Research Article

On the Fixed Points of Correspondences in Modular Spaces

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We give two fixed point results for contractive and nonexpansive correspondences defined on modular spaces.

1. Introduction and Preliminaries

A modular on a real linear space X is a real functional ρ on X which satisfies the conditions:

- (1) $\rho(x) = 0$ if and only if $x = 0$,
- (2) $\rho(x) = \rho(-x)$,
- (3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for all $x, y \in X$ and $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

Then (X, ρ) is called a modular space. Given a modular ρ , a corresponding vector space X_ρ is given as

$$X_\rho = \left\{ x \in X : \lim_{\lambda \rightarrow 0} \rho(\lambda x) = 0 \right\}, \quad (1.1)$$

which is called modular linear space.

It is easy to see that for every modular ρ if $\alpha, \beta \in \mathbb{R}^+$ and $\alpha \leq \beta$, then $\rho(\alpha x) \leq \rho(\beta x)$, for all $x \in X$.

The theory of modular spaces was initiated by Nakano [1] in connection with the theory of ordered spaces. Musielak and Orlicz [2] redefined and generalized the notion of a modular space in order to obtain a generalization of the classical function spaces L^p . Even

if a metric is not defined, many results in metric fixed point theory can be reformulated in modular spaces, we refer, for instance, to [3–5].

In this work, we give some results on the existence of fixed points for contractive and nonexpansive correspondences on modular spaces.

We first recall some basic concepts of modular spaces. We refer to [6, 7] for more details on modular spaces.

Definition 1.1. Let B be a subset of a modular space (X, ρ) .

- (1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ and denoted by $x_n \rightarrow x$, if for every $\epsilon > 0$, there is a positive integer N such that $\rho(x_n - x) < \epsilon$, for all $n > N$.
- (2) The *closure* of B is denoted by $\bar{B}$ and defined as the set of all $x \in X$ such that there is a sequence $\{x_n\}$ of B which is convergent to x . We say that B is *closed* if $B = \bar{B}$.
- (3) A sequence $\{x_n\}$ in X is said to be *Cauchy*, if for every $\epsilon > 0$, there is a positive integer N such that $\rho(x_n - x_m) < \epsilon$, for all $m, n > N$.
- (4) B is said to be *complete* if each Cauchy sequence in B is convergent to a point of B .
- (5) B is said to be *compact* if every sequence in B has a convergent subsequence in B .
- (6) B is called *sequentially bounded*, if for each $\{x_n\} \subset B$ and each real sequence $\{\epsilon_n\}$ converging to zero we have $\epsilon_n x_n \rightarrow 0$, as $n \rightarrow \infty$.

By a correspondence f from a set X to a set Y we mean a relation that assigns to each x in X a nonempty subset $f(x)$ of Y . For any subset C of X and correspondence $f : C \rightarrow X$, an element $x \in C$ is said to be a fixed point if $x \in f(x)$. Also, $f(C) = \bigcup_{c \in C} f(c)$.

Definition 1.2. Let C be a subset of X and let $k \in [0, 1)$. We say that $f : C \rightarrow C$ is a k -contraction if for each $x, y \in C$ and $p \in f(x)$ there is $q \in f(y)$ which satisfies the condition:

$$\rho(p - q) \leq k\rho(x - y). \quad (1.2)$$

Definition 1.3 (see [8]). For a modular space (X, ρ) , the function w_ρ which is called *growth function* is defined on $[0, \infty)$ as follows:

$$w_\rho(t) = \inf\{w : \rho(tx) \leq w\rho(x) : x \in X, 0 < \rho(x)\}. \quad (1.3)$$

2. Main Results

In the sequel, it is assumed that B is a closed subset of complete modular space (X, ρ) and $f : B \rightarrow B$ is a correspondence with compact values.

Lemma 2.1. *Let (X, ρ) be a modular space satisfying $w_\rho(2) < \infty$. Then every convergent sequence in (X, ρ) is a Cauchy sequence in (X, ρ) .*

Proof. Let $\{x_n\}$ be a convergent sequence in (X, ρ) . Then for $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\rho(x_n - x) < \frac{\epsilon}{2w_\rho(2)}, \quad (2.1)$$

for all $n > k$. Thus

$$\begin{aligned}\rho(x_{n+r} - x_n) &\leq w_\rho(2)\rho(x_{n+r} - x) + w_\rho(2)\rho(x_n - x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon,\end{aligned}\tag{2.2}$$

for every $n > k, r \in \mathbb{N}$. Hence $\{x_n\}$ is a Cauchy sequence. $\square$

Definition 2.2. Let C be a subset of a modular space X and let $\epsilon > 0$ be given. A set $M_\epsilon \subseteq X$ is called an ϵ -net for C if for every point $z \in C$ there is a point x of M_ϵ such that $\rho(z - x) < \epsilon$. The set C is said to be *totally bounded* if for every ϵ there is a finite ϵ -net for C .

