Research Article

The Topology of GB-Metric Spaces

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We define the concepts of GB-metric in sets over σ -complete Boolean algebra and obtain some applications of them on the theory of topology. We also study some related properties of them.

1. Introduction

Numerous studies have been made concerning geometries and topologies induced in sets by general distance functions. A formulation of the notion “generalized metric space (or G-metric space)” has been given [1]. In this paper, we begin the elaboration of the topology induced in sets over σ -complete Boolean algebra.

In this paper, B shall always denote a σ -complete Boolean algebra. In B , we denote the operations of join, meet, and complement by $a \vee b$, $a \wedge b$ and a' , respectively.

2. The GB-Metric Spaces

In 1952, a new structure of metric spaces, so called B -metric space was introduced by Ellis and Sprinkle [2], on the set X to Boolean algebra (for details see [3, 4]).

Definition 2.1 (see [2]). A B -metric space is a set X with a map $d : X \times X \rightarrow B$ (B is σ -Boolean algebra) with the properties

- (1) $d(\xi, \eta) = 0$ if and only if $\xi = \eta$,
- (2) $d(\eta, \xi) = d(\xi, \eta)$, (symmetry), and
- (3) $d(\xi, \zeta) \leq d(\xi, \eta) \vee d(\eta, \zeta)$, for all ξ, η, ζ belong to X .

In [1], the present author has introduced a new structure of metric spaces which is a generalized idea of the ordinary metric space. The term generalized metric space is used in [5, 6].

Definition 2.2 (see [7]). Let X be a nonempty set and $G : X \times X \times X \rightarrow [0, \infty)$ be a function satisfying the following properties:

- (G₁) $G(x, y, z) = 0$ if $x = y = z$,
- (G₂) $0 < G(x, x, y)$; for all $x, y \in X$, with $x \neq y$,
- (G₃) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,
- (G₄) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$, (symmetry in all three variables), and
- (G₅) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

2.1. Topology Of GB-Metric Spaces

In this section we define generalized B -metric (or GB -metric) space and introduce some basic notions and results that are used in sequel.

Definition 2.3. Let X be a non empty set and $G_B : X \times X \times X \rightarrow B$, be a function satisfying the following properties

- (GB₁) $G_B(x, y, z) = 0$ if $x = y = z$,
- (GB₂) $0 < G_B(x, x, y)$; for all $x, y \in X$, with $x \neq y$,
- (GB₃) $G_B(x, x, y) \leq G_B(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,
- (GB₄) $G_B(x, y, z) = G_B(x, z, y) = G_B(y, z, x) = \dots$, (symmetry in all three variables), and
- (GB₅) $G_B(x, y, z) \leq G_B(x, a, a) \vee G_B(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then, the function G_B is called a generalized B -metric, or, more specifically, a G_B -metric on X , and the pair (X, G_B) is called a GB -metric space.

Example 2.4. Put $\mathbb{W} = \mathbb{N} \cup \{0\}$. Define a map $G_B : \mathbb{W} \times \mathbb{W} \times \mathbb{W} \rightarrow \{0, 1\}$ by

$$G_B(m, n, k) = \begin{cases} 0 & m = n = k, \\ 1 & \text{else.} \end{cases} \quad (2.1)$$

The map G_B is a GB -metric on $\mathbb{W}$.

Remark 2.5. We can show that a GB -metric space is a generalized B -metric space over X .

Proposition 2.6. Every GB -metric space (X, G_B) will define a B -metric (X, d_{G_B}) by $d_{G_B}(x, y) = G_B(x, y, y)$.

Proof. Conditions 1 and 2 of B -metric are clearly and 3 follows from (GB₅). □

Proposition 2.7. Let G_B be GB -metric on a ring X . For an element $a \in X$, the following maps are GB -metrics on the ring X :

- (1) $G_B^*(x, y, z) = G_B(a - x, a - y, a - z)$ and
- (2) $G_B^*(x, y, z) = G_B(x, y, z) \vee G_B(a - x, a - y, a - z)$.

Proposition 2.8. Let (X, d) be a B -metric space. Define a function, $G_B : X \times X \times X \rightarrow B$ by $G_B(x, y, z) = d(x, y) \vee d(y, z) \vee d(z, x)$ for $x, y, z \in X$. The map G_B is a GB -metric on X , and consequently (X, G_B) is a GB -metric space.

