Research Article

Asymptotic Periodicity for a Class of Partial Integro-differential Equations

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Received 1 March 2011; Accepted 27 April 2011

Academic Editor: M. S. Moslehian

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We study the existence of S -asymptotically ω -periodic solutions for a class of abstract partial integro-differential equations and for a class of abstract partial integrodifferential equations with delay. Applications to integral equations arising in the study of heat conduction in materials with memory are shown.

1. Introduction

In this paper we study the existence of S -asymptotically ω -periodic solutions for a class of abstract integrodifferential equations of the form

$$u'(t) = Au(t) + \int_0^t B(t-s)u(s)ds + g(t, u(t)), \quad t \geq 0, \quad (1.1)$$

$$u(0) = x_0, \quad (1.2)$$

where $A : D(A) \subseteq X \rightarrow X$ and $B(t) : D(B(t)) \subseteq X \rightarrow X$ for $t \geq 0$ are densely defined closed linear operators in a Banach space $(X, \|\cdot\|)$. We assume that $D(A) \subset D(B(t))$ for every $t \geq 0$ and that $g : [0, \infty) \times X \rightarrow X$ is a suitable function.

Due to its numerous applications in several branches of science, abstract integrodifferential equations of type (1.1) have received much attention in recent years. Properties of the solutions of (1.1) have been studied from different point of view. We refer the reader to ([1–3]) for well posedness to ([4–6] and references therein) for the existence of mild solutions; to [7] for the existence of asymptotically almost periodic and almost periodic solutions, and to [8] for the existence of asymptotically almost automorphic solutions.

The literature concerning S -asymptotically ω -periodic functions with values in Banach spaces is very recent (see [9–16]). To the best of our knowledge, the study of the existence of S -asymptotically ω -periodic solutions for equations of type (1.1) is a topic not yet considered in the literature. To fill this gap is the main motivation of this paper. To obtain our results, we use the theory of resolvent operators (see [1–3] for details). This theory is related to abstract integrodifferential equations in a similar manner as the semigroup theory is related to first-order linear abstract partial differential equations.

This paper has five sections. In the next section, we consider some definitions, technical aspects and basic properties related with S -asymptotically ω -periodic functions and resolvent operators. In the third section, we establish very general results about the existence of S -asymptotically ω -periodic mild solutions to the problem (1.1)-(1.2). In the fourth section, we present similar results for abstract partial integrodifferential equations with delay. Finally, as an application of our abstract results, in the fourth section, we establish conditions for the existence of S -asymptotically ω -periodic mild solutions of a specific integral equation arising in the study of heat conduction in materials with memory.

2. Preliminaries

In this section, we introduce some notations and results to be used in this paper. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. In this work $C_b([0, \infty); Z)$ denotes the Banach space consisting of all continuous and bounded functions from $[0, \infty)$ into Z endowed with the norm of the uniform convergence which is denoted by $\|\cdot\|_\infty$. As usual, $C_0([0, \infty); Z)$ is the vector space of all functions $z \in C_b([0, \infty); Z)$ such that $\lim_{t \rightarrow \infty} z(t) = 0$. Also, we denote by $C_0([0, \infty) \times W; Z)$ the vector space of all continuous functions $H : [0, \infty) \times W \rightarrow Z$ such that $\lim_{t \rightarrow \infty} H(t, w) = 0$ uniformly for w in compact subsets of W . The notation $\mathcal{L}(Z)$ stands for the Banach space of bounded linear operators from Z into Z . Besides, we denote by $B_r(Z)$ the closed ball with center at 0 and radius r . We begin by recalling the concept of S -asymptotically ω -periodic functions. In the rest of this paper, $\omega > 0$ is a fixed real number.

Definition 2.1 (see [14]). A function $f \in C_b([0, \infty); X)$ is called S -asymptotically ω -periodic if

$$\lim_{t \rightarrow \infty} (f(t + \omega) - f(t)) = 0. \quad (2.1)$$

In this case, we say that ω is an asymptotic period of f .

In this work, $SAP_\omega(X)$ represents the subspace of $C_b([0, \infty); X)$ consisting of all S -asymptotically ω -periodic functions. It is easy to see that $(SAP_\omega(X), \|\cdot\|_\infty)$ is a Banach space.

Definition 2.2 (see [14]). A function $f \in C([0, \infty) \times W; Z)$ is said to be uniformly S -asymptotically ω -periodic on bounded sets if for every bounded set $K \subset W$, the set $\{f(t, x) : t \geq 0, x \in K\}$ is bounded and $\lim_{t \rightarrow \infty} (f(t + \omega, x) - f(t, x)) = 0$, uniformly for $x \in K$.

Definition 2.3 (see [14]). A function $f \in C([0, \infty) \times W; Z)$ is said to be asymptotically uniformly continuous on bounded sets if for every $\epsilon > 0$ and every bounded set $K \subset W$, there are constants $T_{\epsilon, K} \geq 0$ and $\delta_{\epsilon, K} > 0$ such that $\|f(t, x) - f(t, y)\|_Z \leq \epsilon$ for all $t \geq T_{\epsilon, K}$ and every $x, y \in K$ with $\|x - y\|_W \leq \delta_{\epsilon, K}$.

Lemma 2.4 (see [14]). Assume that $f \in C([0, \infty) \times W; Z)$ is uniformly S -asymptotically ω -periodic on bounded sets and asymptotically uniformly continuous on bounded sets. If $u \in \text{SAP}_\omega(W)$, then the function $t \mapsto f(t, u(t))$ belongs to $\text{SAP}_\omega(Z)$.

Let $h : [0, \infty) \rightarrow [1, \infty)$ be a continuous nondecreasing function such that $h(t) \rightarrow \infty$ as $t \rightarrow \infty$. Next, the notation $C_h(Z)$ stands for the space $C_h(Z) = \{u \in C([0, \infty), Z) : \lim_{t \rightarrow \infty} (u(t)/h(t)) = 0\}$ endowed with the norm $\|u\|_h = \sup_{t \geq 0} (\|u(t)\|/h(t))$.

