Research Article

On Removable Sets of the First Boundary-Value Problem for Degenerated Elliptic Equations

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In the paper, the necessary and sufficient condition of compact removability is obtained.

1. Introduction

The questions of compact removability for Laplace equation is studied by Carleson [1]. The uniform elliptic equation of the second order of divergent structure is studied by Moiseev [2]. The compact removability for elliptic and parabolic equations of nondivergent structure is considered by Landis [3]. Gadjiev, Mamedova [4]. The removability condition of compact in the space of continuous functions are constructed in the papers Harvey and Polking [5], Kilpeläinen and Zhong [6]. The different questions of qualitative properties of solutions of uniformly degenerated elliptic equations is studied by Chanillo and Wreeden [7]. In paper [8] the second order uniform divergent elliptic operator is considered.

Let E_n be n dimensional Euclidean space of the points $x = (x_1, \dots, x_n)$. Denote the ball $\{x : |x - x^0| < R\}$ by $B_R(x^0)$ for $R > 0$ and the cylinder $B_R(x^0) \cup (0, T)$ by $Q_T^R(x^0)$. Further, let for $x^0 \in E_n$, $R > 0$ and $k > 0$ $\varepsilon_{r,k}(x^0)$ be an ellipsoid $\{x : \sum_{i=1}^n ((x_i - x_i^0)^2 / R^{a_i}) < (kR)^2\}$. Let D be a bounded domain E_n with the domain $\partial D, 0 \in D$. ε is a such kind of ellipsoid that $\overline{D} \subset \varepsilon$, $\mathfrak{B}(\varepsilon)$ is a set of all functions, satisfying in $\overline{\varepsilon}$ the uniform Lipschitz condition and having zero near the $\partial\varepsilon$.

Denote by α and $(\alpha_1, \dots, \alpha_n)$ the vector $\langle \alpha \rangle = \alpha_1, \dots, \alpha_n$.

Denote by $W_{2,\alpha}^1(D)$ the Banach space of the functions $u(x)$ given on D with the finite norm

$$\|u\|_{W_{2,\alpha}^1(D)} = \left(\int_D \left(u^2 + \sum_{i=1}^n \lambda_i(x) u_i^2 \right) dx \right)^{1/2}, \quad (1.1)$$

where

$$\begin{aligned} u_i &= \frac{\partial u}{\partial x_i}, \quad i = 1, \dots, n. \\ \lambda_i(x) &= (|x|_\lambda)^{\alpha_i}, \quad |x|_\lambda = \sum_{i=1}^n |x_i| \frac{2}{2 + \alpha_i}, \\ 0 &\leq \alpha_i < \frac{2}{n-1}. \end{aligned} \quad (1.2)$$

Further, let $\overset{\circ}{W}_{2,\alpha}^1(D)$ be a degenerated set of all functions from $C_0^\infty(D)$ by the norm of the space $W_{2,\alpha}^1(D)$. Denote by $\mathcal{M}(D)$ the set of all bounded in D functions.

Let $E \subset D$ be some compact. Denote by $A_E(D)$ the totality of all functions $u(x) \in C^\infty(\overline{D})$, each of which there exists some neighbourhood of the compact E in which $u(x) = 0$.

The compact E is called the removable relative to the first boundary value problem for the operator L in the space $\mathcal{M}(D)$ if all generalized solution of the equation $\mathcal{L}u = 0$ in ∂/E formed in zero on ∂D and belonging to the space $\mathcal{M}(D)$ identically equal to zero. We will say that the function $u(x) \in \overset{\circ}{W}_{2,\alpha}^1(\varepsilon)$ is nonnegative on the set $H \subset \varepsilon$, in the sense $\overset{\circ}{W}_{2,\alpha}^1(\varepsilon)$ if there exists the sequence of the functions $\{u_{(m)}(x)\}$, $m = 1, 2, \dots$ such that $u_m(x) \in \mathfrak{B}(\varepsilon)$, $u_m(x) \geq 0$ for $x \in H$ and $\lim_{m \rightarrow \infty} \|u_{(m)} - u\|_{W_{2,\alpha}^1(\varepsilon)} = 0$.

The function $u(x) \in W_{2,\alpha}^1(D)$ is nonnegative and ∂D in sense $W_{2,\alpha}^1(D)$ if there exists the sequence of the functions $\{u_m(x)\}$, $m = 1, 2, \dots$ such that $u_{(m)}(x) \in C^1(D)$, $u_m(x) \geq 0$ for $x \in \partial D$ and $\lim_{m \rightarrow \infty} \|u_{(m)} - u\|_{W_{2,\alpha}^1(\varepsilon)} = 0$. It is easy to determine the inequalities $u(x) \geq \text{const}$, $u(x) \geq v(x)$, $u(x) \leq 0$, and also equality $u(x) = 1$ on the set H in the sense $\overset{\circ}{W}_{2,\alpha}^1(\varepsilon)$ if at the same time $u(x) \geq 1$ and $u(x) \leq 1$ on H , in the sense $\overset{\circ}{W}_{2,\alpha}^1(\varepsilon)$.

Let $\omega(x)$ be measurable function in D , finite and positive for a.e. $x \in D$. Denote by $\mathcal{L}_{p,\omega}(D)$ the Banach space of the functions given on D , with the norm

$$\|u\|_{\mathcal{L}_{p,\omega}(D)} = \left(\int_D (\omega(x))^{p/2} |u|^p dx \right)^{1/p}, \quad 1 < p < \infty. \quad (1.3)$$

Let $W_{p,\alpha}^1(D)$ be a Banach space of the functions given on $u(x)$, with the finite norm D

$$\|u\|_{W_{p,\alpha}^1(D)} = \left(\int_D \left(|u|^p + \sum_{i=1}^n (\lambda_i(x))^{p/2} |u_i|^p \right) dx \right)^{1/p}, \quad 1 < p < \infty. \quad (1.4)$$

Analogously to $\mathring{W}_{2,\alpha}^1(D)$, it is introduced the subspace $\mathring{W}_{p,\alpha}^1(D)$ for $1 < p < \infty$. The space conjugated to $\mathring{W}_{p,\alpha}^1(D)$ we will denote by $\mathring{W}_{p,\alpha}^*(D)$.

We will consider the elliptic operator in the bounded domain $D \subset E_n$

$$\mathcal{L} = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right). \quad (1.5)$$

In assumption that $\|a_{ij}(x)\|$ is a real symmetric matrix with measurable in D elements, moreover, for all $\xi \in E_n$ and a.e. $x \in D$, the condition

$$\gamma \sum_{i=1}^n \lambda_i(x) \xi_i^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \gamma^{-1} \sum_{i=1}^n \lambda_i(x) \xi_i^2. \quad (1.6)$$

Here, $\gamma \in (0, 1]$ is a constant.

The function $u(x) \in W_{2,\alpha}^1(D)$ is called the generalized solution of the equation $\mathcal{L}u = f(x)$ in D , if for any function $\eta(x) \in \mathring{W}_{2,\alpha}^1(D)$ the integral identity

$$\int_D \sum_{i,j=1}^n a_{ij}(x) u_{x_i} \eta_{x_j} dx = \int_D f \eta dx \quad (1.7)$$

be fulfilled.

Here, $f(x)$ is a given function from $\mathcal{L}_2(D)$.

Let $E \subset D$ be some compact. The function $u(x) \in W_{2,\alpha}^1(D \setminus E)$ is called generalized solution of the equation $\mathcal{L}u = f(x)$ in $D \setminus E$ vanishing on ∂D if integral identity (1.7) is fulfilled for any function $\eta(x) \in A_E(D)$.

We will assume that the coefficients of the operator $\mathcal{L}$ continued in $E_n \setminus D$ with saving condition (1.2), (1.6). For this, it is sufficient, for example, to assume that $a_{ij}(x) = \delta_{ij} \lambda_i(x)$ for $x \in E_n \setminus D$, $i, j = 1, \dots, n$, where δ_{ij} is a Croneker symbol.