By a relatively sequentially compact set in a modular space (X, ρ) we mean that its closure is sequentially compact.

The following lemma is a counterpart of totally boundedness in metric spaces and has a same argument which is omitted here.

Lemma 2.3. Let A be a subset of a modular space X satisfying $w_\rho(2) < \infty$. Then,

- (1) if A is totally bounded and X is complete, each sequence in A has a convergent subsequence in X ;
- (2) if A is relatively sequentially compact, A is totally bounded.

Let (X, ρ) be a modular space and let A be a nonempty subset of X . The diameter of A is defined by

$$D(A) = \sup_{a, b \in A} \rho(a - b),\tag{2.3}$$

and the set A is bounded if and only if $D(A) < \infty$.

Definition 2.4. Let (X, ρ) be a modular space and let A be a bounded subset of X . Then the Kuratowski constant $\alpha(A)$ of the set A is defined as the greatest upper bound for $\epsilon > 0$ that A can be covered with a finite number of sets of diameter less than ϵ .

Theorem 2.5. Every k -contraction $f : B \rightarrow B$ with $w_\rho(2)^2 k < 1/2$ has a fixed point.

Proof. Choose $p_0 \in B$ and $p_1 \in f(p_0)$. Since f is k -contraction, there exists $p_2 \in f(p_1)$ for which

$$\rho(p_1 - p_2) \leq k\rho(p_0 - p_1).\tag{2.4}$$

By induction, there exists a sequence $\{p_n\}$ in B such that

$$\begin{aligned}p_{n+1} &\in f(p_n), \\ \rho(p_{n+1} - p_n) &\leq k\rho(p_n - p_{n-1}).\end{aligned}\tag{2.5}$$

Put $P = \{p_n : n \in \mathbb{N}\}$ and choose $l \in (2kw_\rho(2)^2, 1)$. We claim that

$$\alpha(f(P)) \leq l\alpha(P). \quad (2.6)$$

To see this, let $r' > r > 0$ and $l\alpha(P) < r$. By Definition 2.4, there exists a finite number of sets $N_i (i \in \{1, 2, \dots, n\})$ with diameter less than r/l which covers $\{p_n\}$. Let $x_i \in N_i$. By part (2) of Lemma 2.3, the totally boundedness of $f(x_i)$ implies that for

$$\epsilon = \frac{r}{2w_\rho(2)^2} \left(1 - \frac{2kw_\rho(2)^2}{l} \right), \quad (2.7)$$

there exists a finite ϵ -net $M_{\epsilon,i} \subseteq B$ for $f(x_i)$. Let $M_{\epsilon,i} = \{z_1^i, z_2^i, \dots, z_{n(i)}^i\}$ and $Z = \bigcup_{i=1}^n M_{\epsilon,i}$. We will show that

$$f(P) \subseteq \bigcup_{z \in Z} B(z), \quad D(B(z)) < r, \quad (2.8)$$

where

$$B(z) = \left\{ x \in B : \rho(x - z) < \frac{r}{2w_\rho(2)} \right\}. \quad (2.9)$$

Let $y \in f(p_n)$, for some $n \in \mathbb{N}$. Since $P \subseteq \bigcup_{i=1}^n N_i$, then $p_n \in N_i$ for some i . If $p_n, x_i \in N_i$, $y \in f(p_n)$ there exists $x \in f(x_i)$ such that $\rho(y - x) \leq k\rho(x_i - p_n)$. Also there exists $z_j^i (j \in \{1, 2, \dots, n(i)\})$ such that

$$\rho(x - z_j^i) < \frac{r}{2w_\rho(2)^2} \left(1 - \frac{2kw_\rho(2)^2}{l} \right). \quad (2.10)$$

Therefore,

$$\begin{aligned} \rho(y - z_j^i) &\leq w_\rho(2)\rho(y - x) + w_\rho(2)\rho(x - z_j^i) \\ &\leq w_\rho(2)k\rho(p_n - x_i) + w_\rho(2)\rho(x - z_j^i) \\ &< w_\rho(2)k\frac{r}{l} + \frac{rw_\rho(2)}{2w_\rho(2)^2} \left(1 - \frac{2kw_\rho(2)^2}{l} \right) \\ &< \frac{r}{2w_\rho(2)}. \end{aligned} \quad (2.11)$$