Lemma 2.5 (see [17]). A set $K \subseteq C_h(Z)$ is relatively compact in $C_h(Z)$ if it verifies the following conditions.

- (c-1) For all $b > 0$, the set $K_b = \{u|_{[0,b]} : u \in K\}$ is relatively compact in $C([0, b]; Z)$.
- (c-2) $\lim_{t \rightarrow \infty} (\|u(t)\|/h(t)) = 0$ uniformly for $u \in K$.

Now, we include some preliminaries concerning resolvent operators. In the following definition, $[D(A)]$ represents the space $D(A)$ endowed with the graph norm given by $\|x\|_A = \|x\| + \|Ax\|$.

Definition 2.6 (see [18]). A family $\{R(t) : t \geq 0\}$ of continuous linear operators on X is called a resolvent operator for (1.1) if the following conditions are fulfilled.

- (R1) For each $x \in X$, $R(0)x = x$ and $R(\cdot)x \in C([0, \infty); X)$.
- (R2) The map $R : [0, \infty) \rightarrow \mathcal{L}([D(A)])$ is strongly continuous.
- (R3) For each $y \in D(A)$, the function $t \mapsto R(t)y$ is continuously differentiable and

$$\frac{d}{dt} R(t)y = AR(t)y + \int_0^t B(t-s)R(s)y ds = R(t)Ay + \int_0^t R(t-s)B(s)y ds, \quad t \geq 0. \quad (2.2)$$

In what follows, we assume that there exists a resolvent operator for (1.1). The existence of solutions of the problem

$$\begin{aligned} u'(t) &= Au(t) + \int_0^t B(t-s)u(s)ds + f(t), \quad t \geq 0, \\ u(0) &= x_0, \end{aligned} \quad (2.3)$$

has been studied for many authors. Assuming that $f : [0, \infty) \rightarrow X$ is locally integrable and following [2] we affirm that

$$u(t) = R(t)x_0 + \int_0^t R(t-s)f(s)ds, \quad t \geq 0, \quad (2.4)$$

is the mild solution of the problem (2.3).

Motivated by this result, we adopt the following concept of solution.

Definition 2.7 (see [3]). A function $u \in C([0, \infty); X)$ is called a mild solution of (1.1)-(1.2) if

$$u(t) = R(t)x_0 + \int_0^t R(t-s)g(s, u(s))ds, \quad t \geq 0. \quad (2.5)$$

To establish our results, we introduce the following condition.

(H-1) There are positive constants M, μ such that $\|R(t)\| \leq Me^{-\mu t}$ for all $t \geq 0$.

Remark 2.8. For additional details on resolvent operators and applications to partial integrodifferential equations we refer the reader to [2].

3. Existence Results

In this section, we consider the existence and uniqueness of S -asymptotically ω -periodic mild solutions for the problem (1.1)-(1.2). We will assume that there exists a resolvent operator $R(\cdot)$ which satisfies the condition (H-1). Initially we establish a basic property.

Lemma 3.1. *Let $u \in \text{SAP}_\omega(X)$. Then*

$$v(t) = \int_0^t R(t-s)u(s)ds \in \text{SAP}_\omega(X). \quad (3.1)$$

Proof. The estimate $\|v\|_\infty \leq (M/\mu)\|u\|_\infty$ shows that $v \in C_b([0, \infty); X)$. For $\epsilon > 0$, we select $T > 0$ such that $\|u(t+\omega) - u(t)\| \leq \epsilon$ for all $t \geq T$ and $\int_T^\infty e^{-\mu s} ds \leq \epsilon$. We have the following decomposition:

$$\begin{aligned} v(t+\omega) - v(t) &= \int_0^\omega R(t+\omega-s)u(s)ds + \int_\omega^{t+\omega} R(t+\omega-s)u(s)ds - \int_0^t R(t-s)u(s)ds \\ &= \int_t^{t+\omega} R(s)u(t+\omega-s)ds + \int_0^t R(t-s)[u(s+\omega) - u(s)]ds \\ &= \int_t^{t+\omega} R(s)u(t+\omega-s)ds + \int_0^T R(t-s)[u(s+\omega) - u(s)]ds \\ &\quad + \int_T^t R(t-s)[u(s+\omega) - u(s)]ds. \end{aligned} \quad (3.2)$$

Hence, for $t \geq 2T$, we obtain

$$\begin{aligned} \|v(t+\omega) - v(t)\| &\leq M\|u\|_\infty \int_t^{t+\omega} e^{-\mu s} ds + 2M\|u\|_\infty \int_{t-T}^t e^{-\mu s} ds + \epsilon M \int_0^{t-T} e^{-\mu s} ds \\ &\leq M\|u\|_\infty \int_t^{t+\omega} e^{-\mu s} ds + 2M\|u\|_\infty \int_T^t e^{-\mu s} ds + \epsilon M \int_0^t e^{-\mu s} ds \\ &\leq 3M\|u\|_\infty \int_T^\infty e^{-\mu s} ds + \frac{M}{\mu}\epsilon = M\left(3\|u\|_\infty + \frac{1}{\mu}\right)\epsilon, \end{aligned} \quad (3.3)$$

which completes the proof. $\square$

Theorem 3.2. *Assume that $g : [0, \infty) \times X \rightarrow X$ is a uniformly S -asymptotically ω -periodic on bounded sets function that verifies the Lipschitz condition*

$$\|g(t, x) - g(t, y)\| \leq L\|x - y\|, \quad (3.4)$$

for all $x, y \in X$ and every $t \geq 0$. If $LM/\mu < 1$, then the problem (1.1)-(1.2) has a unique S -asymptotically ω -periodic mild solution.