Let $h(x) \in W_{2,\alpha}^1(D)$, $f^0(x) \in h_2(D)$, $f^i(x) \in \mathcal{L}_{2,\lambda^{-1}}(D)$, $i = 1, 2, \dots, n$, be a given functions. Let us consider the first boundary value problem

$$\mathcal{L}u = f^0(x) + \sum_{i=1}^n \frac{\partial f^i(x)}{\partial x_i}, \quad x \in D, \quad (1.8)$$

$$(u(x) - h(x)) \in \mathring{W}_{2,\alpha}^1(D).$$

The function $u(x) \in W_{2,\alpha}^1(D)$ we will call generalized solution of problem (1.8) if for any function $\eta(x) \in \mathring{W}_{2,\alpha}^1(D)$, the integral identity

$$\int_D \sum_{i,j=1}^n a_{ij}(x) u_{x_i} \eta_{x_j} dx = \int_D \left(-f^0 \eta + \sum_{i=1}^n f^i \eta_{x_i} \right) dx \quad (1.9)$$

is fulfilled.

Our aim to get the necessary and sufficient condition of compact removability E in the class of bounded functions.

2. Preliminaries Statements

At first, we introduce some auxiliary statements.

Lemma 2.1. *If relative to the coefficients of the operator $\mathcal{L}$ condition (1.2), (1.6) are fulfilled, then the first boundary value problem (1.8) has a unique generalized solution $u(x)$ at any $h(x) \in W_{2,\alpha}^1(D)$, $f^0(x) \in h_2(D)$, $f^i(x) \in L_{2,\lambda_i^{-1}}(D)$, $i = 1, 2, \dots, n$. At this, there exists $P_0(\alpha, n)$ such that if $p > p_0$, $h(x) \in W_{p,\alpha}^1(D)$, $f^0(x) \in h_p(D)$, $f^i(x) \in L_{2,\lambda_i^{-1}}(D)$, $i = 1, 2, \dots, n$, $\partial D \in C^1$, then solution $u(x)$ is continuous in $\overline{D}$.*

Lemma 2.2. *Let relative to the coefficients of the operator $\mathcal{L}$ conditions (1.2), (1.6) be fulfilled. Then, any generalized solution of the equation $\mathcal{L}u = 0$ in D is continuous by Holder at each strictly internal domain ∂ .*

Lemma 2.3. *Let relative to the coefficients of the operator $\mathcal{L}$, conditions (1.2), (1.6) be fulfilled and $\overline{\varepsilon_{R,1}} < D$. Then, for any positive generalized solution $u(x)$, the equation $\mathcal{L}u = 0$ in D the Harnack inequality is true*

$$\sup_{\varepsilon_{R,1}(0)} u \leq C_1(\gamma, \alpha, n) \inf_{\varepsilon_{R,1}(0)} u. \quad (2.1)$$

If at this $y \in \partial\varepsilon_{R,2}(0)$ and $\overline{\varepsilon_{R,1}}(0) \subset D$, then the inequality of form (2.1) is true in ellipsoid $\varepsilon_{R,1}(y)$.

Lemma 2.4. *Let relative to the coefficients of the operator $\mathcal{L}$ conditions (1.2), (1.6) be fulfilled and $u(x)$ generalized solution of the first boundary-value problem (1.8) at $f^i(x) \equiv 0$, $i = 0, \dots, n$. Then, if $h(x)$ is bounded on ∂D in the sense $W_{2,\alpha}^1(D)$, then for solution $u(x)$ the following maximum principle is true:*

$$\inf_{\partial D} h \leq \inf_D u \leq \sup_D u \leq \sup_{\partial D} h, \quad (2.2)$$

where $\inf_{\partial D} h$ ($\sup_{\partial D} h$) is an exact lower (upper) bound those numbers a , for which $h(x) \geq a$ ($h(x) \leq a$) on ∂D in the sense $W_{2,\alpha}^1(D)$.

These lemmas are proved analogously to paper [7]. Therefore, the proof of these lemmas is not given.

Let $H \subset \varepsilon$ be some compact and V_H a set of all functions $\varphi(x) \in \overset{\circ}{W}_{2,\alpha}^1(\varepsilon)$ such that $\varphi(x) \geq 1$ on H , in the sense $\overset{\circ}{W}_{2,\alpha}^1(\varepsilon)$. Let one considers the functional

$$J_\theta(\varphi) = \int_\varepsilon \sum_{i,j=1}^n a_{ij}(x) \varphi_i \varphi_j dx, \quad \varphi(x) \in V_H, \quad (2.3)$$

$\mathcal{L}$ is a H compact capacity relative to ellipsoid ε and is called the value $\inf_{\varphi \in V_H} J_{\theta}(u)$ and denoted by $\text{cap}_{\mathcal{L}}^{(\varepsilon)}(H)$. In case $\varepsilon = E_n$, the corresponding value is called $\mathcal{L}$ capacity of the compact H and denoted by $\text{cap}_{\mathcal{L}}(H)$.

Lemma 2.5. *There exists the unique function $u(x) \in \mathring{W}_{2,\alpha}^1(\varepsilon)$ such that $u(x) \geq 1$ on H in the sense $\mathring{W}_{2,\alpha}^1(\varepsilon)$ and $\text{cap}_{\mathcal{L}}^{(\varepsilon)}(H) = J_{\mathcal{L}}(u)$.*

Proof. It is easy to see that V_H is convex closed set in $\mathring{W}_{2,\alpha}^1(\varepsilon)$. From the fact that $\mathring{W}_{2,\alpha}^1(\varepsilon)$ is a Hilbert space, it follows the existence of unique function $u(x) \in V_H$, which achieved an exact lower bound of the functional $J_{\mathcal{L}}(\varphi)$. Next,

$$\{u(x)\}^1 = \begin{cases} u(x) & \text{if } u(x) \leq 1, \\ 1 & \text{if } u(x) > 1. \end{cases} \quad (2.4)$$

It is clear that $\{u(x)\}^1 \in \mathring{W}_{2,\alpha}^1(\varepsilon)$. Moreover, $\{u(x)\}^1 \in V_H$. Denote by $A^+ = \{x : x \in \varepsilon, u(x) > 1\}$. We have

$$J_{\mathcal{L}}\{u(x)\}^1 = \left(\int_{A^+} + \int_{\varepsilon \setminus A^+} \right) \sum_{i,j=1}^n a_{ij}(x) \{u\}_i^1 \{u\}_j^1 dx = \int_{\varepsilon \setminus A^+} \sum_{i,j=1}^n a_{ij}(x) u_i u_j dx. \quad (2.5)$$

On the other side, according to (1.2),

$$\int_{A^+} \sum_{i,j=1}^n a_{ij}(x) u_i u_j dx \geq 0. \quad (2.6)$$

From (2.5) and (2.6), we conclude

$$J_{\mathcal{L}}\{u(x)\}^1 \leq J_{\mathcal{L}}(u) = \inf_{\varphi \in V_H} J_{\mathcal{L}}(\varphi), \quad (2.7)$$

that is, $J_{\mathcal{L}}\{u(x)\}^1 = J_{\mathcal{L}}(u)$. From uniqueness extreme function, it follows that $\{u(x)\}^1 = u(x)$, and lemma is proved.

