banach space (Y, N') , and thus this sequence converges to some $Q(x) \in Y$. It means that

$$Q(x) = N' - \lim_{n \rightarrow \infty} \frac{f(a^n x)}{a^{2n}}. \quad (2.20)$$

Furthermore by putting $m = 0$ in (2.17),

$$N'\left(a^{-2n}f(a^n x) - f(x), \sum_{k=1}^n \left(\frac{\alpha^p}{a^2}\right)^k \alpha^{-p} t^p\right) \geq N(\varphi(x), t), \quad (2.21)$$

$$N'\left(a^{-2n}f(a^n x) - f(x), t\right) \geq N\left(\varphi(x), \frac{t^q}{\left(\sum_{k=1}^n (\alpha^p/a^2)^k \alpha^{-p}\right)^q}\right) \quad (\forall x \in X, t > 0). \quad (2.22)$$

Next we will show that Q is quadratic. Let $x, y \in X$, and then we have that

$$\begin{aligned}
& N' \left(Q(ax + by) + Q(ax - by) - \frac{a}{2}Q(x + y) - \frac{a}{2}Q(x - y) - (2a^2 - a)Q(x) - (2b^2 - b)Q(y), t \right) \\
& \geq N' \left(Q(ax + by) + Q(ax - by) - \frac{a}{2}Q(x + y) - \frac{a}{2}Q(x - y) \right. \\
& \quad \left. - (2a^2 - a)Q(x) - (2b^2 - b)Q(y), t \right) \\
& \geq \min \left\{ N' \left(Q(ax + by) - \frac{f(a^n(ax + by))}{a^{2n}}, \frac{t}{7} \right), \right. \\
& \quad N' \left(Q(ax - by) - \frac{f(a^n(ax - by))}{a^{2n}}, \frac{t}{7} \right), \\
& \quad N' \left(\frac{a}{2} \frac{f(a^n(x + y))}{a^{2n}} - \frac{a}{2}Q(x + y), \frac{t}{7} \right), \\
& \quad N' \left(\frac{a}{2} \frac{f(a^n(x - y))}{a^{2n}} - \frac{a}{2}Q(x - y), \frac{t}{7} \right), \\
& \quad N' \left((2a^2 - a) \frac{f(a^n(x))}{a^{2n}} - (2a^2 - a)Q(x), \frac{t}{7} \right), \\
& \quad N' \left((2b^2 - a) \frac{f(a^n(y))}{a^{2n}} - (2b^2 - a)Q(y), \frac{t}{7} \right), \\
& \quad N' \left(\frac{f(a^n(ax + by))}{a^{2n}} + \frac{f(a^n(ax - by))}{a^{2n}} - \frac{a}{2} \frac{f(a^n(x + y))}{a^{2n}} - \frac{a}{2} \frac{f(a^n(x - y))}{a^{2n}} \right. \\
& \quad \left. - (2a^2 - a) \frac{f(a^n(x))}{a^{2n}} - (2b^2 - a) \frac{f(a^n(y))}{a^{2n}}, \frac{t}{7} \right) \Big\}. \tag{2.23}
\end{aligned}$$

The first six terms on the right-hand side of the above inequality tend to 1 as $n \rightarrow \infty$, and the seventh term, by (2.1), is greater than or equal to

$$\min \left\{ N \left(\varphi(x), \left(\frac{a^{2q}}{\alpha} \right)^n \left(\frac{t}{14} \right)^q \right), N \left(\varphi(y), \left(\frac{a^{2q}}{\alpha} \right)^n \left(\frac{t}{14} \right)^q \right) \right\}, \tag{2.24}$$

which tends to 1 as $n \rightarrow \infty$. Therefore

$$\begin{aligned}
& N' \left(Q(ax + by) + Q(ax - by) - \frac{a}{2}Q(x + y) - \frac{a}{2}Q(x - y) \right. \\
& \quad \left. - (2a^2 - a)Q(x) - (2b^2 - a)Q(y), t \right) = 1 \tag{2.25}
\end{aligned}$$

for each $x, y \in X$ and $t > 0$. So by property (N_2) , we have that

$$\begin{aligned} Q(ax + by) + Q(ax - by) - \frac{a}{2}Q(x + y) - \frac{a}{2}Q(x - y) - (2a^2 - a)Q(x) \\ - (2b^2 - a)Q(y) = 0, \quad \forall x, y \in X. \end{aligned} \quad (2.26)$$

Therefore Q is quadratic function. For every $x \in X$ and $t, s > 0$, by (2.22), for large enough n , we have that

$$\begin{aligned} N'(Q(x) - f(x), t) &\geq \min \left\{ N' \left(Q(x) - \frac{f(a^n x)}{a^{2n}}, \frac{t}{2} \right), N' \left(\frac{f(a^n x)}{a^{2n}} - f(x), \frac{t}{2} \right) \right\} \\ &\geq N \left(\varphi(x), \frac{(t/2)^q}{\left(\sum_{k=1}^n (\alpha^k / a^2)^k \alpha^{-p} \right)^q} \right) = N \left(\varphi(x), \left(\frac{(a^2 - \alpha^p)t}{2} \right)^q \right). \end{aligned} \quad (2.27)$$

Let Q' be another quadratic function from X to Y which satisfies (2.3). Since, for each $n \in \mathbb{N}$,

$$Q(a^n x) = a^{2n}Q(x), \quad Q'(a^n x) = a^{2n}Q'(x), \quad (2.28)$$

We have that

$$\begin{aligned} N'(Q(x) - Q'(x), t) &= N'(Q(a^n x) - Q'(a^n x), a^{2n}t) \\ &\geq \min \left\{ N' \left(Q(a^n x) - f(a^n x), \frac{a^{2n}t}{2} \right), N' \left(f(a^n x) - Q'(a^n x), \frac{a^{2n}t}{2} \right) \right\} \\ &\geq N \left(\varphi(x), \left(\frac{a^2 - \alpha^p}{2} \right)^q \frac{a^{2nq} t^q}{\alpha^n 2^q} \right) \end{aligned} \quad (2.29)$$

for each $n \in \mathbb{N}$. Due to $q > 1/2$,

$$\lim_{n \rightarrow \infty} N \left(\varphi(x), \left(\frac{a^2 - \alpha^p}{2} \right)^q \frac{a^{2nq} t^q}{\alpha^n 2^q} \right) = 1 \quad (2.30)$$

for each $x \in X$ and $t > 0$. Therefore $Q = Q'$. □

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