Proof. We define the map Γ on the space $SAP_\omega(X)$ by the expression

$$\Gamma u(t) = R(t)x_0 + \int_0^t R(t-s)g(s, u(s))ds, \quad t \geq 0. \quad (3.5)$$

We next prove that Γ is a contraction from $SAP_\omega(X)$ into $SAP_\omega(X)$. Initially we show that Γ is a map $SAP_\omega(X)$ -valued. Let $u \in SAP_\omega(X)$. We abbreviate the notation by writing

$$v(t) = \int_0^t R(t-s)g(s, u(s))ds \quad (3.6)$$

Since $R(\cdot)x_0 \in SAP_\omega(X)$, it remains to show that the function $v(\cdot)$ given by (3.6) belongs to $SAP_\omega(X)$. Considering that g is asymptotically uniformly continuous on bounded sets and applying the Lemma 2.4, $g(\cdot, u(\cdot)) \in SAP_\omega(X)$. By Lemma 3.1, $v \in SAP_\omega(X)$. On the other hand, if $u_1, u_2 \in SAP_\omega(X)$ we have the estimate

$$\|(\Gamma u_1)(t) - (\Gamma u_2)(t)\| \leq LM \int_0^t e^{-\mu(t-s)} \|u_1(s) - u_2(s)\| ds \leq \frac{LM}{\mu} \|u_1 - u_2\|_\infty. \quad (3.7)$$

The fixed point of Γ is the unique mild solution of (1.1)-(1.2). The proof is complete. $\square$

A similar result can be established when g satisfies a local Lipschitz condition.

Theorem 3.3. Assume that $g : [0, \infty) \times X \rightarrow X$ is a function uniformly S -asymptotically ω -periodic on bounded sets that satisfies the local Lipschitz condition

$$\|g(t, x) - g(t, y)\| \leq L(r)\|x - y\|, \quad (3.8)$$

for all $t \geq 0$ and for all $x, y \in X$ with $\|x\| \leq r$ and $\|y\| \leq r$, where $L : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function. Let $C = \sup_{t \geq 0} \|g(t, R(t)x_0)\|$. If there is $r > 0$ such that

$$\frac{M}{\mu} \left(L(r + M\|x_0\|) + \frac{C}{r} \right) < 1, \quad (3.9)$$

then there is a unique S -asymptotically ω -periodic mild solution of (1.1)-(1.2).

Proof. Let $SAP_\omega^0(X) = \{v \in SAP_\omega(X) : v(0) = 0\}$. It is clear that $SAP_\omega^0(X)$ is a closed vector subspace of $SAP_\omega(X)$. Let $F : SAP_\omega^0(X) \rightarrow SAP_\omega^0(X)$ be the map defined by

$$Fv(t) = \int_0^t R(t-s)g(s, v(s) + R(s)x_0)ds. \quad (3.10)$$

Since g satisfies (3.8) and we have that it is asymptotically uniformly continuous on bounded sets, we can argue as in the proof of the Theorem 3.2 to conclude that F is well defined. For $v_1, v_2 \in \text{SAP}_\omega^0(X)$ with $\|v_1\|_\infty \leq r$ and $\|v_2\|_\infty \leq r$, we obtain that

$$\begin{aligned} \|Fv_1(t) - Fv_2(t)\| &\leq M \int_0^t e^{-\mu(t-s)} L(r + M\|x_0\|) \|v_1(s) - v_2(s)\| ds \\ &\leq \frac{M}{\mu} L(r + M\|x_0\|) \|v_1 - v_2\|_\infty. \end{aligned} \quad (3.11)$$

Hence

$$\|Fv_1 - Fv_2\|_\infty \leq \frac{M}{\mu} L(r + M\|x_0\|) \|v_1 - v_2\|_\infty. \quad (3.12)$$

On the other hand, for $v \in \text{SAP}_\omega^0(X)$ with $\|v\|_\infty \leq r$, we get

$$\begin{aligned} \|Fv\|_\infty &\leq \|Fv - F(0)\|_\infty + \|F(0)\|_\infty \\ &\leq \frac{M}{\mu} L(r + M\|x_0\|) \|v\|_\infty + \frac{CM}{\mu}. \end{aligned} \quad (3.13)$$

Let $r > 0$ be such that

$$\frac{M}{\mu} (L(r + M\|x_0\|)r) + C < r. \quad (3.14)$$

From the above remarks it follows that F is a contraction on $B_r(\text{SAP}_\omega^0(X))$. Thus there is a unique fixed point $v \in B_r(\text{SAP}_\omega^0(X))$ of F . To finish the proof we note that $u(t) = v(t) + R(t)x_0$ is the S -asymptotically ω -periodic mild solution of (1.1)-(1.2). $\square$

We can also avoid the uniform Lipschitz conditions such as (3.4) or (3.8).

Theorem 3.4. Assume that $g : [0, \infty) \times X \rightarrow X$ is a function uniformly S -asymptotically ω -periodic on bounded sets that verifies the Lipschitz condition

$$\|g(t, x) - g(t, y)\| \leq L(t) \|x - y\|, \quad (3.15)$$

for all $x, y \in X$ and every $t \geq 0$, where the function $L(\cdot)$ is locally integrable on $[0, \infty)$. If

$$\Theta = M \sup_{t \geq 0} \int_0^t e^{-\mu(t-s)} L(s) ds < 1, \quad (3.16)$$

then the problem (1.1)-(1.2) has a unique S -asymptotically ω -periodic mild solution.

