Research Article

GCR-Lightlike Product of Indefinite Kaehler Manifolds

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We study geodesic GCR-lightlike submanifolds of indefinite Kaehler manifolds and obtain some necessary and sufficient conditions for a GCR-lightlike submanifold to be a GCR-lightlike product.

1. Introduction

The geometry of CR -submanifolds of Kaehler manifolds was initiated by Bejancu [1], which includes holomorphic and totally real submanifolds as subcases, and further developed by Bejancu [2], Bejancu et al. [3], Blair and Chen [4], Chen [5], Yano and Kon [6, 7], and many others. They all studied the geometry of CR -submanifolds with positive definite metric. Therefore this geometry may not be applicable to the other branches of mathematics and physics, where the metric is not necessarily definite. Thus the geometry of CR -submanifolds with indefinite metric became a topic of chief discussion and Duggal [8, 9] played a very crucial role. Duggal and Bejancu [10] introduced the notion of CR -lightlike submanifolds which exclude the totally real and complex subcases. Then Duggal and Sahin [11] introduced SCR -lightlike submanifolds which contain complex and totally real subcases but there was no inclusion relation between CR and SCR -cases. Thus to find a class of submanifolds which would behave as an umbrella for CR -lightlike and SCR -lightlike submanifolds of an indefinite Kaehler manifold, Duggal and Sahin [12] introduced GCR -lightlike submanifolds of indefinite Kaehler manifolds. This paper starts with a very brief introduction about lightlike geometry and GCR -lightlike submanifolds which will be needed through the paper and then we study geodesic GCR -lightlike submanifolds and obtain some necessary and sufficient conditions for a GCR -lightlike submanifold to be a GCR -lightlike product.

2. Lightlike Submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to the book [8] by Duggal and Bejancu.

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$ -dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) is an m -dimensional submanifold of $\bar{M}$ and g is the induced metric of $\bar{g}$ on M . If $\bar{g}$ is degenerate on the tangent bundle TM of M , then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M

$$TM^\perp = \cup \left\{ u \in T_x \bar{M} : \bar{g}(u, v) = 0, \forall v \in T_x M, x \in M \right\} \quad (2.1)$$

is a degenerate n -dimensional subspace of $T_x \bar{M}$. Thus both $T_x M$ and $T_x M^\perp$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\text{Rad } T_x M = T_x M \cap T_x M^\perp$ which is known as radical (null) subspace. If the mapping

$$\text{Rad } TM : x \in M \longrightarrow \text{Rad } T_x M \quad (2.2)$$

defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\bar{M}$ is called an r -lightlike submanifold and $\text{Rad } TM$ is called the radical distribution on M .

Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $\text{Rad } TM$ in TM , that is,

$$TM = \text{Rad } TM \perp S(TM), \quad (2.3)$$

and $S(TM^\perp)$ is a complementary vector subbundle to $\text{Rad } TM$ in $TM^\perp$. Let $\text{tr}(TM)$ and $\text{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $\bar{M}|_M$ and to $\text{Rad } TM$ in $S(TM^\perp)^\perp$, respectively. Then we have

$$\text{tr}(TM) = \text{ltr}(TM) \perp S(TM^\perp), \quad (2.4)$$

$$T\bar{M}|_M = TM \oplus \text{tr}(TM) = (\text{Rad } TM \oplus \text{ltr}(TM)) \perp S(TM) \perp S(TM^\perp). \quad (2.5)$$

Let u be a local coordinate neighborhood of M and consider the local quasiorthonormal fields of frames of $\bar{M}$ along M , on u as $\{\xi_1, \dots, \xi_r, W_{r+1}, \dots, W_n, N_1, \dots, N_r, X_{r+1}, \dots, X_m\}$, where $\{\xi_1, \dots, \xi_r\}$ and $\{N_1, \dots, N_r\}$ are local lightlike bases of $\Gamma(\text{Rad } TM|_u)$ and $\Gamma(\text{ltr}(TM)|_u)$, and $\{W_{r+1}, \dots, W_n\}$ and $\{X_{r+1}, \dots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^\perp)|_u)$ and $\Gamma(S(TM)|_u)$, respectively. For this quasiorthonormal fields of frames, we have the following.

