Research Article

Another Aspect of Triangle Inequality

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Received 18 February 2011; Accepted 14 March 2011

Academic Editors: Y. Dai and B. Djafari-Rouhani

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We introduce the notion of ψ -norm by considering the fact that an absolute normalized norm on $\mathbb{C}^2$ corresponds to a continuous convex function ψ on the unit interval $[0, 1]$ with some conditions. This is a generalization of the notion of q -norm introduced by Belbachir et al. (2006). Then we show that a ψ -norm is a norm in the usual sense.

1. Introduction

The triangle inequality is one of the most fundamental inequalities in analysis and has been studied by several authors. For example, Kato et al. in [1] showed a sharpened triangle inequality and its reverse one with n elements in a Banach space (see also [2–4]). Here we consider another aspect of the classical triangle inequality $\|x + y\| \leq \|x\| + \|y\|$. For a Hilbert space H , we recall the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad (x, y \in H). \quad (1.1)$$

This implies that the parallelogram inequality

$$\|x + y\|^2 \leq 2(\|x\|^2 + \|y\|^2) \quad (x, y \in H) \quad (1.2)$$

holds. Saitoh in [5] noted the inequality (1.2) may be more suitable than the classical triangle inequality and used the inequality (1.2) to the setting of a natural sum Hilbert space for two arbitrary Hilbert spaces. Motivated by this, Belbachir et al. [6] introduced the notion of q -norm ($1 \leq q < \infty$) in a vector space X over $\mathbb{K} (= \mathbb{R} \text{ or } \mathbb{C})$, where the definition of q -norm is a mapping $\|\cdot\|$ from X into $\mathbb{R}^+ (= \{a \in \mathbb{R} : a \geq 0\})$ satisfying the following conditions:

- (i) $\|x\| = 0 \Leftrightarrow x = 0$,
- (ii) $\|\alpha x\| = |\alpha| \|x\|$ ($x \in X, \alpha \in \mathbb{K}$),
- (iii) $\|x + y\|^q \leq 2^{q-1} (\|x\|^q + \|y\|^q)$ ($x, y \in X$).

We easily show that every norm is a q -norm. Conversely, they proved that for all q with $1 \leq q < \infty$, every q -norm is a norm in the usual sense.

In this paper, we generalize the notion of q -norm, that is, we introduce the notion of ψ -norm by considering the fact that an absolute normalized norm on $\mathbb{R}^2$ corresponds to a continuous convex function ψ on the unit interval $[0, 1]$ with some conditions (cf. [7]). We show that a ψ -norm is a norm in the usual sense.

We recall some properties of absolute normalized norms on $\mathbb{C}^2$. A norm $\|\cdot\|$ on $\mathbb{C}^2$ is called *absolute* if $\|(x, y)\| = \|(|x|, |y|)\|$ for all $(x, y) \in \mathbb{C}^2$ and *normalized* if $\|(1, 0)\| = \|(0, 1)\| = 1$. The ℓ_p -norms $\|\cdot\|_p$ are such examples:

$$\|(x, y)\|_p = \begin{cases} (|x|^p + |y|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{|x|, |y|\} & \text{if } p = \infty. \end{cases} \quad (1.3)$$

Let AN_2 be the family of all absolute normalized norms on $\mathbb{C}^2$. It is well known that the set AN_2 is a one-to-one correspondence with the set Ψ_2 of all continuous convex functions ψ on the unit interval $[0, 1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for t with $0 \leq t \leq 1$ (see [7, 8]). The correspondence is given by the equation $\psi(t) = \|(1-t, t)\|_\psi$. Indeed, for all ψ in Ψ_2 , we define the norm $\|\cdot\|_\psi$ as

$$\|(x, y)\|_\psi = \begin{cases} (|x| + |y|) \psi\left(\frac{|y|}{|x| + |y|}\right) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases} \quad (1.4)$$

Then $\|\cdot\|_\psi \in AN_2$ and satisfies $\psi(t) = \|(1-t, t)\|_\psi$. The functions which correspond to the ℓ_p -norms $\|\cdot\|_p$ on $\mathbb{C}^2$ are $\psi_p(t) = \{(1-t)^p + t^p\}^{1/p}$ if $1 \leq p < \infty$ and $\psi_\infty(t) = \max\{1-t, t\}$ if $p = \infty$.