Proof. We define the map Γ on the space $\text{SAP}_\omega(X)$ by the expression (3.5). For $u \in \text{SAP}_\omega(X)$, let v be the function given by (3.6). Since the function $u(\cdot)$ is bounded, it follows from the Definition 2.2 that $C = \sup_{s \geq 0} \|g(s, u(s))\| < \infty$. Consequently,

$$\|v(t)\| \leq \int_0^t M e^{-\mu(t-s)} \|g(s, u(s))\| ds \leq \frac{CM}{\mu}, \quad t \geq 0, \quad (3.17)$$

which shows that Γu is a bounded continuous function on $[0, \infty)$.

We next prove that Γ is a Θ -contraction from $\text{SAP}_\omega(X)$ into $\text{SAP}_\omega(X)$. Let $u \in \text{SAP}_\omega(X)$. Next we set $B = \{u(t) : t \geq 0\}$. We can write

$$\begin{aligned} v(t+\omega) - v(t) &= \int_0^\omega R(t+\omega-s)g(s, u(s))ds + \int_\omega^{t+\omega} R(t+\omega-s)g(s, u(s))ds \\ &\quad - \int_0^t R(t-s)g(s, u(s))ds \\ &= \int_0^t R(t-s)[g(s+\omega, u(s+\omega)) - g(s, u(s))]ds \\ &\quad + \int_t^{t+\omega} R(s)g(t+\omega-s, u(t+\omega-s))ds \\ &= \int_0^t R(s)[g(t+\omega-s, u(t+\omega-s)) - g(t-s, u(t-s))]ds \\ &\quad + \int_t^{t+\omega} R(s)g(t+\omega-s, u(t+\omega-s))ds \\ &= \int_0^T R(s)[g(t+\omega-s, u(t+\omega-s)) - g(t-s, u(t+\omega-s))]ds \\ &\quad + \int_0^T R(s)[g(t-s, u(t+\omega-s)) - g(t-s, u(t-s))]ds \\ &\quad + \int_T^t R(s)[g(t+\omega-s, u(t+\omega-s)) - g(t-s, u(t-s))]ds \\ &\quad + \int_t^{t+\omega} R(s)g(t+\omega-s, u(t+\omega-s))ds \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (3.18)$$

Below we will estimate each one of the terms I_i , $1 \leq i \leq 4$, of the above expression separately. For $\varepsilon > 0$, let $\varepsilon' = \min\{\mu/M, 1/\Theta\}(\varepsilon/3)$. We choose $T > 0$ such that the following conditions hold:

- (i) $e^{-\mu T} \leq \varepsilon\mu/9CM$,
- (ii) $\|u(t+\omega) - u(t)\| \leq \varepsilon'$,
- (iii) $\|g(t+\omega, x) - g(t, x)\| \leq \varepsilon'$,

for all $t \geq T$ and $x \in B$. Let $t \geq 2T$. Since $t - s \geq t - T \geq T$ for $0 \leq s \leq T$, we get

$$\begin{aligned} \|I_1\| &\leq M\varepsilon' \int_0^T e^{-\mu s} ds \leq \frac{\varepsilon}{3}, \\ \|I_2\| &\leq \varepsilon' M \int_0^T e^{-\mu s} L(t-s) ds \leq \varepsilon' M \int_0^t e^{-\mu s} L(t-s) ds \leq \varepsilon' \Theta, \\ \|I_3\| &\leq M \int_T^t e^{-\mu s} 2C ds \leq \frac{2CM}{\mu} e^{-\mu T}, \\ \|I_4\| &\leq M \int_t^{t+\omega} e^{-\mu s} C ds \leq \frac{CM}{\mu} e^{-\mu T}. \end{aligned} \quad (3.19)$$

Combining these estimates, we find

$$\|v(t+\omega) - v(t)\| \leq \varepsilon, \quad (3.20)$$

for $t \geq 2T$. Hence $v \in \text{SAP}_\omega(X)$.

On the other hand, if $u_1, u_2 \in \text{SAP}_\omega(X)$, and $t \geq 0$, we have

$$\|(\Gamma u_1)(t) - (\Gamma u_2)(t)\| \leq M \int_0^t e^{-\mu(t-s)} L(s) \|u_1(s) - u_2(s)\| ds \leq \Theta \|u_1 - u_2\|_\infty. \quad (3.21)$$

The proof is complete. $\square$

As a consequence of the Lipchitz conditions (3.4), (3.8), or (3.15), our previous results show the existence of solutions of the problem (1.1)-(1.2) for functions g such that $\|g(t, x)\|/\|x\|$ is bounded as $\|x\| \rightarrow \infty$. In what follows, we will show that using properly the stability of the resolvent operator we can establish existence results for functions g with another type of asymptotic behavior at infinity. To establish our result, we consider functions $g : [0, \infty) \times X \rightarrow X$ that satisfies the following boundedness condition.

(H-2) There is a continuous nondecreasing function $W : [0, \infty) \rightarrow [0, \infty)$ such that $\|g(t, x)\| \leq W(\|x\|)$ for all $t \in [0, \infty)$ and $x \in X$.

Theorem 3.5. Assume that $g : [0, \infty) \times X \rightarrow X$ satisfies the hypotheses in the statement of Lemma 2.4 and the assumption (H-2). Suppose, in addition, that the following conditions are fulfilled.

(a) For each $v \geq 0$, $\lim_{t \rightarrow \infty} (1/h(t)) \int_0^t e^{-\mu(t-s)} W(vh(s)) ds = 0$, where h is the function in Lemma 2.5. We set

$$\sigma_v(t) = \|R(\cdot)x_0\|_\infty + M \int_0^t e^{-\mu(t-s)} W(vh(s)) ds, \quad t \geq 0, \quad (3.22)$$

and $\rho(v) = \|\sigma_v\|_h$.

(b) For each $\epsilon > 0$, there is $\delta > 0$ such that for every $u, v \in C_h(X)$, $\|u - v\|_h \leq \delta$ implies that

$$\int_0^t e^{-\mu(t-s)} \|g(s, u(s)) - g(s, v(s))\| ds \leq \epsilon, \quad (3.23)$$

for all $t \geq 0$.