The function $u(x)$, which achieved an exact lower bound of the functional $J_{\mathcal{L}}(\varphi)$ on the set V_H is called $\mathcal{L}$ capacity of the compact potential H relative to the ellipsoid ε . $\square$

Lemma 2.6. *Let $\mathcal{L}$ be a capacity potential $u(x)$ of the compact H relative to ε which is a generalized solution of the equation $\mathcal{L}u = 0$ in $\varepsilon \setminus H$, vanishing on 0 and $\partial\varepsilon$ in 1 on ∂H in the sense $W_{2,\alpha}^1(\varepsilon)$.*

Proof. It is sufficient to show the truth of the first part of assertion of lemma. Let $\eta(x) \in \mathring{W}_{2,\alpha}^1(\varepsilon)$ and $\eta(x) \geq 0$ on H in the sense $\mathring{W}_{2,\alpha}^1(\varepsilon)$. Then, for any $\varepsilon > 0$ ($u(x) + \varepsilon\eta(x)$) $\in V_H$. Therefore,

$$J_{\mathcal{L}}(u + \varepsilon\eta) \geq J_{\mathcal{L}}(u). \quad (2.8)$$

Thus,

$$J_{\mathcal{L}}(u) + \varepsilon^2 J_{\mathcal{L}}(\eta) + 2\varepsilon \int_{\varepsilon} \sum_{i,j=1}^n a_{ij}(x) u_i \eta_j dx \geq J_{\mathcal{L}}(u), \quad (2.9)$$

that is,

$$J_{\mathcal{L}}(u) + 2\varepsilon \int_{\varepsilon} \sum_{i,j=1}^n a_{ij}(x) u_i \eta_j dx \geq 0. \quad (2.10)$$

Tending ε to zero, we conclude

$$\int_{\varepsilon} \sum_{i,j=1}^n a_{ij}(x) u_i \eta_j dx \geq 0. \quad (2.11)$$

It is easy to see as $\eta(x)$ in (2.11), we can take any function from $C^1(\bar{\varepsilon})$ with compact support in $\varepsilon \setminus H$. Then,

$$\int_{\varepsilon \setminus H} \sum_{i,j=1}^n a_{ij}(x) u_i \eta_j dx \geq 0. \quad (2.12)$$

Substituting $\eta(x)$ on $-\eta(x)$, we arrive to

$$\int_{\varepsilon \setminus H} \sum_{i,j=1}^n a_{ij}(x) u_i \eta_j dx = 0. \quad (2.13)$$

Lemma is proved. $\square$

Let μ be a charge of bounded variation, given on ε . We will say that the function $u(x) \in L_1(\varepsilon)$ is a weak solution of the equation $\mathcal{L}u = -\mu$, equaling to zero on $\partial\varepsilon$ if for any function $\varphi(x) \in \overset{\circ}{W}_{2,\alpha}^1(\varepsilon) \cap C(\bar{\varepsilon})$, the integral identity

$$\int_{\varepsilon} u \mathcal{L}\varphi dx = \int_{\varepsilon} \varphi d\mu \quad (2.14)$$

is fulfilled.

According to Lemma 2.1 (at $h = 0$), there exists the continuous linear operator H from $\overset{*}{W}_{2,\alpha}^1(\varepsilon)$ in $\overset{\circ}{W}_{2,\alpha}^1(\varepsilon)$ such that for any functional $T \in \overset{*}{W}_{2,\alpha}^1(\varepsilon)$, the function $u = H(T)$ is unique in $\overset{\circ}{W}_{2,\alpha}^1(\varepsilon)$ generalized solution of the equation $\mathcal{L}u = T$.

The operator H is called Green operator.

By Lemma 2.1, this operator at $p > p_0$ we transform $\overset{*}{W}_{2,\alpha}^1(\varepsilon)$ to $C(\bar{\varepsilon})$. It is easy to see that the function $u(x)$ is weak solution of the equation $\mathcal{L}u = -\mu$, equaling to zero on $\partial\varepsilon$ if and only if for any function $\varphi(x) \in C(\bar{\varepsilon})$ the integral identity

$$\int_{\varepsilon} u\varphi dx = \int_{\varepsilon} H(\varphi) d\mu \quad (2.15)$$

is fulfilled.

By analogy with [8], we can show that for each measure μ on ε , there exists the unique weak solution of the equation $\mathcal{L}u = -\mu$ equaling to zero on $\partial\varepsilon$.

Let us say that the charge $\mu \in \overset{*}{W}_{2,\alpha}^1(\varepsilon)$ if there exists the vector $\bar{f}(x) = (f^0(x), f^1(x), \dots, f^n(x))$, $f^0(x) \in h_2(\varepsilon)$, $f^i(x) \in L_{2,\lambda_i}(\varepsilon)$, $i = 1, 2, \dots, n$, for any function $\varphi(x) \in \overset{\circ}{W}_{2,\alpha}^1(\varepsilon) \cap C(\bar{\varepsilon})$, the integral identity

$$\mu(\varphi) = \int_{\varepsilon} \varphi d\mu = \int_{\varepsilon} \left(f^0 \varphi - \sum_{i=1}^n f^i \varphi_i \right) dx \quad (2.16)$$

is true.

At this, it is evident that

$$\left| \int_{\varepsilon} \varphi d\mu \right| \leq C_2(\bar{f}) \|\varphi\|_{W_{2,\alpha}^1(\varepsilon)}. \quad (2.17)$$

Lemma 2.7. *The weak solution $u(x)$ of the equation $\mathcal{L}u = -\mu$, equaling to zero on $\partial\varepsilon$, belongs to $\overset{\circ}{W}_{2,\alpha}^1(\varepsilon)$, if and only if $\mu \in \overset{*}{W}_{2,\alpha}^1(\varepsilon)$.*

Proof. At first, we will show that if the function $\varphi(x) \in \overset{\circ}{W}_{2,\alpha}^1(\varepsilon)$ satisfies the integral identity

$$\int_{\varepsilon} \sum_{i,j=1}^n a_{ij}(x) u_i \varphi_j dx = - \int_{\varepsilon} \varphi d\mu, \quad (2.18)$$

for any function $\varphi(x) \in \overset{\circ}{W}_{2,\alpha}^1(\varepsilon) \cap C(\bar{\varepsilon})$ zero on $\partial\varepsilon$, then it is weak solution of the equation $\mathcal{L}u = -\mu$, equaling to zero on $\partial\varepsilon$. Really, assuming that $\varphi = H(\psi)$, $\psi(x) \in C(\bar{\varepsilon})$, we obtain

$$\begin{aligned} \int_{\varepsilon} H(\psi) d\mu &= \int_{\varepsilon} \varphi d\mu = - \int_{\varepsilon} \sum_{i,j=1}^n a_{ij}(x) u_i \varphi_j dx = \int_{\varepsilon} u \sum_{i,j=1}^n (a_{ij}(x) \varphi_j)_i dx \\ &= \int_{\varepsilon} u \mathcal{L}\varphi dx = \int_{\varepsilon} u\varphi dx, \end{aligned} \quad (2.19)$$

and now, it is sufficient to use the identity (2.15). We will show that $\mu \in \overset{*}{W}_{2,\alpha}^1(\varepsilon)$. For this, it is sufficient to prove that if $f^i(x) = \sum_{i=1}^n a_{ij}(x)u_i(x)$, then $f^i(x) \in L_{2,\lambda_i^{-1}}(\varepsilon)$, $i = 1, 2, \dots, n$. Assume in condition (2.18) that $\xi_1 = \dots = \xi_{i-1} = \xi_{i+1} = \dots = \xi_n = 0$, $\xi_i = 1/\sqrt{\lambda_i(x)}$.