Theorem 2.1 (see [8]). *Let $(M, g, S(TM), S(TM^\perp))$ be an r -lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exist a complementary vector bundle $\text{ltr}(TM)$ of $\text{Rad } TM$ in $S(TM^\perp)^\perp$ and a basis of $\Gamma(\text{ltr}(TM)|_u)$ consisting of smooth section $\{N_i\}$ of $S(TM^\perp)^\perp|_u$, where u*

is a coordinate neighborhood of M such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad \text{for any } i, j \in \{1, 2, \dots, r\}, \quad (2.6)$$

where $\{\xi_1, \dots, \xi_r\}$ is a lightlike basis of $\Gamma(\text{Rad}(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$; then according to the decomposition (2.5), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \quad (2.7)$$

$$\bar{\nabla}_X U = -A_U X + \nabla_X^\perp U, \quad \forall X \in \Gamma(TM), \quad U \in \Gamma(\text{tr}(TM)), \quad (2.8)$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla_X^\perp U\}$ belong to $\Gamma(TM)$ and $\Gamma(\text{tr}(TM))$, respectively. Here ∇ is a torsion-free linear connection on M , h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, and A_U is a linear operator on M and known as shape operator.

According to (2.4) considering the projection morphisms L and S of $\text{tr}(TM)$ on $\text{ltr}(TM)$ and $S(TM^\perp)$, respectively, then (2.7) and (2.8) become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad (2.9)$$

$$\bar{\nabla}_X U = -A_U X + D_X^l U + D_X^s U, \quad (2.10)$$

where we put $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D_X^l U = L(\nabla_X^\perp U)$, and $D_X^s U = S(\nabla_X^\perp U)$.

As h^l and h^s are $\Gamma(\text{ltr}(TM))$ -valued and $\Gamma(S(TM^\perp))$ -valued, respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M . In particular

$$\bar{\nabla}_X N = -A_N X + \nabla_X^l N + D^s(X, N), \quad (2.11)$$

$$\bar{\nabla}_X W = -A_W X + \nabla_X^s W + D^l(X, W), \quad (2.12)$$

where $X \in \Gamma(TM)$, $N \in \Gamma(\text{ltr}(TM))$, and $W \in \Gamma(S(TM^\perp))$. Using (2.9)–(2.12) we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \quad (2.13)$$

$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) = 0, \quad (2.14)$$

$$\bar{g}(A_N X, N') + \bar{g}(N, A_{N'} X) = 0, \quad (2.15)$$

for any $\xi \in \Gamma(\text{Rad } TM)$, $W \in \Gamma(S(TM^\perp))$, and $N, N' \in \Gamma(\text{ltr}(TM))$.

Let P be the projection morphism of TM on $S(TM)$; then using (2.3), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X PY = \nabla_X^* PY + h^*(X, Y), \quad (2.16)$$

$$\nabla_X \xi = -A_\xi^* X + \nabla_X^{*t} \xi, \quad (2.17)$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\text{Rad } TM)$, where $\{\nabla_X^* PY, A_\xi^* X\}$ and $\{h^*(X, Y), \nabla_X^{*t} \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\text{Rad } TM)$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $\text{Rad } TM$, respectively. h^* and A^* are $\Gamma(\text{Rad } TM)$ -valued and $\Gamma(S(TM))$ -valued bilinear forms and are called as second fundamental forms of distributions $S(TM)$ and $\text{Rad } TM$, respectively.

From the geometry of Riemannian submanifolds and nondegenerate submanifolds, it is known that the induced connection ∇ on a nondegenerate submanifold is a metric connection. Unfortunately, this is not true for lightlike submanifolds. Indeed considering $\bar{\nabla}$ a metric connection, we have

$$(\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y), \quad (2.18)$$

for any $X, Y, Z \in \Gamma(TM)$. From [8, page 171], using the properties of linear connection, we have

$$\begin{aligned} (\nabla_X h^l)(Y, Z) &= \nabla_X^l(h^l(Y, Z)) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z), \\ (\nabla_X h^s)(Y, Z) &= \nabla_X^s(h^l(Y, Z)) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z). \end{aligned} \quad (2.19)$$

Barros and Romero [13] defined indefinite Kaehler manifolds as follows.