2. ψ -Norm

Definition 2.1. Let X be a vector space and $\psi \in \Psi_2$. Then a mapping $\|\cdot\| : X \rightarrow \mathbb{R}^+$ is called ψ -norm on X if it satisfies the following conditions:

- (i) $\|x\| = 0 \Leftrightarrow x = 0$,
- (ii) $\|\alpha x\| = |\alpha| \|x\|$ ($x \in X, \alpha \in \mathbb{K}$),
- (iii) $\|x + y\| \leq (1/\min_{0 \leq t \leq 1} \psi(t)) (\|x\|, \|y\|)_\psi$ ($x, y \in X$).

Note that for all q with $1 \leq q < \infty$, any ψ_q -norm $\|\cdot\|$ is just a q -norm. Indeed, since the function ψ_q takes the minimum at $t = 1/2$ and

$$\psi_q\left(\frac{1}{2}\right) = \left(\left(\frac{1}{2}\right)^q + \left(\frac{1}{2}\right)^q\right)^{1/q} = 2^{1/q-1}, \quad (2.1)$$

the condition (iii) of Definition 2.1 implies

$$\|x + y\| \leq \frac{1}{\psi_q(1/2)} \|(\|x\|, \|y\|)\|_{\psi_q} = 2^{1-1/q} (\|x\|^q + \|y\|^q)^{1/q}. \quad (2.2)$$

Thus we have $\|x + y\|^q \leq 2^{q-1} (\|x\|^q + \|y\|^q)$ and so $\|\cdot\|$ becomes a q -norm.

If $\psi = \psi_1$, then the condition (iii) of Definition 2.1 is just a triangle inequality. Thus we suppose that $\psi \neq \psi_1$.

Proposition 2.2. *Let X be a vector space and $\psi \in \Psi_2$ with $\psi \neq \psi_1$. Then every norm on X in the usual sense is a ψ -norm.*

To do this, we need the following lemma given in [7].

Lemma 2.3 (see [7]). *Let $\varphi, \psi \in \Psi_2$ and $\varphi \geq \psi$. Put*

$$M = \max_{0 \leq t \leq 1} \frac{\varphi(t)}{\psi(t)}. \quad (2.3)$$

Then

$$\|\cdot\|_{\varphi} \leq \|\cdot\|_{\psi} \leq M \|\cdot\|_{\varphi}. \quad (2.4)$$

Proof of Proposition 2.2. Let $\|\cdot\|$ be a norm on X and $x, y \in X$. Since $\psi \leq \psi_1$, we have by Lemma 2.3,

$$\begin{aligned} \|x + y\| &\leq \|x\| + \|y\| = \|(\|x\|, \|y\|)\|_1 \\ &\leq \max_{0 \leq t \leq 1} \frac{1}{\psi(t)} \|(\|x\|, \|y\|)\|_{\psi} \\ &= \frac{1}{\min_{0 \leq t \leq 1} \psi(t)} \|(\|x\|, \|y\|)\|_{\psi}. \end{aligned} \quad (2.5)$$

Thus $\|\cdot\|$ is a ψ -norm on X . □

We will show that every ψ -norm is a norm in the usual sense. To do this, we need the following lemma given in [6].

Lemma 2.4 (see [6]). *Let X be a vector space. Let $\|\cdot\| : X \rightarrow \mathbb{R}^+$ be a mapping satisfying the conditions (i) and (ii) in Definition 2.1. Then $\|\cdot\|$ is a norm if and only if the set $B_X = \{x \in X : \|x\| \leq 1\}$ is convex.*

Proof. Suppose that B_X is convex. For every $x, y \in X$ such that $x \neq 0, y \neq 0$, we have

$$\left\| \frac{x}{\|x\| + \|y\|} + \frac{y}{\|x\| + \|y\|} \right\| = \left\| \frac{\|x\|}{\|x\| + \|y\|} \frac{x}{\|x\|} + \frac{\|y\|}{\|x\| + \|y\|} \frac{y}{\|y\|} \right\| \leq 1. \quad (2.6)$$

This completes the proof. $\square$

Since every φ_1 -norm is just a usual norm, we suppose that $\varphi \in \Psi_2$ with $\varphi \neq \varphi_1$. Put t_0 with $0 < t_0 < 1$ such that $\min_{0 \leq t \leq 1} \varphi(t) = \varphi(t_0)$. Then we have the following lemma.