It implies that $f(P) \subseteq \bigcup_{z \in P} B(z)$. Also for each $z', z'' \in B(z)$, we have

$$\begin{aligned} \rho(z' - z'') &\leq w_\rho(2)\rho(z' - z) + w_\rho(2)\rho(z'' - z) \\ &\leq w_\rho(2)\frac{r}{2w_\rho(2)} + w_\rho(2)\frac{r}{2w_\rho(2)} \\ &< r, \end{aligned} \quad (2.12)$$

therefore, $D(B(z)) < r'$, that is, $\alpha(f(P)) < r'$. Hence $\alpha(f(P)) \leq l\alpha(P)$. Since for each n , $p_{n+1} \in f(p_n)$, we get $\alpha(P) \leq \alpha(f(P))$. Consequently, for every $n \in \mathbb{N}$

$$\alpha(P) \leq \alpha(f(P)) \leq l\alpha(P) \leq l\alpha(f(P)) \leq \dots \leq l^n \alpha(P). \quad (2.13)$$

It implies that $\alpha(P) = 0$ and hence P is totally bounded. By Lemma 2.3, $\{p_n\}$ has a convergent subsequence $\{p_{n_l}\}$. Let $\lim_{l \rightarrow \infty} p_{n_l} = p$. We have

$$\rho(p_{n_{l+1}} - p_{n_l}) \leq k^{n_l} \rho(p_1 - p_0), \quad (2.14)$$

also

$$\rho(p_{n_{l+1}} - p) \leq w_\rho(2)\rho(p_{n_{l+1}} - p_{n_l}) + w_\rho(2)\rho(p_{n_l} - p), \quad (2.15)$$

therefore, $\lim_{l \rightarrow \infty} \rho(p_{n_{l+1}} - p) = 0$. On the other hand, for each l , there exists an $x_{n_l} \in f(p)$ such that

$$\rho(p_{n_{l+1}} - x_{n_l}) \leq k\rho(p_{n_l} - p), \quad (2.16)$$

therefore, $\lim_{l \rightarrow \infty} \rho(p_{n_{l+1}} - x_{n_l}) = 0$. Since $f(p)$ is a compact set, $\{x_{n_l}\}$ has a convergent subsequence. Let $\lim_{i \rightarrow \infty} x_{n_{l_i}} = x$. Given $\epsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that for $i > i_0$,

$$\begin{aligned} \rho\left(\frac{x-p}{3}\right) &\leq \rho(x - x_{n_{l_i}}) + \rho(x_{n_{l_i}} - p_{n_{l_i+1}}) + \rho(p_{n_{l_i+1}} - p) \\ &< \epsilon. \end{aligned} \quad (2.17)$$

It shows that $x = p$ which completes the proof. $\square$

Definition 2.6. A sequence $\{y_n\}$ in C is said to be *approximate fixed point sequence* of $f : C \rightarrow C$, if for every $n \in \mathbb{N}$ there exists $p_n \in f(y_n)$ such that $\rho(y_n - p_n) \rightarrow 0$, as $n \rightarrow \infty$.

We recall that a subset C of a vector space is called *star shaped*, if there exists $z \in C$ (the center of C) such that $\lambda x + (1 - \lambda)z \in C$, for every $x \in C$, and $\lambda \in [0, 1]$.

Lemma 2.7. Suppose that ρ satisfies $w_\rho(\beta)w_\rho^2(2) < 1/2$ for every $\beta \in (0, 1)$, B is star shaped with the center z , and $f : B \rightarrow B$ is nonexpansive, that is, for each $x, y \in B$ and $p \in f(x)$ there exists $q \in f(y)$ such that

$$\rho(p - q) \leq \rho(x - y). \quad (2.18)$$

Then,

- (1) for every $\alpha, \beta \in \mathbb{R}^+$, where $\alpha + \beta = 1$, there exist $p_0 \in B$ such that $p_0 \in \alpha z + \beta f(p_0)$;
- (2) if $f(B)$ is sequentially bounded, then f has an approximate fixed point sequence.

Proof. (1) Let $\alpha, \beta \in \mathbb{R}^+$, with $\alpha + \beta = 1$. We define $g : B \rightarrow B$ by $g(x) = \alpha z + \beta f(x)$. If $x, y \in B$ and $p \in f(x)$, there exists $q \in f(y)$ such that $\rho(p - q) \leq \rho(x - y)$. This implies that $\alpha z + \beta q \in g(y)$ and

$$\rho(\alpha z + \beta p - \alpha z - \beta q) = \rho(\beta(p - q)) \leq w_\rho(\beta)\rho(x - y). \quad (2.19)$$

Since $w_\rho(\beta)w_\rho^2(2) < 1/2$, Theorem 2.5 implies that g has a fixed point.