Definition 2.2. Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then $\bar{M}$ is called an indefinite Kaehler manifold if $\bar{J}$ is parallel with respect to $\bar{\nabla}$, that is,

$$(\bar{\nabla}_X \bar{J})Y = 0, \quad \forall X, Y \in \Gamma(T\bar{M}). \quad (2.20)$$

3. Generalized Cauchy-Riemann Lightlike Submanifolds

Definition 3.1. Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g}, \bar{J})$, then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied.

(A) There exist two subbundles D_1 and D_2 of $\text{Rad } (TM)$ such that

$$\text{Rad } (TM) = D_1 \oplus D_2, \quad \bar{J}(D_1) = D_1, \quad \bar{J}(D_2) \subset S(TM). \quad (3.1)$$

(B) There exist two subbundles D_0 and D' of $S(TM)$ such that

$$S(TM) = \{\bar{J}D_2 \oplus D'\} \perp D_0, \quad \bar{J}(D_0) = D_0, \quad \bar{J}(D') = L_1 \perp L_2, \quad (3.2)$$

where D_0 is a nondegenerate distribution on M , and L_1 and L_2 are vector bundle of $\text{ltr}(TM)$ and $S(TM)^\perp$, respectively.

Then the tangent bundle TM of M is decomposed as

$$TM = D \perp D', \quad D = \text{Rad}(TM) \oplus D_0 \oplus \bar{J}D_2. \quad (3.3)$$

M is called a proper GCR-lightlike submanifold if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$, $D_0 \neq \{0\}$, and $L_2 \neq \{0\}$.

Let Q , P_1 , and P_2 be the projections on D , $\bar{J}(L_1) = M_1$ and $\bar{J}(L_2) = M_2$, respectively. Then for any $X \in \Gamma(TM)$ we have

$$X = QX + P_1X + P_2X, \quad (3.4)$$

applying $\bar{J}$ to (3.4), we obtain

$$\bar{J}X = TX + wP_1X + wP_2X, \quad (3.5)$$

and we can write (3.5) as

$$\bar{J}X = TX + wX, \quad (3.6)$$

where TX and wX are the tangential and transversal components of $\bar{J}X$, respectively.

Similarly

$$\bar{J}V = BV + CV, \quad (3.7)$$

for any $V \in \Gamma(\text{tr}(TM))$, where BV and CV are the sections of TM and $\text{tr}(TM)$, respectively.

Differentiating (3.5) and using (2.9)–(2.12) and (3.7) we have

$$\begin{aligned} D^s(X, wP_1Y) &= -\nabla_X^s wP_2Y + wP_2\nabla_X Y - h^s(X, TY) + Ch^s(X, Y), \\ D^l(X, wP_2Y) &= -\nabla_X^l wP_1Y + wP_1\nabla_X Y - h^l(X, TY) + Ch^l(X, Y). \end{aligned} \quad (3.8)$$

Using Kaehlerian property of $\bar{\nabla}$ with (2.11) and (2.12), we have the following lemmas.

Lemma 3.2. *Let M be a GCR-lightlike submanifold of an indefinite Kaehlerian manifold $\bar{M}$. Then one has*

$$(\nabla_X T)Y = A_{wY}X + Bh(X, Y), \quad (3.9)$$

$$(\nabla_X^t w)Y = Ch(X, Y) - h(X, TY), \quad (3.10)$$

where $X, Y \in \Gamma(TM)$ and

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \quad (3.11)$$

$$(\nabla_X^t w)Y = \nabla_X^t wY - w\nabla_X Y. \quad (3.12)$$

Lemma 3.3. *Let M be a GCR-lightlike submanifold of an indefinite Kaehlerian manifold $\overline{M}$. Then one has*

$$\begin{aligned} (\nabla_X B)V &= A_{CV}X - TA_VX, \\ (\nabla_X^t C)V &= -wA_VX - h(X, BV), \end{aligned} \quad (3.13)$$

where $X \in \Gamma(TM)$, $V \in \Gamma(\text{tr}(TM))$, and

$$\begin{aligned} (\nabla_X B)V &= \nabla_X BV - B\nabla_X^t V, \\ (\nabla_X^t C)V &= \nabla_X^t CV - C\nabla_X^t V. \end{aligned} \quad (3.14)$$

Duggal and Sahin [12] investigated the conditions to define totally geodesic foliations by the distributions D and D' in M as follows.