(c) For all $a \geq 0$ and $r > 0$, the set $\{g(s, x) : 0 \leq s \leq a, x \in X, \|x\| \leq r\}$ is relatively compact in X .

(d) $\liminf_{\xi \rightarrow \infty} (\xi/\rho(\xi)) > 1$.

Then the problem (1.1)-(1.2) has an S -asymptotically ω -periodic mild solution.

Proof. Let $\Gamma : C_h(X) \rightarrow C([0, \infty); X)$ be the map defined by the expression (3.5). Next, we prove that Γ has a fixed point in $SAP_\omega(X)$. We divide the proof in several steps.

(i) For $u \in C_h(X)$, we have that

$$\frac{\|\Gamma u(t)\|}{h(t)} \leq \frac{M}{h(t)} \|x_0\| + \frac{M}{h(t)} \int_0^t e^{-\mu(t-s)} W(\|u\|_h h(s)) ds. \quad (3.24)$$

It follows from the condition (a) that $\Gamma : C_h(X) \rightarrow C_h(X)$.

(ii) The map Γ is continuous from $C_h(X)$ into $C_h(X)$. In fact, for $\epsilon > 0$, let $\delta > 0$ be the constant involved in the condition (b). For $u, v \in C_h(X)$, $\|u - v\|_h \leq \delta$, taking into account that $h(t) \geq 1$, we get

$$\frac{\|\Gamma u(t) - \Gamma v(t)\|}{h(t)} \leq \frac{M}{h(t)} \int_0^t e^{-\mu(t-s)} \|g(s, u(s)) - g(s, v(s))\| ds \leq M\epsilon, \quad (3.25)$$

which implies that $\|\Gamma u - \Gamma v\|_h \leq M\epsilon$. Since $\epsilon > 0$ is arbitrary, this shows the assertion.

(iii) We next show that Γ is completely continuous. Let $V = \Gamma(B_r(C_h(X)))$. We set $v = \Gamma(u)$ for $u \in B_r(C_h(X))$. Initially, we prove that $V(t) = \{v(t) : v \in V\}$ is a relatively compact subset of X for each $t \geq 0$. From the mean value theorem,

$$v(t) = R(t)x_0 + \int_0^t R(s)g(t-s, u(t-s))ds \in R(t)x_0 + \overline{tc(K)}, \quad (3.26)$$

where $c(K)$ denotes the convex hull of K and $K = \{R(s)g(\xi, x) : 0 \leq s \leq t, 0 \leq \xi \leq t, \|x\| \leq r\}$. Combining the fact that $R(\cdot)$ is strongly continuous with the property (c), we infer that K is a relatively compact set, and $V(t) \subseteq R(t)x_0 + \overline{tc(K)}$ is also a relatively compact set. Let $b > 0$. We next show that the set $V_b = \{v|_{[0, b]} : v \in V\}$ is equicontinuous. In fact, for $t \geq 0$ fixed we can decompose $v(t+s) - v(t)$ as

$$\begin{aligned} v(t+s) - v(t) &= (R(t+s) - R(t))x_0 + \int_t^{t+s} R(t+s-\xi)g(\xi, u(\xi))d\xi \\ &\quad + \int_0^t (R(\xi+s) - R(\xi))g(t-\xi, u(t-\xi))d\xi. \end{aligned} \quad (3.27)$$

For each $\epsilon > 0$, we can choose $\delta_1 > 0$ such that

$$\left\| \int_t^{t+s} R(t+s-\xi)g(\xi, u(\xi))d\xi \right\| \leq M \int_t^{t+s} e^{-\mu(t+s-\xi)} W(rh(\xi))d\xi \leq \frac{\epsilon}{3}, \quad (3.28)$$

for $s \leq \delta_1$. Moreover, since $\{g(t - \xi, u(t - \xi)) : 0 \leq \xi \leq t, u \in B_r(C_h(X))\}$ is a relatively compact set and $R(\cdot)$ is strongly continuous, we can choose $\delta_2 > 0$ and $\delta_3 > 0$ such that $\|(R(t + s) - R(t))x_0\| \leq \epsilon/3$, for $s \leq \delta_2$ and $\|(R(\xi + s) - R(\xi))g(t - \xi, u(t - \xi))\| \leq \epsilon/3(b + 1)$, for $s \leq \delta_3$. Combining these estimates, we get $\|v(t + s) - v(t)\| \leq \epsilon$ for $|s| \leq \min\{\delta_1, \delta_2, \delta_3\}$ with $t + s \geq 0$ and for all $u \in B_r(C_h(X))$.

Finally, applying condition (a), we can show that

$$\frac{\|v(t)\|}{h(t)} \leq \frac{M\|x_0\|}{h(t)} + \frac{M}{h(t)} \int_0^t e^{-\mu(t-s)} W(rh(s)) ds \longrightarrow 0, \quad t \longrightarrow \infty, \quad (3.29)$$

and this convergence is independent of $u \in B_r(C_h(X))$. Hence V satisfies the conditions (c-1) and (c-2) of Lemma 2.5, which completes the proof that V is a relatively compact set in $C_h(X)$.