Lemma 2.5. *Let $\|\cdot\|$ be a φ -norm on X . Then, for every $x, y \in B_X$ we have $(1 - t_0)x + t_0y \in B_X$.*

Proof. Let $x, y \in B_X$. We may assume that $x \neq y$ and $x, y \neq 0$. From the definition of a φ -norm and Lemma 1 in [8], we have

$$\begin{aligned} \|(1 - t_0)x + t_0y\| &\leq \frac{1}{\varphi(t_0)} \|((1 - t_0)\|x\|, t_0\|y\|)\|_{\varphi} \\ &\leq \frac{1}{\varphi(t_0)} \|(1 - t_0, t_0)\|_{\varphi} = 1, \end{aligned} \quad (2.7)$$

which implies $(1 - t_0)x + t_0y \in B_X$. $\square$

Here we define the set A_n for all $n = 1, 2, \dots$, by

$$A_0 = \{0, 1\}, \quad A_n = \{(1 - t_0)a + t_0b : a, b \in A_{n-1}\} \quad (n = 1, 2, \dots). \quad (2.8)$$

Put $A = \bigcup_{n=0}^{\infty} A_n$. It is clear that $\overline{A} = [0, 1]$. We also define a function f by $f(x, y, t) = (1 - t)x + ty$ for all $x, y \in B_X$ and all $t \in [0, 1]$.

Lemma 2.6. *For every $x, y \in B_X$, we have $f(x, y, t) \in B_X$ for all $t \in A$.*

Proof. Let $x, y \in B_X$. It is clear that $f(x, y, t) \in B_X$ for all $t \in A_0$. We suppose that $f(x, y, t) \in B_X$ for all $t \in A_{n-1}$. Then, for all $t \in A_n$, there exist $a, b \in A_{n-1}$ such that $t = (1 - t_0)a + t_0b$. Hence

$$\begin{aligned} f(x, y, t) &= (1 - t)x + ty \\ &= (1 - ((1 - t_0)a + t_0b))x + ((1 - t_0)a + t_0b)y \\ &= (1 - t_0)((1 - a)x + ay) + t_0((1 - b)x + by) \\ &= (1 - t_0)f(x, y, a) + t_0f(x, y, b). \end{aligned} \quad (2.9)$$

Since $f(x, y, a)$ and $f(x, y, b)$ are in B_X , we have from Lemma 2.5, $f(x, y, t) \in B_X$ for all $t \in A_n$. Thus $f(x, y, t) \in B_X$ for all $t \in A$. $\square$

Theorem 2.7. *Let X be a vector space and $\varphi \in \Psi_2$ with $\varphi \neq \varphi_1$. Then every φ -norm on X is a norm in the usual sense.*

Proof. Let $x, y \in B_X$ and λ with $0 < \lambda < 1$. Let $z = (1 - \lambda)x + \lambda y$. Take a strictly decreasing sequence $\{r_n\}$ in A such that $r_n \searrow \lambda$. For each n , we define $\beta_n = (1 - r_n)/(1 - \lambda)$. Then $0 < \beta_n < 1$ and $\beta_n \nearrow 1$. Since $0 < \lambda\beta_n/r_n < 1$, we have $(\lambda\beta_n/r_n)y \in B_X$. By Lemma 2.6,

$$\begin{aligned}\beta_n z &= (1 - \lambda)\beta_n x + \lambda\beta_n y \\ &= (1 - r_n)x + r_n \frac{\lambda\beta_n}{r_n} y \\ &= f\left(x, \frac{\lambda\beta_n}{r_n} y, r_n\right) \in B_X.\end{aligned}\tag{2.10}$$

Since $\beta_n \|z\| = \|\beta_n z\| \leq 1$, we get $z \in B_X$. Thus B_X is convex. By Lemma 2.4, $\|\cdot\|$ becomes a norm. This completes the proof. $\square$

Acknowledgments

Kichi-Suke Saito was supported in part by Grants-in-Aid for Scientific Research (No. 20540158), Japan Society for the Promotion of Science. Runling An was supported by Program for Top Young Academic Leaders of Higher Learning Institutions of Shanxi (TYAL) and a grant from National Foundation of China (No. 11001194).

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