Theorem 3.4 (see [12]). *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the distribution D defines a totally geodesic foliation in M if and only if $Bh(X, Y) = 0$, for any $X, Y \in \Gamma(D)$.*

Theorem 3.5 (see [12]). *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the distribution D' defines a totally geodesic foliation in M if and only if $A_{wY}X \in \Gamma(D')$, for any $X, Y \in \Gamma(D')$.*

4. Geodesic GCR-Lightlike Submanifolds

Definition 4.1. A GCR-lightlike submanifold of an indefinite Kaehler manifold is called mixed geodesic GCR-lightlike submanifold if its second fundamental form h satisfies $h(X, Y) = 0$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.

Definition 4.2. A GCR-lightlike submanifold of an indefinite Kaehler manifold is called D geodesic GCR-lightlike submanifold if its second fundamental form h satisfies $h(X, Y) = 0$ for any $X, Y \in \Gamma(D)$.

Definition 4.3. A GCR-lightlike submanifold of an indefinite Kaehler manifold is called D' geodesic GCR-lightlike submanifold if its second fundamental form h satisfies $h(X, Y) = 0$ for any $X, Y \in \Gamma(D')$.

Theorem 4.4. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then M is D -geodesic if and only if*

$$\begin{aligned} g(A_W X, Y) &= \overline{g}(D^l(X, W), Y), \\ \nabla_X^* \overline{J} \xi &\notin \Gamma(D_0 \perp \overline{J} L_1), \quad A_\xi^* Y \notin \Gamma(\overline{J} L_1), \quad h^l(X, \xi) \notin \Gamma(L_1), \end{aligned} \quad (4.1)$$

for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(\text{Rad}(TM))$, and $W \in \Gamma(S(TM^\perp))$.

Proof. Using the definition of GCR-lightlike submanifolds, M is D -geodesic, if and only if

$$\begin{aligned} \overline{g}(h^l(X, Y), \xi) &= 0, \\ \overline{g}(h^s(X, Y), W) &= 0, \end{aligned} \quad (4.2)$$

for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(\text{Rad}(TM))$, and $W \in \Gamma(S(TM^\perp))$. Thus for $X, Y \in \Gamma(D)$, first part of the assertion follows from (2.13).

Now for $X, Y \in \Gamma(D)$, $\xi \in \Gamma(\text{Rad}(TM))$ using (2.16), we have

$$\begin{aligned} \overline{g}(h^l(X, Y), \xi) &= \overline{g}(\overline{\nabla}_X Y, \xi) \\ &= -\overline{g}(\overline{J} Y, \overline{\nabla}_X \overline{J} \xi) \\ &= -g(\overline{J} Y, \nabla_X \overline{J} \xi) - \overline{g}(\overline{J} Y, h^l(X, \overline{J} \xi)) \\ &= -g(\overline{J} Y, \nabla_X^* \overline{J} \xi) - \overline{g}(\overline{J} Y, h^l(X, \overline{J} \xi)). \end{aligned} \quad (4.3)$$

Since $Y \in \Gamma(D)$, this implies that $Y \in \Gamma(D_0)$, $Y \in \Gamma(D_1)$, $Y \in \Gamma(D_2)$, or $Y \in \Gamma(\overline{J} D_2)$. If $Y \in \Gamma(D_0)$ or $Y \in \Gamma(D_2)$, then we have

$$\overline{g}(\overline{J} Y, h^l(X, \overline{J} \xi)) = 0, \quad (4.4)$$

and if $Y \in \Gamma(D_1)$ or $Y \in \Gamma(\overline{J} D_2)$, then we have

$$\overline{g}(\overline{J} Y, h^l(X, \overline{J} \xi)) = g(A_\xi^* X, \overline{J} \xi) + \overline{g}(h^l(X, \xi'), \overline{J} \xi), \quad (4.5)$$

for any $\xi' = \overline{J} Y \in \Gamma(\text{Rad}(TM))$.