- (iv) If $u^\lambda(\cdot)$ is a solution of the equation $u^\lambda = \lambda\Gamma(u^\lambda)$ for some $0 < \lambda < 1$, we have the estimate $\|u^\lambda\|_h / \rho(\|u^\lambda\|_h) \leq 1$ and, combining this estimate with the condition (d), we conclude that the set $K = \{u^\lambda : u^\lambda = \lambda\Gamma(u^\lambda), \lambda \in (0, 1)\}$ is bounded.
- (v) Since $\text{SAP}_\omega(X) \subset C_h(X)$, it follows from Lemmas 2.4 and 3.1 that $\Gamma(\text{SAP}_\omega(X)) \subseteq \text{SAP}_\omega(X)$ and, consequently, we can consider $\Gamma : \overline{\text{SAP}_\omega(X)}^{C_h(X)} \rightarrow \overline{\text{SAP}_\omega(X)}^{C_h(X)}$, where $\overline{B}^{C_h(X)}$ denotes the closure of a set B in the space $C_h(X)$. We have that this map is completely continuous. Applying (iv) and the Leray-Schauder alternative theorem ([19, Theorem 6.5.4]), we deduce that the map Γ has a fixed point $u \in \overline{\text{SAP}_\omega(X)}^{C_h(X)}$. Let $(u_n)_n$ be a sequence in $\text{SAP}_\omega(X)$ such that $u_n \rightarrow u$ in the norm of $C_h(X)$. For $\epsilon > 0$, let $\delta > 0$ be the constant in (b), there is $n_0 \in \mathbb{N}$ so that $\|u_n - u\|_h \leq \delta$, for all $n \geq n_0$. We observe that for $n \geq n_0$

$$\begin{aligned} \|\Gamma u_n - \Gamma u\|_\infty &= \sup_{t \geq 0} \left\| \int_0^t R(t-s) [g(s, u_n(s)) - g(s, u(s))] ds \right\| \\ &\leq M \sup_{t \geq 0} \int_0^t e^{-\mu(t-s)} \|g(s, u_n(s)) - g(s, u(s))\| ds \leq \epsilon. \end{aligned} \quad (3.30)$$

Hence $(\Gamma u_n)_n$ converges to $\Gamma u = u$ uniformly in $[0, \infty)$. This implies that $u \in \text{SAP}_\omega(X)$ and completes the proof. $\square$

4. Existence Results for Functional Equations

In this section, we apply the results established in the Section 3 to study the existence of S -asymptotically ω -periodic mild solutions for abstract functional integrodifferential equations. We keep the notations and the standing hypotheses considered in the Section 3. Initially we are concerned with the initial value problem

$$\begin{aligned} u'(t) &= Au(t) + \int_0^t B(t-s)u(s)ds + g(t, u_t), \quad t \geq 0, \\ u_0 &= \varphi, \end{aligned} \quad (4.1)$$

here $\varphi \in C([-r, 0]; X)$, the history of the function $u(\cdot)$ is given by $u_t : [-r, 0] \rightarrow X$, $u_t(\theta) = u(t + \theta)$, and $g : [0, \infty) \times C([-r, 0]; X) \rightarrow X$ is a continuous function. The following property is an immediate consequence of our definitions.

Lemma 4.1. *Let $u : [-r, \infty) \rightarrow X$ be a continuous function. If $u|_{[0, \infty)} \in \text{SAP}_\omega(X)$. Then the function $[0, \infty) \rightarrow C([-r, 0]; X)$, $t \mapsto u_t$, is S -asymptotically ω -periodic.*

Definition 4.2. A function $u \in C([-r, \infty); X)$ is said to be a mild solution of (4.1) if $u_0 = \varphi$ and the integral equation

$$u(t) = R(t)\varphi(0) + \int_0^t R(t-s)g(s, u_s)ds, \quad t \geq 0, \quad (4.2)$$

is verified.

The following result is an immediate consequence of the Lemma 4.1 and the Theorem 3.2.

Corollary 4.3. *Assume that $g : [0, \infty) \times C([-r, 0]; X) \rightarrow X$ is a uniformly S -asymptotically ω -periodic on bounded sets function that verifies the Lipschitz condition*

$$\|g(t, \varphi_1) - g(t, \varphi_2)\| \leq L\|\varphi_1 - \varphi_2\|_\infty, \quad (4.3)$$

for all $\varphi_1, \varphi_2 \in C([-r, 0]; X)$ and every $t \geq 0$. If $LM/\mu < 1$, then the problem (4.1) has a unique S -asymptotically ω -periodic mild solution.

For this type of problems we can also establish existence results similar to Theorems 3.3 and 3.4. For the sake of brevity we omit the details. On the other hand, proceeding in a similar way, we can study existence of solutions for equations with infinite delay. Specifically, in what follows we will be concerned with the problem (4.1) when the history u_t is given by $u_t : (-\infty, 0] \rightarrow X$, $u_t(\theta) = u(t + \theta)$. To model this problem we assume that u_t belongs to some phase space $\mathcal{B}$ which satisfies appropriate conditions. We will employ the axiomatic definition of the phase space $\mathcal{B}$ introduced in [20]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and verifying the following axioms.

(A) If $x : (-\infty, \sigma + a) \rightarrow X$, $a > 0$, $\sigma \in \mathbb{R}$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$ the following hold:

- (i) x_t is in $\mathcal{B}$.
- (ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$.
- (iii) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \rightarrow [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in (A), the function $t \rightarrow x_t$ is continuous from $[\sigma, \sigma + a)$ into $\mathcal{B}$.

(B) The space $\mathcal{B}$ is complete.

(C-2) If $(\psi^n)_{n \in \mathbb{N}}$ is a uniformly bounded sequence of continuous functions with compact support and $\psi^n \rightarrow \psi$, $n \rightarrow \infty$, in the compact-open topology, then $\psi \in \mathcal{B}$ and $\|\psi^n - \psi\|_{\mathcal{B}} \rightarrow 0$ as $n \rightarrow \infty$.

We introduce the space $\mathcal{B}_0 = \{\psi \in \mathcal{B} : \psi(0) = 0\}$ and the operator $S(t) : \mathcal{B} \rightarrow \mathcal{B}$ given by

$$[S(t)\psi](\theta) = \begin{cases} \psi(0), & -t \leq \theta \leq 0, \\ \psi(t + \theta), & -\infty < \theta < -t. \end{cases} \quad (4.4)$$

It is well known that $(S(t))_{t \geq 0}$ is a C_0 -semigroup ([20]).