We will obtain

$$\gamma \leq \frac{a_{ii}(x)}{\lambda_i(x)} \leq \gamma^{-1}; \quad i = 1, \dots, n. \quad (2.20)$$

Let $i \neq j$. Assuming that $\xi_k = 0$ at $k \neq j$ and $k \neq i$, $\xi_i = 1/\sqrt{\lambda_i(x)}$, $\xi_j = 1/\sqrt{\lambda_j(x)}$, we will obtain

$$2\gamma \leq \frac{a_{ii}(x)}{\lambda_i(x)} + \frac{a_{jj}(x)}{\lambda_j(x)} + \frac{2a_{ij}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}} \leq 2\gamma^{-1}. \quad (2.21)$$

Using (2.20), we conclude

$$\frac{|a_{ij}(x)|}{\sqrt{\lambda_i(x)\lambda_j(x)}} \leq \gamma^{-1} - \gamma; \quad i, j = 1, \dots, n; \quad i \neq j. \quad (2.22)$$

From (2.20) and (2.22), it follows that

$$\frac{|a_{ij}(x)|}{\sqrt{\lambda_i(x)\lambda_j(x)}} \leq \gamma^{-1}; \quad i, j = 1, \dots, n. \quad (2.23)$$

Thus, from (2.23), take out for $j = 1, \dots, n$

$$\int_{\varepsilon} \frac{1}{\lambda_j(x)} (f^j)^2 dx = \int_{\varepsilon} \frac{1}{\lambda_j(x)} \left(\sum_{i=1}^n a_{ij}(x)u_i \right)^2 dx \leq \gamma^{-2} n \sum_{i=1}^n \int_{\varepsilon} \lambda_i(x) u_i^2 dx < \alpha. \quad (2.24)$$

So, $\mu \in \overset{*}{W}_{2,\alpha}^1(\varepsilon)$. Inversely, if $u(x)$ is a weak solution of the equation $\mathcal{L}u = -\mu$, vanishing on $\partial\varepsilon$, then there exists $\mu \in \overset{*}{W}_{2,\alpha}^1(\varepsilon)$ such that

$$\begin{aligned} \left(f^\circ \varphi - \sum_{i=1}^n f^i \varphi_i \right) dx &= \int_{\varepsilon} \varphi d\mu = \int_{\varepsilon} u \mathcal{L}\varphi dx = \int_{\varepsilon} u \sum_{i,j=1}^n (a_{ij}(x)\varphi_j)_i dx \\ &= - \int_{\varepsilon} \sum_{i,j=1}^n a_{ij}(x) u_i \varphi_j dx, \end{aligned} \quad (2.25)$$

for any function $\varphi(x) \in \overset{*}{W}_{2,\alpha}^1(\varepsilon) \cap C(\overline{\varepsilon})$, $\mathcal{L}\varphi(x) \in C(\overline{\varepsilon})$.

Then, from Lemma 2.1, we obtain that $u(x) \in \overset{\circ}{W}_{2,\alpha}^1(\varepsilon)$. The lemma is proved. $\square$

Let now $\delta(x)$ be Dirac measure, concentrated at the point 0, and y an arbitrary fixed point ε .

The weak solution $g(x, y)$ of the equation $\mathcal{L}y = -\delta(x - y)$, vanishing on $\partial\varepsilon$, is called Green function of the operator $\mathcal{L}$ in ε .

In case $\varepsilon = E_n$ the corresponding function is called the fundamental solution of the operator $\mathcal{L}$ and denoted by $G(x, y)$.

According to the above proved, if $\varphi(x)$ is an arbitrary function from $C(\bar{\varepsilon})$, then the generalized solution $\varphi(x) \in \mathring{W}_{2,\alpha}^1(\varepsilon)$ of the equation $\mathcal{L}\varphi = -\varphi$ can be introduced in the following form:

$$\varphi(y) = \int_{\varepsilon} g(x, y) \varphi(x) dx. \quad (2.26)$$

We can show that $g(x, y)$ is nonnegative in $\varepsilon \times \varepsilon$ moreover; $g(x, y) = g(y, x)$.

Lemma 2.8. *For any charge of bounded variation on ε , the integral*

$$u(x) = \int_{\varepsilon} g(x, y) d\mu(y) \quad (2.27)$$

exists, finite a.e. in ε and is weak solution of the equation $\mathcal{L}u = -\mu$, equaling to zero on $\partial\varepsilon$.

Proof. Without losing generality, we will assume that the charge μ is the measure in ε . Let $\varphi(x) \in C(\bar{\varepsilon})$, $\varphi(x) \geq 0$ in ε . Denote by $\varphi(x) \in \mathring{W}_{2,\alpha}^1(\varepsilon)$ the generalized solution of the equation $\mathcal{L}\varphi = -\varphi(x)$. Then, $\varphi(x) \in C(\bar{\varepsilon})$ according to Lemma 2.1 and $\varphi(x) \geq 0$ according to Lemma 2.4. At this,

$$\varphi(y) = \int_{\varepsilon} g(x, y) \varphi(x) dx. \quad (2.28)$$

Then, by the Fubini theorem, we conclude that the integral $\int_{\varepsilon} g(x, y) d\mu(y)$ there exists for almost all $x \in \varepsilon$; moreover,

$$\int_{\varepsilon} H(\varphi) d\mu(y) = \int_{\varepsilon} \varphi(y) d\mu(y) = \iint_{\varepsilon \times \varepsilon} g(x, y) \varphi(x) dx d\mu(y) = \int_{\varepsilon} \varphi(x) u(x) dx. \quad (2.29)$$

Let us note that (2.29) is fulfilled for weak nonnegative and continuous in $\bar{\varepsilon}$ function $\varphi(x)$. Now, it is sufficient to remember the identity (2.15) and lemma is proved.

Let us consider now $\mathcal{L}$ -capacity of the potential $u(x)$ of the compact H relative to the ellipsoid ε . Before, it was proved that $u(x)$ satisfies (2.11) at any nonnegative on H the function $\eta(x) \in C_0^\infty(\varepsilon)$. By the Schwartz theorem [9, 10], there exists the measure μ on H such that

$$\int_{\varepsilon} \sum_{i,j=1}^n a_{ij}(x) u_i \eta_j dx = \int_{\varepsilon} \eta d\mu. \quad (2.30)$$

Further, since $u = 1$ on H in the sense $\overset{\circ}{W}_{2,\alpha}^1(\varepsilon)$, then the carrier of the measure μ is situated on ∂H . The measure μ is called $\mathcal{L}$ -capacity of the compact H .

According to Lemma 2.8, $\mathcal{L}$ -capacity potential $u(x)$ is weak solution of the equation $\mathcal{L}u = -\mu$, equaling to zero on $\partial\varepsilon$ and can be represented in the following form:

$$u(x) = \int_{\varepsilon} g(x, z) d\mu(z). \quad (2.31)$$

On the other side, there exists the sequence of the functions $\{\eta^{(m)}(x)\}; m = 1, 2, \dots$ such that $\eta^{(m)}(x) \in \mathfrak{B}(\varepsilon)$, $\eta^{(m)}(x) = 1$ for $x \in H$ and $\lim_{m \rightarrow \infty} \|\eta^{(m)} - u\|_{W_{2,\alpha}^1(\varepsilon)} = 0$. Assuming in (2.15) $\eta^{(m)}(x)$ instead of $\eta^{(m)}$, we conclude that it first part is equal to $\mu(H)$ at any natural m , while the left part tends to $\text{cap}_{\mathcal{L}}^{(\varepsilon)}(H)$ as $m \rightarrow \infty$. Thus,

$$\text{cap}_{\mathcal{L}}^{(\varepsilon)}(H) = \mu(H). \quad (2.32)$$

□

Lemma 2.9. *Let relative to coefficients of the operator $\mathcal{L}$ conditions (1.2)–(1.6), $y \in \partial\varepsilon_{R,2}(0)$, $\bar{\varepsilon}_{R,1}(0) \subset D$, $x \in \partial\varepsilon_{R,1}(y)$ be fulfilled. Then, for the Green function $g(x, y)$ the following estimations are true:*

$$C_3(\gamma, \alpha, n) \left[\text{cap}_{\mathcal{L}}^{(\varepsilon)}(\bar{\varepsilon}_{R,1}(y)) \right]^{-1} \leq g(x, y) \leq C_4(\gamma, \alpha, n) \left[\text{cap}_{\mathcal{L}}^{(\varepsilon)}(\bar{\varepsilon}_{R,1}(y)) \right]^{-1}. \quad (2.33)$$

If $\bar{\varepsilon}_{R,1}(0) \subset D$, $x \in \partial\varepsilon_{R,1}(0)$ then,

$$C_3 \left[\text{cap}_{\mathcal{L}}^{(\varepsilon)}(\bar{\varepsilon}_{R,1}(0)) \right]^{-1} \leq g(x, 0) \leq C_4 \left[\text{cap}_{\mathcal{L}}^{(\varepsilon)}(\bar{\varepsilon}_{R,1}(0)) \right]^{-1}. \quad (2.34)$$