Proof. Conditions (GB_1) , (GB_2) , and (GB_4) are clear. We show that (GB_3) and (GB_5) are valid too.

$$\begin{aligned} G_B(x, x, y) &= d(x, x) \vee d(x, y) \vee d(x, y) < d(x, z) \vee d(z, y) < d(x, z) \vee d(z, y) \vee d(x, y) \\ &= G_B(x, y, z). \end{aligned} \tag{2.2}$$

And for (GB_5) , we have

$$\begin{aligned} G_B(x, y, z) &= d(x, y) \vee d(y, z) \vee d(z, x) < d(x, a) \vee d(y, a) \vee d(z, a) \vee d(y, z) \\ &= d(x, a) \vee G_B(a, y, z) \\ &= G_B(a, a, x) \vee G_B(a, y, z). \end{aligned} \tag{2.3}$$

□

Proposition 2.9. Let (X, G_B) be a GB -metric space, then for all $x, y, z, a \in X$, it follows that:

- (1) $G_B(x, y, z) \leq G_B(x, x, y) \vee G_B(x, x, z)$,
- (2) $G_B(x, y, y) = G_B(x, x, y)$,
- (3) $G_B(x, y, z) \leq G_B(x, a, z) \vee G_B(a, y, z)$, and
- (4) $G_B(x, y, z) \leq G_B(x, a, a) \vee G_B(y, a, a) \vee G_B(z, a, a)$.

Proposition 2.10. Let (X, G_B) be a GB -metric space, then the following are equivalent:

- (1) $G_B(x, y, y) \leq G_B(x, y, a)$ for all $x, y, z, a \in X$,
- (2) $G_B(x, y, z) \leq G_B(x, y, a) \vee G_B(z, y, b)$ for all $x, y, z, a \in X$.

Proof. Use (2) of Proposition 2.9. □

Definition 2.11. Let (X, G_B) be a GB -metric. For all $x_0 \in X, r \in B \setminus \{0\}$, the GB -ball with center x_0 and radius r is $B_{GB}(x_0, r) = \{y \in X \mid G_B(x_0, y, y) < r\}$. For any GB -ball, we can define $\overline{B_{GB}}(x_0, r) = \{y \in X \mid G_B(x_0, y, y) \leq r\}$.

Proposition 2.12. Let (X, G_B) be a G -metric space. Then for all $x_0 \in X$ and $r \in B \setminus \{0\}$, we have

- (1) If $G_B(x_0, x, y) < r$, then $x, y \in B_{GB}(x_0, r)$ and
- (2) if $y \in B_{GB}(x_0, r)$, then there exists one element $\delta \in B \setminus \{0\}$ such that, $B_{G_B}(y, \delta) \subseteq B_{GB}(x_0, r)$.

Proof. (1) follows directly $G_B(x, x, y) \leq G_B(x, y, z)$. Put $\delta = r - G_B(x, y, y)$ and use (GB_5) , then deduce (2). □

It follows from (2) of the above proposition that the family of all GB-balls

$$\beta = \{B_{GB}(x, r) \mid x \in X, r \in B \setminus \{0\}\} \quad (2.4)$$

is the base of topology τ_{GB} on X , we call the GB-metric topology.

Definition 2.13. Let (X, G_B) be a GB-metric space. The sequence $\{x_n\} \subseteq X$ is GB-convergent to x if it converges to x in the GB-metric topology, τ_{GB} .

Remark 2.14. Another topological notions as GB-Hausdorff, GB-compact, GB-normal, and ... define similarly as usual.

Definition 2.15. Let (X, G_B) and (X', G'_B) be GB-metric spaces, a function $f : X \rightarrow X'$ is GB-continuous at a point $x_0 \in X$ if $f^{-1}(B_{G'_B}(f(x_0), r)) \in \tau(G)$ for all $r \in B \setminus \{0\}$. We say f is GB-continuous if it is GB-continuous at all points of X , that is, continuous as a function from X with the τ_{GB} -topology to X' with the τ'_{GB} -topology.

Theorem 2.16. Let (X, G_B) be a GB-metric space and f be a self map of X into itself. Suppose f is GB-continuous at $x_0 \in X$, if there is a point $x \in X$ such that the sequence of iterates $\{f^n(x)\}$ converges to x_0 , then $f(x_0) = x_0$.