(2) Let $\{k_n : n \in \mathbb{N}\} \subset (0, 1)$, $k_n \rightarrow 1$. By part (1), for each $n \in \mathbb{N}$, there exist $y_n \in B$ and $p_n \in f(y_n)$ such that $y_n = (1 - k_n)z + k_n p_n$. Since $f(B)$ is sequentially bounded, $\rho(2(1 - k_n)z) \rightarrow 0$ and $\rho(2(1 - k_n)p_n) \rightarrow 0$, as $n \rightarrow \infty$. Therefore,

$$\rho(y_n - p_n) \leq \rho(2(1 - k_n)p_n) + \rho(2(1 - k_n)z) \rightarrow 0, \quad (2.20)$$

as $n \rightarrow \infty$, that is, f has an approximate fixed point sequence. $\square$

Theorem 2.8. Let X_ρ be a complete modular linear space, and let B be a compact and star shaped subset of X_ρ . If ρ satisfies $w_\rho(\beta)w_\rho^2(2) < 1/2$ for every $\beta \in (0, 1)$, and $f : B \rightarrow B$ is nonexpansive, then f has a fixed point.

Proof. First, we show that B is sequentially bounded. To see this, if λ_k 's are real numbers converging to zero and $\{p_k\} \subseteq B$, then every subsequence of $\{\lambda_k p_k\}$ has a convergent subsequence to zero. Choose subsequence $\{\lambda_{k_i} p_{k_i}\}$ of $\{\lambda_k p_k\}$. Since B is relatively compact, there exist $p \in X_\rho$ and a subsequence $\{k_{n_i}\}$ of $\{k_n\}$ such that $\rho(p_{k_{n_i}} - p) \rightarrow 0$ as $i \rightarrow \infty$. Taking i so large that $2\lambda_{k_{n_i}} < 1$, we obtain

$$\begin{aligned} \rho(\lambda_{k_{n_i}} p_{k_{n_i}}) &\leq \rho(2\lambda_{k_{n_i}}(p_{k_{n_i}} - p)) + \rho(2\lambda_{k_{n_i}} p) \\ &\leq \rho(p_{k_{n_i}} - p) + \rho(2\lambda_{k_{n_i}} p). \end{aligned} \quad (2.21)$$

Therefore, $\lim_{i \rightarrow \infty} \lambda_{k_{n_i}} p_{k_{n_i}} = 0$. It implies that $\lim_{k \rightarrow \infty} \lambda_k p_k = 0$. Otherwise there exists $\epsilon > 0$ and a subsequence $\{\lambda_{k_j} p_{k_j}\}$ of $\{\lambda_k p_k\}$ such that $\rho(\lambda_{k_j} p_{k_j}) > \epsilon$, for all j . This contradicts the fact that $\{\lambda_{k_j} p_{k_j}\}$ has a convergent subsequence to zero.

By part (2) of Lemma 2.7, f has an approximate fixed point sequence, that is, there exist $\{y_n\}$ and $\{p_n\}$ in B such that $p_n \in f(y_n)$ and $\rho(p_n - y_n) \rightarrow 0$. The sequences $\{p_n\}$ and

$\{y_n\}$ have convergent subsequences $\{p_{n_k}\}$ and $\{y_{n_k}\}$, say $\lim_{k \rightarrow \infty} p_{n_k} = p$, and $\lim_{k \rightarrow \infty} y_{n_k} = y$. Since

$$\rho\left(\frac{p-y}{3}\right) \leq \rho(p-p_{n_k}) + \rho(p_{n_k}-y_{n_k}) + \rho(y_{n_k}-y), \quad (2.22)$$

so $y = p$. The nonexpansivity of f implies that for each k there exists $z_{n_k} \in f(y)$ such that

$$\rho(z_{n_k}-p_{n_k}) \leq \rho(y_{n_k}-y). \quad (2.23)$$

Since $f(y)$ is compact, consider a convergent subsequence of $\{z_{n_k}\}$, $z_{n_{k_i}} \rightarrow z$. Again,

$$\begin{aligned} \rho\left(\frac{y-z}{3}\right) &\leq \rho(y-p_{n_{k_i}}) + \rho(p_{n_{k_i}}-z_{n_{k_i}}) + \rho(z_{n_{k_i}}-z) \\ &\leq \rho(y-p_{n_{k_i}}) + \rho(y_{n_{k_i}}-y) + \rho(z_{n_{k_i}}-z). \end{aligned} \quad (2.24)$$

As $i \rightarrow \infty$, we get $p \in f(p)$. □

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