Research Article

Fourier Transform of Lookback Option Price

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The Fourier transform of the damped price of Lookback option under B-S model is presented. Thus, the Lookback option across a range of strikes can be simultaneously priced via FFT algorithm. FFT algorithm is more efficient than both Monte Carlo simulation method and the integral of the usual pricing formula. In addition, by FFT algorithm, investors can easily capture the sensitivity of option prices when the strike prices vary as to make reasonable investment decisions.

1. Introduction

Recently, the Fourier transform of option prices is of great interest to many researchers. Carr and Madan [1] use damped option price method to get the Fourier integral representation of standard European call and put option value. Motivated by the work of Carr and Madan [1], we use similar methods to give the Fourier transform of the damped price of Lookback option.

Although the Lookback option price has the explicit integral formula, the FFT algorithm is more efficient than the usual integral computations. In Section 1, we first get the characteristic function of the log maximum of stock prices on a time interval by a large amount of calculations, then we use the characteristic function to obtain Fourier transform of the damped price of Lookback option. Although the formulation is somewhat cumbersome, it is composed only of elementary functions, and it is readily applied on desktop computers. In Section 2, FFT algorithm is outlined to calculate the inversion of the Fourier transform obtained in Section 1. Thus, the Lookback option prices for a range of strikes can be obtained by only one FFT computation. In Section 3, we make a simple numerical experiment.

Here, we need the characteristic function of the log maximum of stock prices on a time interval, so we only consider the Black-Scholes model.

As usual, we assume that the stock price under the equivalent martingale measure (EMM) Q satisfies

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \quad (1.1)$$

where r is the risk-free rate, σ is the volatility, and $W(t)$ is a Q -Brownian motion. The payoff X at maturity T of a Lookback call option with fixed strike K is

$$X = \max(S^* - K, 0), \quad (1.2)$$

where $S^* = \max_{0 \leq t \leq T} S(t)$.

We first give a lemma on Brownian motion which will be used in the next section.

Lemma 1.1 (see [2]). *Suppose that $\{W(t)\}_{t \geq 0}$ is a standard Q -Brownian motion, a is a constant, and $(at + W(t))^* = \max_{0 \leq t \leq T} (at + W(t))$, then the joint density function of $(at + W(t), (at + W(t))^*)$ is*

$$f(x, y) = e^{ax - (1/2)a^2T} \sqrt{\frac{2}{\pi}} \frac{2y - x}{T^{3/2}} e^{-(2y-x)^2/2T}, \quad x < y. \quad (1.3)$$

2. Fourier Transform of Damped Option Value

From (1.1),

$$S(t) = S(0)e^{(r - (1/2)\sigma^2)t + \sigma W(t)}. \quad (2.1)$$

So, $\ln S(t) = \ln S(0) + (r - (1/2)\sigma^2)t + \sigma W(t)$ and $\ln S^* = \ln S(0) + ((r - (1/2)\sigma^2)t + \sigma W(t))^*$, where $((r - (1/2)\sigma^2)t + \sigma W(t))^* = \max_{0 \leq t \leq T} ((r - (1/2)\sigma^2)t + \sigma W(t))$.

To obtain the characteristic function of $\ln S^*$, we first calculate the density function of $((r - (1/2)\sigma^2/\sigma)t + W(t))^*$.

For simplicity of writing, we use a to denote $(r - (1/2)\sigma^2)/\sigma$, Φ , φ , and ϕ to denote the distribution function, density function, and characteristic function of standard normal distribution $N(0, 1)$, respectively, in the following.

Lemma 2.1. *The density function of $(at + W(t))^*$ is*

$$g(y) = \sqrt{\frac{2}{T\pi}} e^{2ya} \left(e^{-(y+Ta)^2/2T} - a\sqrt{2\pi T} \Phi\left(\frac{-y-Ta}{\sqrt{T}}\right) \right). \quad (2.2)$$

Proof. From Lemma 1.1, in the instruction, the density function of $(at + W(t))^*$ is

$$\begin{aligned} g(y) &= \sqrt{\frac{2}{\pi}} \frac{1}{T^{2/3}} \int_{-\infty}^y e^{ax - (1/2)a^2T} (2y - x) e^{-(2y-x)^2/2T} dx \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{T^{2/3}} \left[2ye^{2ya} \int_{-\infty}^y e^{-[x - (2y+Ta)]^2/2T} dx - e^{2ya} \int_{-\infty}^y x e^{-[x - (2y+Ta)]^2/2T} dx \right]. \end{aligned} \quad (2.3)$$