Proof. Without loss of generality, we can assume that the coefficients of the operator $\mathcal{L}$ are continuously differentiable in $\bar{\varepsilon}$. The general case is obtained by means of limit passage. Then, at $x \neq y$, the function $g(x, y)$ is continuous by x and y moreover,

$$\lim_{x \rightarrow y} g(x, y) = \infty. \quad (2.35)$$

Let a be a positive number, which will be chosen later, $K_a = \{x : g(x, y) \geq a\}$, where y is an arbitrary fixed point on $\partial\varepsilon_{R,2}(0)$. From (2.35), it follows that y is internal point of the compact K_a . Then, $\mathcal{L}$ is capacity potential K_a , represented in the form (2.31), so it means that it equal to zero in it. Thus,

$$1 = \int_{\varepsilon} y(y, z) d\mu_a(z), \quad (2.36)$$

where μ is a $\mathcal{L}$ -capacity distribution of the compact K_a . Allowing for the carrier of the measure μ_a is situated on ∂K_a , where $g(y, z) = a$ and using (2.32), we obtain

$$\mu_a(K_a) = \text{cap}_{\mathcal{L}}^{(\varepsilon)}(K_a) = \frac{1}{a}. \quad (2.37)$$

Let us assume now that $a = \inf_{x \in \partial \varepsilon_{R,1}(y)} g(x, y)$. According to maximum principle, $\bar{\varepsilon}_{R,1}(y) \subset K_a$. Therefore, from (2.37), we conclude

$$\text{cap}_{\mathcal{L}}^{(\varepsilon)}(\bar{\varepsilon}_{R,1}(y)) \leq \text{cap}_{\mathcal{L}}^{(\varepsilon)}(K_a) = \frac{1}{\inf_{x \in \partial \varepsilon_{R,1}(y)} g(x, y)}. \quad (2.38)$$

If we will assume $b = \sup_{x \in \partial \varepsilon_{R,1}(y)} g(x, y)$, then $\bar{\varepsilon}_{R,1}(y) \subset K_b$; that is,

$$\text{cap}_{\mathcal{L}}^{(\varepsilon)}(\bar{\varepsilon}_{R,1}(y)) \leq \text{cap}_{\mathcal{L}}^{(\varepsilon)}(K_b) = \frac{1}{\sup_{x \in \partial \varepsilon_{R,1}(y)} g(x, y)}. \quad (2.39)$$

From (2.38) and (2.39), follows that

$$\inf_{x \in \partial \varepsilon_{R,1}(y)} g(x, y) \leq \left[\text{cap}_{\mathcal{L}}^{(\varepsilon)}(\bar{\varepsilon}_{R,1}(y)) \right]^{-1} \leq \sup_{x \in \partial \varepsilon_{R,1}(y)} g(x, y). \quad (2.40)$$

On the other side, according to Lemma 2.3,

$$\sup_{x \in \partial \varepsilon_{R,1}(y)} g(x, y) \leq C_5(\gamma, \alpha, n) \inf_{x \in \partial \varepsilon_{R,1}(y)} g(x, y). \quad (2.41)$$

Now, the required estimations (2.33) follows from (2.40) and (2.41). Absolutely analogously the truth of (2.34) is proved. $\square$

Corollary 2.10. *Let the conditions of the lemma and $y \in \partial \varepsilon_{R,2}(0)$ be fulfilled, $\bar{\varepsilon}_{R,1}(0) \subset D$, $x \in \partial \varepsilon_{R,1}(0)$ or $y = 0$, $\bar{\varepsilon}_{R,1}(0) \subset D$, $x \in \partial \varepsilon_{R,1}(0)$. Then, for fundamental solution $G(x, y)$, the estimations*

$$C_3 \left[\text{cap}_{\mathcal{L}}^{(\varepsilon)}(\bar{\varepsilon}_{R,1}(0)) \right]^{-1} \leq G(x, y) \leq C_4 \left[\text{cap}_{\mathcal{L}}^{(\varepsilon)}(\bar{\varepsilon}_{R,1}(0)) \right]^{-1} \quad (2.42)$$

are true.

3. Removability Criterion of the Compact in the Space $M(D)$

Theorem 3.1. *Let relative to the coefficients of the operator $\mathcal{L}$, conditions (1.2)–(1.6) be fulfilled. Then, for removability of the compact $E \subset D$ relative to the first boundary value problem for the operator $\mathcal{L}$ in the space $\mathcal{M}(D)$, it is necessary and sufficient that*

$$\text{cap}_{\mathcal{L}}(E) = 0. \quad (3.1)$$

Proof. Let the ellipsoid ε have the same sense as above. It is easy to see that if condition (3.1) is fulfilled, then

$$\text{cap}_{\mathcal{L}}^{(\varepsilon)}(E) = 0. \quad (3.2)$$

Not losing generality, we can limit the case, when the coefficients of the operator $\mathcal{L}$ is continuously differentiable in $\bar{\varepsilon}$. Let us fix an arbitrary $\varepsilon > 0$ and $x^0 \in D \setminus E$. By virtue of (3.1), there exists the neighbourhood H of the compact E such that

$$\text{cap}_{\mathcal{L}}^{(\varepsilon)}(\overline{H}) < \varepsilon. \quad (3.3)$$

At this, we can assume that ε is such small that

$$\text{dist}(x^0, \overline{H}) \geq \frac{1}{2} \text{dist}(x^0, E). \quad (3.4)$$

Denote by $V_H(x)$ and μ_H the $\mathcal{L}$ -capacity potential of the compact $\overline{H}$ relative to the ellipsoid ε and $\overline{\mathcal{L}}$ -capacity of the distribution $\overline{H}$, respectively. According to above proved,

$$V_H(x) = \int_{\varepsilon} g(x, y) d\mu_H(y), \quad (3.5)$$

moreover, the function $V_H(x)$ is the generalized solution of the equation $\mathcal{L}V_H = 0$ in $\varepsilon \setminus \overline{H}$, vanishing on 0 and in $\partial\varepsilon$ on 1 in ∂H in the sense $W_{2,\alpha}^1(\varepsilon)$. Let now $u(x) \in \mathcal{M}(D)$ be an arbitrary solution of the equation $\mathcal{L}u = 0$ in $D \setminus E$, vanishing on ∂D , $M = \sup_D |u|$. It is easy to see that the function $V_H(x)$ is nonnegative on ∂D , in the sense $W_{2,\alpha}^1(D)$. Hence, it follows, that the function $u(x) - MV_H(x)$ is generalized solution of the equation $\mathcal{L}u = 0$ in $D \setminus \overline{H}$, is non-positive on $\partial(D \setminus \overline{H})$. According to Lemma 2.4, $u(x) - MV_H(x) \leq 0$ and $D \setminus \overline{H}$ in particular

$$u(x^0) \leq MV_H(x^0) \leq M \sup_{y \in \partial H} g(x^0, y) \mu_H(\overline{H}) = M \sup_{y \in \partial H} g(x^0, y) \text{cap}_{\mathcal{L}}^{(\varepsilon)}(\overline{H}). \quad (3.6)$$

By virtue of continuity of the function $g(x, y)$ at $x \neq y$ and (3.4), we obtain

$$\sup_{y \in \partial H} g(x^0, y) \leq C_6(\gamma, \alpha, n, x^0, E). \quad (3.7)$$

Thus, from (3.3) and (3.6), we conclude

$$u(x^0) \leq MC_6\varepsilon. \quad (3.8)$$

Using an arbitrariness ε , we have

$$u(x^0) \leq 0. \quad (3.9)$$

Making analogous considerations with the function $u(x) + MV_H(x)$, we obtain

$$u(x^0) \geq 0. \quad (3.10)$$

From (3.8)-(3.9) and an arbitrariness of the point x^0 it follows that $u(x) \equiv 0$ in $D \setminus E$. Thereby, the sufficiency of condition (3.1) is proved. Let us prove its necessity. Let us assume that $\text{cap}_{\mathcal{L}}(E) > 0$. Denote by ε' the ellipsoid such that $\bar{\varepsilon}' \subset \delta$, $E \subset \varepsilon'$. Assume $D = \varepsilon$ potential of the compact E relative to the ellipsoid ε' and $\mathcal{L}$ -capacity distribution E , respectively. Following to [11], we can give the equivalent definition of Vallee-Poussin type of $\mathcal{L}$ -capacity of the compact E , relative to the ellipsoid ε' . Let $g(x, y)$ be a Green function of the operator $\mathcal{L}$ in ε' . Let us call the measure μ on E , $\mathcal{L}$ -admissible if $\mu \subset E$ and

$$V_{\mu}^E(x) = \int_{\varepsilon'} g(x, y) d\mu(y) \leq 1 \quad \text{for } x \in \sup p\mu. \quad (3.11)$$

The value $\sup \mu(E) = \text{cap}_{\mathcal{L}}^{(\varepsilon')}(E)$, where an exact upper boundary is taken on all $\mathcal{L}$ -admissible measures and is called $\mathcal{L}$ -capacity of the compact E relative to the ellipsoid ε' .