Now using (4.4) and (4.5) in (4.3), we obtain

$$\overline{g}(h^l(X, Y), \xi) = -g(\overline{J} Y, \nabla_X^* \overline{J} \xi) - g(A_\xi^* X, \overline{J} \xi) - \overline{g}(h^l(X, \xi'), \overline{J} \xi). \quad (4.6)$$

Hence the second part of the assertion follows from (4.6). $\square$

Theorem 4.5. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then M is D' -geodesic if and only if $A_W X$ and $A_\xi^* X$ have no components in $M_2 \perp \overline{J}D_2$ for any $X \in \Gamma(D')$, $\xi \in \Gamma(\text{Rad}(TM))$, and $W \in \Gamma(S(TM^\perp))$.*

Proof. For $X, Y \in \Gamma(D')$ and $W \in \Gamma(S(TM^\perp))$ using (2.13), we obtain

$$\overline{g}(h^s(X, Y), W) = g(A_W X, Y), \quad (4.7)$$

and for $\xi \in \Gamma(\text{Rad}(TM))$ using (2.14) and (2.17) we obtain

$$\overline{g}(h^l(X, Y), \xi) = g(A_\xi^* X, Y). \quad (4.8)$$

Hence the assertion follows from (4.7) and (4.8). $\square$

Theorem 4.6. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then M is mixed geodesic if and only if*

$$\begin{aligned} A_\xi^* X &\in \Gamma(D_0 \perp \overline{J}L_1), \\ A_W X &\in \Gamma(D_0 \perp \text{Rad}(TM) \perp \overline{J}L_1), \end{aligned} \quad (4.9)$$

for any $X \in \Gamma(D)$, $\xi \in \Gamma(\text{Rad}(TM))$, and $W \in \Gamma(S(TM^\perp))$.

Proof. For any $X \in \Gamma(D)$, $Y \in \Gamma(D')$, and $\xi \in \Gamma(\text{Rad}(TM))$ using (2.14) and (2.17) we obtain

$$\overline{g}(h^l(X, Y), \xi) = g(A_\xi^* X, Y), \quad (4.10)$$

and for $W \in \Gamma(S(TM^\perp))$ with (2.13), we obtain

$$\overline{g}(h^s(X, Y), W) = g(A_W X, Y). \quad (4.11)$$

Hence the result follows from (4.10) and (4.11). $\square$

Theorem 4.7. *Let M be a mixed geodesic GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then one has*

$$A_\xi^* X \in \Gamma(\overline{J}D_2), \quad (4.12)$$

for any $X \in \Gamma(D')$ and $\xi \in \Gamma(D_2)$.

Proof. For $X \in \Gamma(D')$ and $\xi \in \Gamma(D_2)$ we have

$$\begin{aligned} h(X, \bar{J}\xi) &= \bar{\nabla}_X \bar{J}\xi - \nabla_X \bar{J}\xi \\ &= \bar{J} \bar{\nabla}_X \xi - \nabla_X \bar{J}\xi \\ &= \bar{J} \nabla_X \xi - \bar{J} h(X, \xi) - \nabla_X \bar{J}\xi. \end{aligned} \quad (4.13)$$

Since M is mixed geodesic, therefore

$$\bar{J} \nabla_X \xi = \nabla_X \bar{J}\xi. \quad (4.14)$$

Using (2.16) and (2.17) we obtain

$$-TA_\xi^* X - wA_\xi^* X + \bar{J} \nabla_X^{*t} \xi = \nabla_X^* \bar{J}\xi + h^*(X, \bar{J}\xi). \quad (4.15)$$

Equating the transversal components we have

$$wA_\xi^* X = 0. \quad (4.16)$$

Thus

$$A_\xi^* X \in \Gamma(\bar{J}D_2 \perp D_0). \quad (4.17)$$

Now, for $Z \in \Gamma(D_0)$ and $\xi \in \Gamma(D_2)$ we have

$$\begin{aligned} \bar{g}(A_\xi^* X, Z) &= \bar{g}(-\nabla_X \xi + \nabla_X^{*t} \xi, Z) \\ &= -\bar{g}(\nabla_X \xi, Z) \\ &= -\bar{g}(\bar{\nabla}_X \xi, Z) \\ &= \bar{g}(\xi, \bar{\nabla}_X Z) \\ &= \bar{g}(\xi, \nabla_X Z + h(X, Z)) \\ &= 0. \end{aligned} \quad (4.18)$$

If $A_\xi^* X \in \Gamma(D_0)$, then using the nondegeneracy of D_0 for any $Z \in \Gamma(D_0)$, we have $\bar{g}(A_\xi^* X, Z) \neq 0$. Therefore $A_\xi^* X \notin \Gamma(D_0)$. Hence the assertion is proved. $\square$