Since

$$\begin{aligned} \int_{-\infty}^y x e^{-[x-(2y+Ta)]^2/2T} dx &= \int_{-\infty}^y [x - (2y + Ta)] e^{-[x-(2y+Ta)]^2/2T} dx \\ &+ \int_{-\infty}^y (2y + Ta) e^{-[x-(2y+Ta)]^2/2T} dx, \end{aligned} \quad (2.4)$$

so

$$\begin{aligned} g(y) &= -\sqrt{\frac{2}{T\pi}} \frac{1}{T^{2/3}} e^{2ya} \left[\int_{-\infty}^y [x - (2y + Ta)] e^{-[x-(2y+Ta)]^2/2T} dx + Ta \int_{-\infty}^y e^{-[x-(2y+Ta)]^2/2T} dx \right] \\ &= \sqrt{\frac{2}{T\pi}} e^{2ya} \left[e^{-(y+Ta)^2/2T} - a\sqrt{2\pi T} \Phi\left(\frac{-y-Ta}{\sqrt{T}}\right) \right]. \end{aligned} \quad (2.5)$$

The proof is completed. $\square$

Proposition 2.2. *The characteristic function of $(at + W(t))^*$ is*

$$2\left(1 - \frac{a}{iu + 2a}\right) e^{-i(u-2ai)Ta} \phi\left(\sqrt{T}(u - 2ai)\right). \quad (2.6)$$

Proof. From Lemma 2.1, the characteristic function of $(at + W(t))^*$ is

$$\begin{aligned} &\sqrt{\frac{2}{T\pi}} \int_{-\infty}^{\infty} e^{iuy} e^{2ya} \left[e^{-(y+Ta)^2/2T} - a\sqrt{2\pi T} \Phi\left(\frac{-y-Ta}{\sqrt{T}}\right) \right] dy \\ &= \sqrt{\frac{2}{T\pi}} \left[\int_{-\infty}^{\infty} e^{i(u-2ai)y} e^{-(y+Ta)^2/2T} dy - a\sqrt{2\pi T} \int_{-\infty}^{\infty} e^{i(u-2ai)y} \Phi\left(\frac{-y-Ta}{\sqrt{T}}\right) dy \right]. \end{aligned} \quad (2.7)$$

Since the first term in the brackets is

$$\int_{-\infty}^{\infty} e^{i(u-2ai)y} e^{-(y+Ta)^2/2T} dy = \sqrt{2\pi T} e^{-i(u-2ai)Ta} \phi\left(\sqrt{T}(u - 2ai)\right), \quad (2.8)$$

and the second term in the brackets is

$$\begin{aligned} &a\sqrt{2\pi T} \int_{-\infty}^{\infty} e^{i(u-2ai)y} \Phi\left(\frac{-y-Ta}{\sqrt{T}}\right) dy \\ &= a\sqrt{2\pi T} \left[\frac{1}{i(u-2ai)} e^{i(u-2ai)y} \Phi\left(\frac{-y-Ta}{\sqrt{T}}\right) \Big|_{-\infty}^{\infty} \right. \\ &\quad \left. - \int_{-\infty}^{\infty} \frac{1}{i(u-2ai)} e^{i(u-2ai)y} \varphi\left(\frac{-y-Ta}{\sqrt{T}}\right) \left(-\frac{1}{\sqrt{T}}\right) dy \right] \\ &= a\sqrt{2\pi T} \frac{1}{i(u-2ai)} \int_{-\infty}^{\infty} e^{i(u-2ai)y} \frac{1}{\sqrt{T}} \varphi\left(\frac{-y-Ta}{\sqrt{T}}\right) dy \\ &= a\sqrt{2\pi T} \frac{1}{i(u-2ai)} e^{-i(u-2ai)Ta} \phi\left(\sqrt{T}(u - 2ai)\right), \end{aligned} \quad (2.9)$$

so the characteristic function of $(at + W(t))^*$ is

$$\begin{aligned} & \sqrt{\frac{2}{T\pi}} \left[\sqrt{2\pi T} e^{-i(u-2ai)Ta} \phi\left(\sqrt{T}(u-2ai)\right) - a\sqrt{2\pi T} \frac{1}{i(u-2ai)} e^{-i(u-2ai)Ta} \phi\left(\sqrt{T}(u-2ai)\right) \right] \\ &= 2\left(1 - \frac{a}{iu + 2a}\right) e^{-i(u-2ai)Ta} \phi\left(\sqrt{T}(u-2ai)\right). \end{aligned} \quad (2.10)$$

The proof is completed. $\square$

Recall that $a = (r - (1/2)\sigma^2)/\sigma$ and $\ln S^* = \ln S(0) + ((r - (1/2)\sigma^2)t + \sigma W(t))^*$, then from Proposition 2.2, we can easily obtain the characteristic function of $\ln S^*$ (denoted by ϕ_*) as stated in the following theorem.

Theorem 2.3. *The characteristic function of $\ln S^*$ is*

$$\phi_*(u) = 2e^{iu \ln S_0} \left(1 - \frac{a}{i\sigma u + 2a}\right) e^{-i(\sigma u - 2ai)Ta} \phi\left(\sqrt{T}(\sigma u - 2ai)\right). \quad (2.11)$$

Let k denote the log strike price, that is, $k = \ln K$, and $C(k)$ the Lookback call option price at time-0. To obtain a square-integrable function, one uses the damped option price [1]; that is, let

$$c(k) = e^{\alpha k} C(k) = e^{\alpha k - rT} E_Q \left[\left(S^* - e^k \right)^+ \right], \quad (2.12)$$

for $\alpha > 0$. The discussion for the choice of α in Carr and Madan [1] is applicable here. In Section 3, one makes a simple numerical experiment for α .