Analogously, the $\mathcal{L}$ -capacity $\text{cap}_{\mathcal{L}}(E)$ is determined. At this, by the standard method, we show that there exists the unique measure on which an exact upper boundary of the functional $\mu(E)$ is reached by the set of all $\mathcal{L}$ -admissible measures μ . This measure is $\mathcal{L}$ -capacity distribution of the compact E .

According to the above proved, the function $u_E(x)$ is generalized solution of the equation $\mathcal{L}u_E = 0$ in $\varepsilon' \setminus E$, equaling to zero on $\partial\varepsilon'$. Besides, from (3.10) and maximum principle, it follows that $u_E(x) \in M(\varepsilon')$. On the other side, $u_E(x) \not\equiv 0$, as $V_H(E) > 0$. Theorem is proved. $\square$

Lemma 3.2. *Let relative to the coefficients of the operator $\mathcal{L}$ condition (1.2) be fulfilled. Then, if $y \in \partial\varepsilon_{R,2}(0)$, then*

$$C_7(\gamma, \alpha, n)R^{n+((\alpha)/2)-2} \leq \text{cap}_{\mathcal{L}}(\bar{\varepsilon}_{R,1}(y)) \leq C_8(\gamma, \alpha, n)R^{n+((\alpha)/2)-2}. \quad (3.12)$$

Proof. Let $\mathcal{L}_0 = \sum_{i=1}^n (\partial/\partial x_i)(\lambda_i(x)(\partial/\partial x_i))$. Then, according to (1.2),

$$\gamma \text{cap}_{\mathcal{L}_0}(\bar{\varepsilon}_{R,1}(y)) \leq \text{cap}_{\mathcal{L}}(\bar{\varepsilon}_{R,1}(y)) \leq \gamma^{-1} \text{cap}_{\mathcal{L}_0}(\bar{\varepsilon}_{R,1}(y)). \quad (3.13)$$

Let $u(x) \in C_0^\infty(\varepsilon_{R,(3/2)}(y))$, $u(x) = 1$ for $\varepsilon_{R,1}(y)$; moreover,

$$|u_i(x)| \leq \frac{C_9(\lambda, n)}{R^{1+(\alpha_i/2)}}, \quad i = 1, \dots, n. \quad (3.14)$$

Then,

$$\text{cap}_{\mathcal{L}_0}(\bar{\varepsilon}_{R,1}(\mathbf{y})) \leq \int_{\varepsilon_{R,(3/2)}(\mathbf{y})} \sum_{i=1}^n \lambda_i(x) u_i^2 dx. \quad (3.15)$$

On the other side, as $\mathbf{y} \in \partial \varepsilon_{R,2}(0)$, then $\sum_{i=1}^n (y_i^2 / R^{\alpha_i}) = 4R^2$, and thereby

$$|y_i| \leq 2R^{1+(\alpha_i/2)}; \quad i = 1, \dots, n. \quad (3.16)$$

Besides, as $x \in \varepsilon_{R,(3/2)}(\mathbf{y})$, then

$$|x_i - y_i| \leq \frac{3}{2}R^{1+(\alpha_i/2)}; \quad i = 1, \dots, n. \quad (3.17)$$

Thus,

$$|x_i| \leq |y_i| + |x_i - y_i| \leq \frac{7}{2}R^{1+(\alpha_i/2)}; \quad i = 1, \dots, n. \quad (3.18)$$

Hence, it follows that

$$|x|_\alpha \leq R \sum_{i=1}^n \left(\frac{7}{2}\right)^{2/(2+\lambda_i)}. \quad (3.19)$$

Therefore,

$$\lambda_i(x) \leq C_{10}^{\alpha_i} R^{\alpha_i} \leq C_{10}^{\alpha^+} R^{\alpha_i}; \quad i = 1, \dots, n. \quad (3.20)$$

where $\alpha^+ = \max\{\alpha_1, \dots, \alpha_n\}$.

Allowing for (3.14) and (3.20) in (3.15), we obtain

$$\text{cap}_{\mathcal{L}_0}(\bar{\varepsilon}_{R,1}(\mathbf{y})) \leq C_{10}(\alpha, n) R^{-2} \text{mes}(\varepsilon_{R,(3/2)}(\mathbf{y})) = C_{11}(\alpha, n) R^{n+((\alpha)/2)-2}, \quad (3.21)$$

and by virtue of (3.13), the estimation from upper in (3.11) is proved.

For showing the truth of the estimations from lower in (3.11), we note that

$$\text{cap}_{\mathcal{L}_0}(\bar{\varepsilon}_{R,1}(\mathbf{y})) \geq \text{cap}_{\mathcal{L}_0}(\bar{\varepsilon}_{R,(1/2\sqrt{n})}(\mathbf{y})). \quad (3.22)$$

Besides, considering the same as in [8], we conclude

$$\text{cap}_{\mathcal{L}_0}(\bar{\varepsilon}_{R,(1/2\sqrt{n})}(\mathbf{y})) \geq C_{12}(\alpha, n) \text{cap}_{\mathcal{L}_0}^{(\varepsilon_0)}(\bar{\varepsilon}_{R,(1/2\sqrt{n})}(\mathbf{y})), \quad (3.23)$$

where $\varepsilon_0 = \varepsilon_{R,(1/\sqrt{n})}(\mathbf{y})$.

Let $W = \{u(x) : u(x)C_0^\infty(\varepsilon_0), u(x) = 1 \text{ for } x \in \varepsilon_{R(1/2\sqrt{n})}(y)\}$. Then,

$$\text{cap}_{\mathcal{L}_0}^{(\varepsilon_0)}(\bar{\varepsilon}_{R(1/2\sqrt{n})}(y)) = \inf_{u \in W} \int_{\varepsilon_0} \sum_{i=1}^n \lambda_i(x) u_i^2 dx. \quad (3.24)$$

On the other side, if $y \in \partial \varepsilon_{R,2}(0)$, then we can find $i_0, 1 \leq i_0 \leq n$ such that $y_{i_0}^2 \geq 4R^{2+\alpha_{i_0}}/n$, that is;

$$|y_{i_0}| \geq \frac{4R^{1+(\alpha_{i_0}/2)}}{\sqrt{n}}. \quad (3.25)$$

Besides, as $x \in \varepsilon_0$, then

$$|x_{i_0} - y_{i_0}| \leq \frac{R^{1+(\alpha_{i_0}/2)}}{\sqrt{n}}. \quad (3.26)$$

Therefore,

$$|x_{i_0}| \geq |y_{i_0}| - |x_{i_0} - y_{i_0}| \geq \frac{R^{1+(\alpha_{i_0}/2)}}{\sqrt{n}}. \quad (3.27)$$

Thereby,

$$\lambda_i(x) \geq n^{-1/(2+\alpha_{i_0})} R; \quad i = 1, \dots, n, \quad (3.28)$$

where $\alpha^- = \min\{\alpha_1, \dots, \alpha_n\}$.