Theorem 4.8. *Let M be a mixed geodesic GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the transversal section $V \in \Gamma(\bar{J}D')$ is D -parallel if and only if $\nabla_X \bar{J}V \in \Gamma(D)$, for any $X \in \Gamma(D)$.*

Proof. Let $Y \in \Gamma(D')$ such that $\bar{J}Y = wY = V \in \Gamma(L_1 \perp L_2)$ and $X \in \Gamma(D)$; then using hypothesis in (3.9) we have $T\nabla_X Y = -A_{wY}X = -A_V X$. Now $\nabla_X^t V = \bar{\nabla}_X V + A_V X = \bar{\nabla}_X \bar{J}Y - T\nabla_X Y$. Since $\bar{\nabla}$ is a Kaehlerian connection and M is mixed geodesic, therefore we have $\nabla_X^t V = w\nabla_X Y$ or consequently $\nabla_X^t V = -w\nabla_X \bar{J}V$, which clearly proves the theorem. $\square$

Theorem 4.9. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ such that $D^\perp(X, V) \in \Gamma(L_2^\perp)$. Then $A_{\bar{J}V}X = \bar{J}A_V X$ for any $X \in \Gamma(D)$ and $V \in \Gamma(L_1^\perp)$.*

Proof. For $X \in \Gamma(D)$, $Y \in \Gamma(D')$, and $V \in \Gamma(L_1^\perp)$, we have

$$\begin{aligned}
 g(A_{\bar{J}V}X - \bar{J}A_V X, Y) &= g(A_{\bar{J}V}X, Y) - g(\bar{J}A_V X, Y) \\
 &= g(A_{\bar{J}V}X, Y) + g(A_V X, \bar{J}Y) \\
 &= -g(\bar{\nabla}_X \bar{J}V, Y) - g(\bar{\nabla}_X V, \bar{J}Y) \\
 &= -g(\bar{J}\bar{\nabla}_X V, Y) + g(\bar{J}\bar{\nabla}_X V, Y) \\
 &= 0.
 \end{aligned} \tag{4.19}$$

For $X \in \Gamma(D)$, $Z \in \Gamma(D_0)$, and $V \in \Gamma(L_1^\perp)$, we have

$$\begin{aligned}
 g(A_{\bar{J}V}X - \bar{J}A_V X, Z) &= g(A_{\bar{J}V}X, Z) - g(\bar{J}A_V X, Z) \\
 &= g(A_{\bar{J}V}X, Z) + g(A_V X, \bar{J}Z) \\
 &= -g(\bar{\nabla}_X \bar{J}V, Z) - g(\bar{\nabla}_X V, \bar{J}Z) \\
 &= -g(\bar{J}\bar{\nabla}_X V, Z) + g(\bar{J}\bar{\nabla}_X V, Z) \\
 &= 0.
 \end{aligned} \tag{4.20}$$

For $X \in \Gamma(D)$, $N \in \Gamma(\text{ltr}(TM))$, and $V \in \Gamma(L_1^\perp)$, we have

$$\begin{aligned}
 g(A_{\bar{J}V}X - \bar{J}A_V X, N) &= g(A_{\bar{J}V}X, N) - g(\bar{J}A_V X, N) \\
 &= g(A_{\bar{J}V}X, N) + g(A_V X, \bar{J}N) \\
 &= -g(\bar{\nabla}_X \bar{J}V, N) - g(\bar{\nabla}_X V, \bar{J}N) \\
 &= -g(\bar{J}\bar{\nabla}_X V, N) + g(\bar{J}\bar{\nabla}_X V, N) \\
 &= 0.
 \end{aligned} \tag{4.21}$$

For $X \in \Gamma(D)$, $\bar{J}N \in \Gamma(\bar{J}L_1)$, and $V \in \Gamma(L_1^\perp)$, we have

$$\begin{aligned}
 g(A_{\bar{J}V}X - \bar{J}A_VX, \bar{J}N) &= g(A_{\bar{J}V}X, \bar{J}N) - g(\bar{J}A_VX, \bar{J}N) \\
 &= g(A_{\bar{J}V}X, \bar{J}N) - g(A_VX, N) \\
 &= -g(\bar{\nabla}_X \bar{J}V, \bar{J}N) + g(\bar{\nabla}_X V, N) \\
 &= -g(\bar{J} \bar{\nabla}_X V, \bar{J}N) + g(\bar{\nabla}_X V, N) \\
 &= -g(\bar{\nabla}_X V, N) + g(\bar{\nabla}_X V, N) \\
 &= 0.
 \end{aligned} \tag{4.22}$$

Hence the assertion follows from (4.19)–(4.22). $\square$

5. GCR-Lightlike Product

Definition 5.1. A GCR-lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ is called a GCR-lightlike product if both the distributions D and D' define totally geodesic foliations in M .