We write $\Psi(u)$ as the Fourier transform of $c(k)$, $\rho(s)$ as the density function of $\ln S^*$, and $\phi_*(u)$ as the characteristic function of $\ln S^*$. Then, we have

$$\begin{aligned} \Psi(u) &= \int_{-\infty}^{\infty} e^{iuk} c(k) dk \\ &= e^{-rT} \int_{-\infty}^{\infty} e^{iuk} e^{\alpha k} \int_{-k}^{\infty} (e^s - e^k) \rho(s) ds dk \\ &= e^{-rT} \int_{-\infty}^{\infty} \rho(s) \int_{-\infty}^s [e^{s+\alpha k} - e^{(1+\alpha)k}] e^{iuk} dk \\ &= \frac{e^{-rT} \phi_*(u - (\alpha + 1)i)}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u}. \end{aligned} \quad (2.13)$$

From Theorem 2.3, we have

$$\Psi(u) = \frac{2e^{-rT} e^{i(u-(\alpha+1)i) \ln S_0} A e^{-i(\sigma(u-(\alpha+1)i)-2ai)Ta} \phi\left(\sqrt{T}(\sigma(u-(\alpha+1)i)-2ai)\right)}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u}, \quad (2.14)$$

where $A = (i\sigma(u - (\alpha + 1)i) + a) / (i\sigma(u - (\alpha + 1)i) + 2a)$.

3. Using FFT to Price Lookback Option

$C(k)$ thus can be calculated by taking the Fourier inversion transform

$$\begin{aligned} C(k) &= \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \Psi(u) du \\ &= \frac{e^{-\alpha k}}{\pi} \operatorname{Re} \left(\int_0^{\infty} e^{-iuk} \Psi(u) du \right), \end{aligned} \quad (3.1)$$

where Re denotes the real part of a complex number and the second equality is due to that $\Psi(u)$ is odd in its imaginary part and even in its real part.

The above integral can be computed using FFT. A numerical approximation for $C(k)$ is

$$C(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^N e^{-iu_j k} \Psi(u_j) \Delta u, \quad (3.2)$$

where $u_j = j\Delta u$, $j = 1, 2, \dots, N$. Discussions on the errors in the numerical computing are presented in Lee [3].

The FFT is an efficient algorithm that computes the sum of the following form:

$$y(n) = \sum_{j=1}^N e^{-i(2\pi/N)jn} x(j), \quad n = 1, \dots, N. \quad (3.3)$$

The interesting values of the strike price K is around the forward price, that is, $e^{rT} S_0$, so we calculate the sum (3.2) for $k = rT + \ln S_0 + m\Delta k$, $m = -(N/2) + 1, \dots, N/2$.

For (3.2) to be transformed to the form of (3.3), we let $n = m + (N/2)$ (then n ranges from 1 to N), $\Delta u \Delta k = 2\pi/N$, and N be an integer power of 2.

Then, (3.2) turns to be

$$C(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^N e^{-i(2\pi/N)jn} e^{-ij\Delta u(rT + \ln S_0) + ij\pi} \Psi(u_j) \Delta u, \quad n = 1, \dots, N, \quad (3.4)$$

where $e^{-ij\Delta u(rT + \ln S_0) + ij\pi} \Psi(u_j) \Delta u$ is the counterpart of $x(j)$ in (3.3). So, (3.4) can be calculated by FFT quickly.

4. Simple Numerical Experiment

In our example, $T = 1$; $r = 0.05$; $\sigma = 0.3$; $S_0 = 5$; $N = 10000$; $\Delta u = 0.01$, and we use matlab software as a computing tool.

Figure 1: Option prices $C(k)$ against strikes K .

Figure 2: The results of FFT and Monte Carlo.

Motivated by Lord et al. [4], we let α ranges from 1 to 3, increasing 0.1 each step. Then, we calculated the option prices $C(k)$ against strikes $K = e^k$ around the forward future price $S_0 e^{rT}$ for every α . Fix $N = 2^{14}$, and we find that for each α , the curves of option prices against strikes are almost the same curve, as Figure 1 shows. It shows the results are relatively stable for different α .

We take $\alpha = 2$ and compare the results with Monte Carlo method. See Figure 2, where $N = 2^{14}$ in FFT algorithm and Monte Carlo method do 10^4 simulations. Although the results are similar, FFT is much more efficient than Monte Carlo simulation.

5. Conclusion

Although the Lookback option price has the explicit integral formula, the FFT algorithm is more efficient than the usual integral computations. Also, using FFT calculation once, practitioners can directly capture the price sensitivity of an option with varying strike prices. Using the same technique above, we can also obtain the Fourier transform of Lookback call and put options with floating strikes under Black-Scholes model. For asset prices under Lévy processes, Feng and Linetsky [5] take the asset prices on discrete time points to get the approximate price of Lookback options; Kou [6] give a survey of several discretization method to price Barrier and Lookback options.

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