Allowing for (3.28) in (3.24), we obtain

$$\text{cap}_{\mathcal{L}_0}^{(\varepsilon_0)}(\bar{\varepsilon}_{R(1/2\sqrt{n})}(y)) = C_{13}(\alpha, n) \inf_{u \in W} \int_{\varepsilon_0} \sum_{i=1}^n R^{\alpha_i} u_i^2 dx. \quad (3.29)$$

Denote by $B_R(z)$ the ball $\{x : |x-z| < R\}$. Let us make in (3.30) the substitution of the variables $v_i = x_i/R^{1+(\alpha_i/2)}$; $i = 1, \dots, n$, and let $\tilde{y}$ be an image of the point y , where $\tilde{W} = \{\tilde{u}(v) : \tilde{u}(\tau)C_0^\infty(B_0), \tilde{u}(\tau) = 1 \text{ for } v \in B_{(1/2\sqrt{n})}(\tilde{y})\}$. Then, from (3.30), we deduce $B_0 = B_{(1/2\sqrt{n})}(\tilde{y})$ where by (3.30),

$$\begin{aligned} \text{cap}_{\mathcal{L}_0}^{(\varepsilon_0)}(\bar{\varepsilon}_{R(1/2\sqrt{n})}(y)) &\geq C_{13}R^{n+(\alpha/2)-2} \inf_{\tilde{u} \in \tilde{W}} \int_{B_0} \sum_{i=1}^n \left(\frac{\partial \tilde{u}}{\partial v_i} \right)^2 d\tau \\ &= C_{13}R^{n+(\alpha/2)-2} \text{cap}^{(B_0)}(\bar{B}_{(1/2\sqrt{n})}(\tilde{y})), \end{aligned} \quad (3.30)$$

we will denote by $\text{cap}^{(B_0)}(\bar{B}_{(1/2\sqrt{n})}(\tilde{y}))$ Wiener capacity of the compact $\bar{B}_{(1/2\sqrt{n})}(\tilde{y})$, relative to the ball B_0 . Now, it is sufficient to note that $\text{cap}^{(B_0)}(\bar{B}_{(1/2\sqrt{n})}(\tilde{y})) = C_{14}(n)$, and required estimation follows from (3.22), (3.23), and (3.31). Lemma is proved. $\square$

Lemma 3.3. *Let relative to the coefficients of the operator $\mathcal{L}$ condition (1.2) be fulfilled. Then,*

$$C_{15}(\gamma, \alpha, n) R^{n+(\langle \alpha \rangle / 2)-2} \leq \text{cap}_{\mathcal{L}}(\bar{\varepsilon}_{R,1}(\gamma)) \leq C_{16}(\gamma, \alpha, n) R^{n+(\langle \alpha \rangle / 2)-2}. \quad (3.31)$$

Upper estimation in (3.32) is proved analogously to the estimation in (3.11). For the proofing of the lower estimation, it is sufficient to note that $\varepsilon_{R,(1/4)}(\bar{\gamma}) \subset \varepsilon_{R,1}(0)$, that is,

$$\text{cap}_{\mathcal{L}}(\bar{\varepsilon}_{R,(1/4)}(\bar{\gamma})) < \text{cap}_{\mathcal{L}}(\bar{\varepsilon}_{R,1}(0)), \quad (3.32)$$

where $\bar{\gamma} = ((1/2)R^{1+(\alpha/2)}, 0, \dots, 0)$ and repeat the consideration of the proofing of the previous lemma.

Corollary 3.4. *If conditions (1.2)–(1.6) $\gamma \in \partial \varepsilon_{R,2}(0)$ be fulfilled, then for any $\rho \in (0, R]$, the estimation*

$$\text{cap}_{\mathcal{L}}(\bar{\varepsilon}_{\rho,1}(\bar{\gamma})) \leq C_{17}(\gamma, \alpha, n) \rho^{n+(\langle \alpha \rangle / 2)-2} \left(1 + \sum_{i=1}^n \left(\frac{R}{\rho} \right)^{\alpha_i} \right) \quad (3.33)$$

is true.

Then, $v(x) \in C_0^\infty(\varepsilon_{\rho,(3/2)}(\gamma))$, $v(x) = 1$ for $x \in \varepsilon_{\rho,1}(\gamma)$

$$|v_i(x)| \leq \frac{C_{18}(\alpha, n)}{\rho^{1+(\alpha_i/2)}}; \quad i = 1, \dots, n, \quad (3.34)$$

$$\text{cap}_{\mathcal{L}_0}(\bar{\varepsilon}_{\rho,1}(\bar{\gamma})) = \gamma^{-1} C_{18}^2 \rho^{-2} \int_{\varepsilon_{\rho,(3/2)}(\gamma)} \sum_{i=1}^n \lambda_i(x) \rho^{-\alpha_i} dx.$$

On the other side, arguing the same, as well as in the proof of Lemma 3.2, we have

$$\lambda_i(x) < C_{19}(\alpha, n) (R + \rho)^{\alpha_i}, \quad x \in \varepsilon_{\rho,(3/2)}(\gamma); \quad i = 1, \dots, n. \quad (3.35)$$

Now, it is sufficient to take into account that

$$\sum_{i=1}^n \left(1 + \frac{R}{\rho} \right)^{\alpha_i} \leq \sum_{i=1}^n \left[1 + \left(\frac{R}{\rho} \right)^{\alpha_i} \right] \leq n \left(1 + \sum_{i=1}^n \left(\frac{R}{\rho} \right)^{\alpha_i} \right), \quad (3.36)$$

and from (3.34)–(3.35) follows the required estimation (3.33).

Corollary 3.5. *If conditions (1.2)–(1.6) $\gamma \neq 0$ are fulfilled, then at $x \in \varepsilon_{d|\gamma|_a,1}(\gamma)$, $x \neq \gamma$ for the fundamental solution $G(x, \gamma)$, the estimation*

$$G(x, \gamma) \geq C_{20}(\gamma, \alpha, n) \frac{(|x - \gamma|_a)^{2-n-(\langle \alpha \rangle / 2)}}{1 + \sum_{i=1}^n (|\gamma|_a / |x - \gamma|_a)^{\alpha_i}} \quad (3.37)$$

is true.

If $y = 0$, then estimation (3.37) is true for all $x \neq 0$. Here, $d = 1/n2^{2/(2+\alpha)}$.
For proving, at first, let us show that if $y \neq 0$, then $y \notin \varepsilon_{d|y|_d, 2}(0)$. Really, as

$$|y|_\alpha = \sum_{i=1}^n |y_i|^{2/(2+\alpha i)}, \quad (3.38)$$

then there exists i_0 , $1 \leq i_0 \leq n$ such that

$$|y_0|^{2/(2+\alpha i_0)} \geq \frac{|y|_\alpha}{n}. \quad (3.39)$$

Thus,

$$\frac{|y_{i_0}^2|}{(|y|_\alpha)^{\alpha i_0}} \geq \frac{(|y|_\alpha)^2}{n^{2+\alpha i_0}}. \quad (3.40)$$

Thereby,

$$\sum_{i=1}^n \frac{y_i^2}{(d|y|_\alpha)^{\alpha i}} \geq \frac{y_{i_0}^2}{(d|y|_\alpha)^{\alpha i_0}} \geq \frac{(d|y|_\alpha)^2}{(dn)^{2+\alpha i_0}} = \frac{4(d|y|_\alpha)^2}{(2^{2/(2+\alpha i_0)} dn)^{2+\alpha i_0}}. \quad (3.41)$$

Now, it is sufficient to note that $2^{2/(2+\alpha i_0)} dn \leq 2^{2/(2+\alpha)} dn = 1$, and the required assertion is proved.
On the other side from (3.38), it follows that for all i , $1 \leq i \leq n$

$$|y_i|^{2/(2+\alpha i)} \leq |y|_\alpha, \quad (3.42)$$

that is,

$$\sum_{i=1}^n \frac{y_i^2}{(|y|_\alpha)^{\alpha i}} \leq n(|y|_\alpha)^2. \quad (3.43)$$

So, one will show that $\varepsilon_{|y|_\alpha, \sqrt{n}}(0)$ if only $y \neq 0$.