Lemma 5.2. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$; then the distribution D' defines a totally geodesic foliation in M .*

Proof. For any $X, Y \in \Gamma(D')$, (3.9) implies that $T\nabla_X Y = -A_{wY}X - Bh(X, Y)$; then for $Z \in \Gamma(D_0)$ we have

$$\begin{aligned}
 g(T\nabla_X Y, Z) &= -g(A_{wY}X, Z) - g(Bh(X, Y), Z) \\
 &= g(\bar{\nabla}_X wY, Z) = g(\bar{\nabla}_X \bar{J}Y, Z) = -g(\bar{\nabla}_X Y, \bar{J}Z) = -g(\bar{\nabla}_X Y, Z') \\
 &= g(Y, \nabla_X Z'),
 \end{aligned} \tag{5.1}$$

where $Z' = \bar{J}Z \in \Gamma(D_0)$. Since $X \in \Gamma(D')$ and $Z \in \Gamma(D_0)$, then from (3.8) we have $wP\nabla_X Z = h(X, TZ) - Ch(X, Z) = Hg(X, TZ) - CHg(X, Z) = 0$, therefore $wP\nabla_X Z = 0$, and this implies that $\nabla_X Z \in \Gamma(D)$. Therefore (5.1) implies that $g(T\nabla_X Y, Z) = 0$; then the nondegeneracy of D_0 implies that $T\nabla_X Y = 0$. Hence $\nabla_X Y \in \Gamma(D')$, for any $X, Y \in \Gamma(D')$. Thus the result follows. $\square$

Theorem 5.3. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is a GCR-lightlike product if and only if $Bh(X, Y) = 0$, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.*

Proof. Let M be a GCR-lightlike product; therefore the distributions D and D' define a totally geodesic foliation in M . Therefore using Theorem 3.4, $Bh(X, Y) = 0$ for any $X, Y \in \Gamma(D)$. Now let $X \in \Gamma(D')$ and $Y \in \Gamma(D)$; then $Bh(X, Y) = g(X, Y)BH = 0$. Hence $Bh(X, Y) = 0$, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.

Conversely, let $X, Y \in \Gamma(D)$; then $Bh(X, Y) = 0$ implies that D defines a totally geodesic foliation in M . Let $X, Y \in \Gamma(D')$; then (3.9) and (3.11) imply that $A_{wY}X = -T\nabla_X Y - Bh(X, Y)$. Using Lemma 5.2, we obtain $TA_{wY}X + wA_{wY}X = -h(X, Y)$, we compare the tangential components, we get $TA_{wY}X = 0$, and this implies that $A_{wY}X \in \Gamma(D')$. Hence using Theorem 3.5, the distribution D' defines a totally geodesic foliation in M . Consequently, M is a GCR-lightlike product of an indefinite Kaehler manifold. $\square$

Theorem 5.4. *Let M be a totally geodesic GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Suppose that there exists a transversal vector bundle of M , which is parallel along D' with respect to the Levi-Civita connection on M ; that is, one has $\overline{\nabla}_X V \in \Gamma(\text{tr}(TM))$, for any $V \in \Gamma(\text{tr}(TM))$ and $X \in \Gamma(D')$. Then M is a GCR-lightlike product.*

Proof. Since M is a totally geodesic GCR-lightlike submanifold, therefore $Bh(X, Y) = 0$, for any $X, Y \in \Gamma(D)$. Hence the distribution D defines a totally geodesic foliation in M . Next, since $\overline{\nabla}_X V \in \Gamma(\text{tr}(TM))$ for any $V \in \Gamma(\text{tr}(TM))$ and $X \in \Gamma(D')$, therefore using (2.8), we have $A_V X = 0$ then using (3.9), we obtain $T\nabla_X Y = 0$ for any $X, Y \in \Gamma(D')$ and this implies that $\nabla_X Y \in \Gamma(D')$. Hence the distribution D' defines a totally geodesic foliation in M . Thus M is a GCR-lightlike product. $\square$

Definition 5.5. A lightlike submanifold M of a semi-Riemannian manifold is said to be an irrotational submanifold if $\overline{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma\text{Rad}(TM)$. Thus M is an irrotational lightlike submanifold if and only if $h^l(X, \xi) = 0$, $h^s(X, \xi) = 0$.