Definition 4.4. The phase space $\mathcal{B}$ is said to be a fading memory space if $\|S(t)\psi\|_{\mathcal{B}} \rightarrow 0$ as $t \rightarrow \infty$ for every $\psi \in \mathcal{B}_0$.

Remark 4.5. Since $\mathcal{B}$ verifies axiom (C-2), the space $C_b((-\infty, 0], X)$ consisting of continuous and bounded functions $\psi : (-\infty, 0] \rightarrow X$ is continuously included in $\mathcal{B}$. Thus, there exists a constant $Q \geq 0$ such that $\|\psi\|_{\mathcal{B}} \leq Q\|\psi\|_{\infty}$, for every $\psi \in C_b((-\infty, 0], X)$ ([20, Proposition 7.1.1]).

Moreover, if $\mathcal{B}$ is a fading memory space, then K, M are bounded functions and we can choose $K = Q$ (see [20, Proposition 7.1.5]).

Example 4.6. The phase space $C_r \times L^p(\rho, X)$

Let $r \geq 0, 1 \leq p < \infty$ and let $\tilde{\rho} : (-\infty, -r] \rightarrow \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5)-(g-6) in the terminology of [20]. Briefly, this means that $\tilde{\rho}$ is locally integrable and there exists a nonnegative locally bounded function γ on $(-\infty, 0]$ such that $\tilde{\rho}(\xi + \theta) \leq \gamma(\xi)\tilde{\rho}(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_{\xi}$, where $N_{\xi} \subseteq (-\infty, -r)$ is a set whose Lebesgue measure zero.

The space $\mathcal{B} = C_r \times L^p(\tilde{\rho}, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \mapsto X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $\tilde{\rho}\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(\tilde{\rho}, X)$ is defined as follows:

$$\|\varphi\|_{\mathcal{B}} = \sup\{\|\varphi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} \tilde{\rho}(\theta) \|\varphi(\theta)\|^p d\theta \right)^{1/p}. \quad (4.5)$$

The space $\mathcal{B} = C_r \times L^p(\tilde{\rho}, X)$ satisfies axioms (A), (A-1), and (B). Moreover, when $r = 0$ and $p = 2$, it is possible to choose $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \left(\int_{-t}^0 \tilde{\rho}(\theta) d\theta \right)^{1/2}$ for $t \geq 0$ (see [20, Theorem 1.3.8] for details). Note that if conditions (g-6)-(g-7) of [20] hold, then $\mathcal{B}$ is a fading memory space ([20, Example 7.1.8]).

For fading memory spaces the following property holds ([15, Lemma 2.10]).

Lemma 4.7. Assume that $\mathcal{B}$ is a fading memory space. Let $u : \mathbb{R} \rightarrow X$ be a function with $u_0 \in \mathcal{B}$ and $u|_{[0, \infty)} \in \text{SAP}_{\omega}(X)$. Then the function $t \rightarrow u_t$ belongs to $\text{SAP}_{\omega}(\mathcal{B})$.

Next we assume that $\varphi \in \mathcal{B}$ and that $g : [0, \infty) \times \mathcal{B} \rightarrow X$ is a continuous function.

Definition 4.8. A function $u \in C(\mathbb{R}; X)$ is said to be a mild solution of (4.1) if $u_0 = \varphi$ and the integral equation

$$u(t) = R(t)\varphi(0) + \int_0^t R(t-s)g(s, u_s)ds, \quad t \geq 0, \quad (4.6)$$

is verified.

The following result is an immediate consequence of the Lemma 4.7 and the Theorem 3.2.

Corollary 4.9. Assume that $g : [0, \infty) \times \mathcal{B} \rightarrow X$ is a uniformly S -asymptotically ω -periodic on bounded sets function that verifies the Lipschitz condition

$$\|g(t, \varphi_1) - g(t, \varphi_2)\| \leq L\|\varphi_1 - \varphi_2\|_{\mathcal{B}}, \quad (4.7)$$

for all $\varphi_1, \varphi_2 \in \mathcal{B}$ and every $t \geq 0$. If $LMQ/\mu < 1$, then the problem (4.1) has a unique S -asymptotically ω -periodic mild solution.

As was mentioned for the problem (4.1) with finite delay, in this case we can also establish results similar to Theorems 3.3 and 3.4.

5. Applications to the Heat Conduction

Let Ω be a bounded open connected subset of $\mathbb{R}^3$ with C^∞ boundary; let α and β be in $C^2([0, \infty), \mathbb{R})$ with $\alpha(0)$ and $\beta(0)$ positive, and let $a : [0, \infty) \rightarrow \mathbb{R}$ and $b : H_0^1(\Omega) \rightarrow L^2(\Omega)$ be functions.

Let us consider the following equation that arises in the study of heat conduction in materials with memory (see [2, 21–23])

$$\theta''(t) + \beta(0)\theta'(t) = \alpha(0)\Delta\theta(t) - \int_0^t \beta'(t-s)\theta'(s)ds + \int_0^t \alpha'(t-s)\Delta\theta(s)ds + a(t)b(\theta(t)), \quad (5.1)$$

for $t \geq 0$, where Δ is the Laplacian on Ω .

We consider (5.1) with initial condition

$$\theta(0) = \theta_0, \quad \theta'(0) = \eta_0. \quad (5.2)$$

To model this problem, we consider the space $X = H_0^1(\Omega) \times L^2(\Omega)$ and the linear operators

$$A = \begin{pmatrix} 0 & I \\ \alpha(0)\Delta & -\beta(0)I \end{pmatrix} \quad (5.3)$$

on the domain $D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, and $B(t) = F(t)A$, where $F(t) = [F_{ij}(t)] : X \rightarrow X$, $t \geq 0$, is defined by $F_{11}(t) = F_{12}(t) = 0$, $F_{21}(t) = -\beta'(t)I + \beta(0)(\alpha'(t)/\alpha(0))I$, $F_{22}(t) = (\alpha'(t)/\alpha(0))I$.