Let now for $y \neq 0$, $x \in \varepsilon_{d|y|_d, 1}(y)$ and $x \neq y$. Denote by $|x - y|_\alpha$ the ρ . It is easy to see that there exists i_1 , $1 \leq i_1 \leq n$ such that

$$|x_{i_1} - y_{i_1}|^{2/(2+\alpha i_1)} \geq \frac{\rho}{n}. \quad (3.44)$$

Hence, it follows that

$$\sum_{i=1}^n \frac{(x_i - y_i)^2}{\rho^{\alpha i}} \geq \frac{(x_{i_1} - y_{i_1})^2}{\rho^{\alpha i_1}} \geq \frac{\rho^2}{n^{2+\alpha i_1}} \geq \frac{\rho^2}{n^{2+\alpha}}. \quad (3.45)$$

Thus, $x \notin \varepsilon_{\rho, d_1}(y)$, where $d_1 = 1/n^{1+(\alpha/2)}$. Analogously, it is proved that $x \in \varepsilon_{\rho, \sqrt{n}}(y)$. Now, the required estimation (3.37) at $y \neq 0$ follows from (2.42) and Corollary 3.4 from Lemma 3.2. If $y = 0$, then (3.37) immediately follows from (2.42) and Lemma 2.7.

Let $F(x, y)$ be a positive function, determined in $E_n \times E_n$, continuous at $x \neq y$, moreover $\lim_{x \rightarrow y} F(x, y) = \infty$ (condition (A)).

Further, let $E \subset E_n$ be some compact. Let us call the measure μ on E $[F]$ admissible if $\sup p\mu \subset E$ and $V_\mu^E(x) = \int_E F(x, y) d\mu(y) \leq 1$, for $x \in \sup p\mu$.

The value $\sup \mu(E) = \text{cap}_{[F]}(E)$, where an exact upper boundary is taken by all $[F]$ admissible measures, is called $[F]$ -capacity of the compact E .

Theorem 3.6. Let relative to the coefficients of the operator $\mathcal{L}$ conditions (1.2)–(1.6) be fulfilled. Then, for removability of the compact $E \subset D$ relative to the first boundary-value problem for the operator $\mathcal{L}$ in the space $\mathcal{M}(D)$ it is sufficient that

$$\text{cap}_{[F_1]}(E) = 0, \quad (3.46)$$

where $F_1(x, y) = [1 + \sum_{i=1}^n (|y|_\alpha / |x - y|_\alpha)^{\alpha_i}]^{-1} (|x - y|_\alpha)^{2-n-(\langle \alpha \rangle / 2)}$.

Proof. We will use the following assertion, which is proved in [11]. Let the function $F(x, y)$ be satisfied condition (A), the compact E has zero $[F]$ -capacity, μ zero measure concentrated on E . Then, there exists the point $x^\circ \in \sup p\mu$, such that $V_\mu^E(x^\circ) = \infty$. At this the potential of the measure $\sup p\mu$ cannot be bounded on any portion B , that is, for any open set B at $E' \in \sup p\mu \cap B$; the potential $V_\mu^{E'}(x)$ is not bound B . In particular, if B is an arbitrary neighborhood of the point x° that $V_\mu^{E'}(x^\circ) = \infty$.

Let the condition (3.46) be fulfilled, μ an arbitrary measure, concentrated on E , $x^\circ \in \sup p\mu$ is a point, corresponding to the above-stated assertion at $F = F_1$. Let us assume at first that $x^\circ \neq 0$. Then, $|x^\circ|_\alpha = v > 0$. Further, let B be such small neighborhood of the point x° that if $E' \in \sup p\mu \cap B$, then

$$\sup_{y \in E'} |y|_\alpha \leq (1 + \varepsilon)r, \quad \inf_{y \in E'} |y|_\alpha \geq (1 + \varepsilon)r, \quad (3.47)$$

where the number $\varepsilon > 0$ will be chosen later. Let us consider the ellipsoids $\varepsilon_{d|y|_{d'}, 1}(y)$ at $y \in E'$. Let us choose ε such small than $x^0 \in \varepsilon_{d|y|_{d'}, 1}(y)$ for all $y \in E'$. Then, according to Corollary 3.5 from Lemma 2.7, we obtain

$$\begin{aligned} V_\mu^E(x^0) &= \int_E G(x^0, y) d\mu(y) \geq \int_{E'} G(x^0, y) d\mu(y) \geq C_{20} \int_E F_1(x^0, y) d\mu(y) \\ &= C_{20} V_\mu^E(x^0) = \infty. \end{aligned} \quad (3.48)$$

Hence, it follows that any zero measure μ , concentrated on E cannot be $\mathcal{L}$ admissible. Thus, $\text{cap}_{\mathcal{L}}(E) = 0$, and the required assertion follows from Theorem 3.1.

Let now $x^\circ = 0$. Then, using the equality $G(x, y) = G(y, x)$ and Corollary 3.5 from Lemma 2.7, we conclude

$$\begin{aligned} V_\mu^E(0) &= \int_E G(0, y) d\mu(y) = \int_E G(y, 0) d\mu(y) \geq C_{20} \int_E F_1(y, 0) d\mu(y) \\ &= C_{20} \int_E F_1(0, y) d\mu(y) = C_{20} V_\mu^E(0) = \infty. \end{aligned} \quad (3.49)$$

Theorem is proved. $\square$

Remark 3.7. Let conditions of the real theorem be fulfilled and the compact $E \subset D$ removable relative to the first boundary-value problem for the operator $\mathcal{L}$ in the space $\mathcal{M}(D)$. Then, $\text{mes}(E) = 0$.

At first, let us note for proofing that the discussion are the same, as at the conclusion of estimation (3.37), we can show that at $x \in \varepsilon_{d|y|_d, 1}(y)$, $x \neq y$ ($y \neq 0$), and at $x \neq y$ ($y = 0$) the estimations

$$G(x, y) \leq C_{21}(\gamma, \alpha, n)(|x - y|_d)^{2-n-\langle \alpha \rangle/2} \quad (3.50)$$

are true.

Further, analogously to Theorem 3.1, it is shown that if the compact E is removable, then according to $\text{cap}_{[-F_2]}(E) = 0$, where $F_2(x, y) = (|x - y|_d)^{2-n-\langle \alpha \rangle/2}$.

Hence, it follows that if $\text{mes}(E) > 0$, then there exists the point $x^2 \in E$, such that $V^E(x^1) = \infty$, where

$$V^E(x) = \int_E F_2(x, y) dy. \quad (3.51)$$

Moreover, if B' is an arbitrary neighborhood of the point $E' = B' \cap E$, then the potential $V^{E'}(x)$ is not bounded on E' . Let us consider the case $x^1 \neq 0$. Choose small neighborhood B' of the point x^1 that at all $x \in E'$, $y \in E'$ the inequality $|x_i - y_i| \leq 1$; $i = 1, \dots, n$ are fulfilled. For $x \in E'$, we have

$$\begin{aligned} V^{E'}(x) &= \int_{E'} \left(\sum_{i=1}^n |x_i - y_i|^{(2/2+\alpha i)} \right)^{2-n-\langle \alpha \rangle/2} dy \leq \int_{E'} \left(\sum_{i=1}^n |x_i - y_i| \right)^{2-n-\langle \alpha \rangle/2} dy \\ &\leq \int_{E'} |x - y|^{2-n-\langle \alpha \rangle/2} dy \leq \int_{B''} |z|^{2-n-\langle \alpha \rangle/2} dy, \end{aligned} \quad (3.52)$$

where B'' is a ball of the radius $\sqrt{n}$ with the center origin of the coordinate. Now, it is sufficient to note that according to condition (1.6) $\langle \alpha \rangle/2 \leq n/(n-1) \leq 3/2$, and the assertion the corollary is proved.

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