Theorem 5.6. *Let M be an irrotational lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then M is a GCR-lightlike product if the following conditions are satisfied:*

- (A) $\overline{\nabla}_X U \in \Gamma(S(TM^\perp))$ for all $X \in \Gamma(TM)$ and $U \in \Gamma\text{tr}(TM)$,
- (B) $A_\xi^* Y \in \Gamma(\overline{J}L_2)$ for all $Y \in \Gamma(D)$.

Proof. Using (2.11) and (2.12) with (A), we get $A_W X = 0$, $D^l(X, W) = 0$, and $\nabla_X^l W = 0$ for any $X \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\perp))$. Therefore using (2.13) we have $g(h^s(X, Y), W) = 0$; then nondegeneracy of $S(TM^\perp)$ implies that $h^s(X, Y) = 0$. Hence $Bh^s(X, Y) = 0$. Now, let $X, Y \in \Gamma(D)$ and $\xi \in \Gamma(\text{Rad}(TM))$; then using (B), we have $g(h^l(X, Y), \xi) = -g(\nabla_X \xi, Y) = g(A_\xi^* X, Y) = 0$. Then using (2.6), we get $h^l(X, Y) = 0$. Hence $Bh^l(X, Y) = 0$. Thus the distribution D defines a totally geodesic foliation in M .

Next, let $X, Y \in \Gamma(D')$; then $JY = wY \in \Gamma(L_1 \perp L_2) \subset \text{tr}(TM)$. Using (3.9) we obtain $T\nabla_X Y = -Bh(X, Y)$, comparing the components along D we get $T\nabla_X Y = 0$, and this implies that $\nabla_X Y \in \Gamma(D')$. Thus the distribution D' defines a totally geodesic foliation in M . Hence M is a GCR-lightlike product. $\square$

Theorem 5.7. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then M is a GCR-lightlike product if and only if $(\nabla_X T)Y = 0$, for any $X, Y \in \Gamma(D)$ or $X, Y \in \Gamma(D')$.*

Proof. Let $(\nabla_X T)Y = 0$, for any $X, Y \in \Gamma(D)$ or $X, Y \in \Gamma(D')$. Let $X, Y \in \Gamma(D)$, then $wY = 0$ and (3.9) gives that $Bh(X, Y) = 0$. Hence using Theorem 3.4, the distribution D defines a totally geodesic foliation in M . Next, let $X, Y \in \Gamma(D')$. Since $BV \in \Gamma(D')$ for any $V \in \Gamma(\text{tr}(TM))$, then (3.9) implies that $A_{wY}X \in \Gamma(D')$. Hence using Theorem 3.5, the distribution D' defines a totally geodesic foliation in M . Since both the distributions D and D' define totally geodesic foliations in M , hence M is a GCR-lightlike product.

Conversely, let M be a GCR-lightlike product; therefore the distribution D defines a totally geodesic foliation in M . Using Kaehlerian property of $\bar{\nabla}$, for any $X, Y \in \Gamma(D)$ we have $\bar{\nabla}_X \bar{J}Y = \bar{J} \bar{\nabla}_X Y$; then comparing transversal components, we obtain $h(X, \bar{J}Y) = \bar{J}h(X, Y)$ and then $(\nabla_X T)Y = \nabla_X TY - T \nabla_X Y = \bar{\nabla}_X \bar{J}Y - h(X, \bar{J}Y) - \bar{J} \bar{\nabla}_X Y + h(X, \bar{J}Y) = 0$, that is, $(\nabla_X T)Y = 0$, for any $X, Y \in \Gamma(D)$. Let D' defines a totally geodesic foliation in M , and using Kaehlerian property of $\bar{\nabla}$, we have $\bar{\nabla}_X \bar{J}Y = \bar{J} \bar{\nabla}_X Y$; then comparing tangential components on both sides, we obtain $-A_{wY}X = Bh(X, Y)$; then (3.9) implies that $(\nabla_X T)Y = 0$, which completes the proof. $\square$

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