Proof. From (GB_5) , we derive $G_B(f(x_0), x_0, x_0) \leq G_B(f(x_0), f^n(x), f^n(x)) \vee G_B(f^n(x), x_0, x_0) = G_B(f(x_0), f(f^{n-1}(x)), f(f^{n-1}(x))) \vee G_B(f^n(x), x_0, x_0)$ as $n \rightarrow \infty$ we deduce, $G_B(f(x_0), f(f^{n-1}(x)), f(f^{n-1}(x))) \vee G_B(f^n(x), x_0, x_0) \rightarrow 0$, so the above equation will be $G_B(f(x_0), x_0, x_0) \leq 0$, therefore, $f(x_0) = x_0$. $\square$

Theorem 2.17. Let (X, G_B) be a GB-metric space and $f : X \rightarrow X$ be GB-continuous. If for $x \in X$, the sequence $\{f^n(x)\}$ has a convergent subsequence $\{f^{n_i}(x)\}$ GB-converges to p , and $G_B(f^{n_i}(x), f^{n_i+1}(x), f^{n_i+1}(x)) \rightarrow 0$, then f has a fixpoint.

Proof. Since $\{f^{n_i}(x)\} \rightarrow p$, then by GB-continuity of f we have $f(f^{n_i}(x)) = f^{n_i+1}(x) \rightarrow f(p)$, and we have

$$\begin{aligned} G_B(p, f(p), f(p)) &\leq G_B(p, f^{n_i}(x), f^{n_i}(x)) \vee G_B(f^{n_i}(x), f^{n_i+1}(x), f^{n_i+1}(x)) \\ &\vee G_B(f^{n_i+1}(x), f(p), f(p)). \end{aligned} \quad (2.5)$$

As $i \rightarrow \infty$ we get $G_B(p, f(p), f(p)) \leq 0$, which implies that $f(p) = p$. $\square$

Theorem 2.18. A GB-metric space X is GB-Hausdorff.

Proof. Consider two elements x, y belong to X that $x \neq y$, and two subsets $A = \{x \in X \mid G_B(x, x, z) < G_B(x, x, y)\}$ and $B = \{x \in X \mid G_B(x, x, y) < G_B(x, x, z)\}$. These are two disjoint neighbourhoods of x and y , respectively. It is sufficient to prove that the subsets A and B are open. Suppose $t \in A$. Put $G_B(t, t, z) = r_1$, $G_B(t, t, y) = r_2$ and $\delta = r_2 - r_1$. If $s \in B_{GB}(t, \delta)$, then $G_B(s, s, z) < G_B(t, t, z) \vee G_B(s, s, t) < r_2 = G_B(t, t, y) < G_B(t, t, s) \vee G_B(s, s, y) = G_B(s, s, y)$, therefore, $s \in A$ and $B_{GB}(t, \delta) \subset A$. This proves X is Hausdorff. $\square$

Theorem 2.19. A GB-metric space X is GB-normal.

Proof. Let $G_B(x, x, D) = \bigwedge \{G_B(x, x, y) \mid y \in D\}$. Consider two closed disjoint subsets E and F of X . Put $A = \{x \in X \mid (x, x, F) < G_B(x, x, E)\}$ and $B = \{x \in X \mid G_B(x, x, E) < G_B(x, x, F)\}$. It is clear that A, B are disjoint open subsets of X , $E \subset A$, and $F \subset B$. $\square$

Corollary 2.20. *A GB-metric space X is GB-regular.*

2.2. Monoid Invariant GB-Metric

Definition 2.21. A GB-metric g_b on a $(X, \odot)$ is m -invariant (or monoid invariant) when $G_B(x \odot a, y \odot a, z \odot a) \leq G_B(x, y, z)$ and $G_B(a \odot x, a \odot y, a \odot z) \leq G_B(x, y, z)$ are valid for any $x, y, z, a \in X$.

Proposition 2.22. *Let $(X, \odot)$ be a monoid. If G_B is an m -invariant GB-metric on X , then*

- (1) $G_B(x \odot a, y \odot b, z \odot c) \leq G_B(x, y, z) \vee G_B(a, b, c)$ and
- (2) $G_B(x_1 \odot x_2 \odot x_3 \odot \cdots \odot x_n, y_1 \odot y_2 \odot y_3 \odot \cdots \odot y_n, z_1 \odot z_2 \odot z_3 \odot \cdots \odot z_n) \leq G_B(x_1, y_1, z_1) \vee G_B(x_2, y_2, z_2) \vee G_B(x_3, y_3, z_3) \vee \cdots \vee G_B(x_n, y_n, z_n)$.

Proof. (1) We have

$$\begin{aligned}
 g_b(x \odot a, y \odot b, z \odot c) &\leq g_b(x \odot a, y \odot a, y \odot a) \vee g_b(y \odot a, y \odot b, z \odot c) \\
 &\leq g_b(x, y, y) \vee g_b(y \odot a, y \odot b, y \odot c) \vee g_b(y \odot c, y \odot c, z \odot c) \\
 &\leq g_b(x, y, y) \vee g_b(a, b, c) \vee g_b(y, y, z) \\
 &\leq g_b(x, y, z) \vee g_b(x, y, z) \vee g_b(a, b, c) \\
 &= g_b(x, y, z) \vee g_b(a, b, c).
 \end{aligned} \tag{2.6}$$

(2) It concludes by induction. $\square$

Put $(X, \odot)$ is a monoid and G_B be a GB-metric on X . We call G_B is left effective on X if $G_B(a \odot x, a \odot y, a \odot z) = G_B(x, y, z)$ for all $x, y, z \in X$ then $a = e$. And we say G_B is left free GB-metric on X if for a triple $(x, y, z) \in X \times X \times X$, $G_B(a \odot x, a \odot y, a \odot z) = G_B(x, y, z)$ then $a = e$. The right effective (free) is defined samely, and we call G_B is effective (free) if it is left and right effective (free) on X . Immediately we deduce, if G_B is free GB-metric on $(X, \odot)$ then it is effective GB-metric.

Proposition 2.23. *Put $f : (X, \odot) \rightarrow (X', \otimes)$ is a monoid homomorphism and G_B' is a GB-metric on X' the following results are valid:*

- (1) $G_B : X \times X \times X \rightarrow B, (x, y, z) \mapsto G_B'(f(x), f(y), f(z))$ is a GB-metric on $(X, \odot)$.
- (2) If G_B' is m -invariant, then G_B is m -invariant.
- (3) The GB-metric G_B on X is free (or effective) if G_B' is free (or effective) on X' .

Let E and F be subsets of X . The set E is congruent to F , written $E \approx F$, provided there exist a map $f : E \rightarrow F$ that for any $x, y, z \in E$, $G_B(x, y, z) = G_B(f(x), f(y), f(z))$ and that F is congruent to E by the map $f^{-1} : F \rightarrow E$. The map f (or f^{-1}) is called a congruence between E and F . Clearly every single set $\{x\}$, $x \in X$ and each map f , $\{x\}$ congruent to $\{f(x)\}$.

Definition 2.24. A motion of X is a congruence of X with itself.

Proposition 2.25. Let (X, G_B) be a GB-metric space. The set $\mathcal{M}(X) = \{f : X \rightarrow X \mid f \text{ is motion}\}$ is a group.

Proof. The identity map belongs to $\mathcal{M}(X)$ so $\mathcal{M}(X)$ is not empty. Group action is composition of maps, so the identity map is the identity element. If $g \in \mathcal{M}(X)$, then $G_B(g^{-1}(x), g^{-1}(y), g^{-1}(z)) = G_B(g(g^{-1}(x)), g(g^{-1}(y)), g(g^{-1}(z))) = (x, y, z)$, so $g^{-1} \in \mathcal{M}(x)$. $\square$

Theorem 2.26. Let (X, G_B) be a GB-metric space and $f : X \rightarrow X$ is a motion of X , with the properties: if $\{f^n\} \rightarrow x$, then $G_B(f^{n_i+1}(x), f^{n_i+1}(x), f^n(x)) \rightarrow 0$, then the set of fixpoints of f (or $F(f)$) is equal to the set of fixpoints of f^n (or $F(f^n)$) for all $n \in \mathbb{N}$.

Proof. Clearly $F(f) \subseteq F(f^n)$. Suppose $x \in F(f^n)$, so the sequence $\{f^{nk}(x)\}_{k=1}^{\infty} \rightarrow x$, because $f^{kn}(x) = x$, for all $k \in \mathbb{N}$. Now by Theorem 2.16, we conclude $f(x) = x$. $\square$

Proposition 2.27. Let $(X, *)$ be a monoid and G_B be a GB-metric on X .

- (1) If for all $a \in X$, L_a and R_a be motions of X , then G_B is m -invariant.
- (2) L_a (or R_a) ($a \neq e$) cannot be a motion of X , if G_B is effective.

Example 2.28. The GB-metric defined in Example 2.4 is invariant on the monoid $(\mathbb{W}, +)$, but it is not effective. Clearly for all $k \in \mathbb{N}$, $f_k : \mathbb{W} \rightarrow \mathbb{W}$, $n \mapsto n + k$ is a motion.

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