Introducing the variable

$$u(t) = \begin{pmatrix} \theta(t) \\ \theta'(t) \end{pmatrix} \in X, \quad (5.4)$$

and defining

$$g(t, u(t)) = \begin{pmatrix} 0 \\ a(t)b(\theta(t)) \end{pmatrix}, \quad (5.5)$$

the equation (5.1) with initial condition

$$u(0) = \begin{pmatrix} \theta_0 \\ \eta_0 \end{pmatrix} \quad (5.6)$$

takes the abstract form (1.1)-(1.2).

It follows from [24] that A generates a C_0 -semigroup $(T(t))_{t \geq 0}$ such that $\|T(t)\| \leq \widetilde{M}e^{-\gamma t}$ for all $t \geq 0$ and some constants $\widetilde{M}, \gamma > 0$. Assume that $\alpha'(t)e^{\gamma t}$, $\alpha''(t)e^{\gamma t}$, $\beta'(t)e^{\gamma t}$, and $\beta''(t)e^{\gamma t}$ are bounded and uniformly continuous functions on $[0, \infty)$, and that for all $t \geq 0$,

$$\begin{aligned} |\beta'(t)| + \max\{\beta(0), 1\} \frac{|\alpha'(t)|}{\alpha(0)} &\leq \frac{\gamma e^{-\gamma t}}{2\widetilde{M}}, \\ |\beta''(t)| + \max\{\beta(0), 1\} \frac{|\alpha''(t)|}{\alpha(0)} &\leq \frac{\gamma^2 e^{-\gamma t}}{4\widetilde{M}^2}. \end{aligned} \quad (5.7)$$

Then, by Grimmer [2, Theorem 4.1], there is a resolvent operator $R(t)$ associated to the operators A and $B(\cdot)$ and satisfying

$$\|R(t)\| \leq \widetilde{M}e^{(-\gamma/2)t}, \quad t \geq 0. \quad (5.8)$$

In addition, suppose that $a \in \text{SAP}_\omega(\mathbb{R})$ and $b : H_0^1(\Omega) \rightarrow L^2(\Omega)$ satisfies

$$\|b(\theta_1) - b(\theta_2)\|_{L^2(\Omega)} \leq L_b \|\theta_1 - \theta_2\|_{H_0^1(\Omega)}, \quad (5.9)$$

for all $\theta_1, \theta_2 \in H_0^1(\Omega)$.

We claim that for each $\theta_0 \in H_0^1(\Omega)$ and $\eta_0 \in L^2(\Omega)$ the problem (5.1)-(5.2) satisfies the assumptions of the Theorem 3.2. In fact, the assumption (H-1) follows from (5.8). It is immediate also that g satisfies the Lipschitz condition (3.4) with $L = \|a\|_\infty L_b$. In addition, estimates

$$\begin{aligned} \|g(t, u)\|_{H_0^1(\Omega) \times L^2(\Omega)} &\leq \|a\|_\infty \left(L_b \|u\|_{H_0^1(\Omega) \times L^2(\Omega)} + \|b(0)\|_{L^2(\Omega)} \right), \\ \|g(t + \omega, u) - g(t, u)\|_{H_0^1(\Omega) \times L^2(\Omega)} &\leq |a(t + \omega) - a(t)| \left(L_b \|u\|_{H_0^1(\Omega) \times L^2(\Omega)} + \|b(0)\|_{L^2(\Omega)} \right), \end{aligned} \quad (5.10)$$

which are verified for $t \geq 0$ and $u \in H_0^1(\Omega) \times L^2(\Omega)$ show that $g : [0, \infty) \times H_0^1(\Omega) \times L^2(\Omega) \rightarrow H_0^1(\Omega) \times L^2(\Omega)$ is a function uniformly S -asymptotically ω -periodic on bounded sets. As a consequence of Theorem 3.2, we obtain the following result.

Proposition 5.1. *Under the above conditions, if $(2\widetilde{M}/\gamma)\|a\|_\infty L_b < 1$, then the problem (5.1)-(5.2) has a unique S -asymptotically ω -periodic mild solution.*

To establish our next result, we assume the following conditions.

(H-3) Let $a \in \text{SAP}_\omega(\mathbb{R})$, $\tau \in (0, 1)$ and let $b : H_0^1(\Omega) \rightarrow L^2(\Omega)$ be a function that satisfies the Hölder type condition

$$\|b(\theta_1) - b(\theta_2)\|_{L^2(\Omega)} \leq L_b \|\theta_1 - \theta_2\|_{H_0^1(\Omega)}^\tau, \quad (5.11)$$

for all $\theta_1, \theta_2 \in H_0^1(\Omega)$.

(H-4) There is a continuous nondecreasing function $h : [0, \infty) \rightarrow [1, \infty)$ such that $h(t) \rightarrow \infty$ as $t \rightarrow \infty$ and

$$\begin{aligned} (1) \quad & \lim_{t \rightarrow \infty} (1/h(t)) \int_0^t e^{(-\gamma/2)(t-s)} h(s)^\tau ds = 0, \\ (2) \quad & \sup_{t \geq 0} \int_0^t e^{(-\gamma/2)(t-s)} |a(s)| h(s)^\tau ds < \infty. \end{aligned}$$

For a concrete example, take $0 < \lambda < \gamma/2(1 - \tau)$, $a(t) = e^{-\lambda t}$, and $h(t) = e^{\lambda t}$.

From Theorem 3.5, we can deduce the following result.

Proposition 5.2. *Suppose that assumptions (H-3), (H-4) hold. Then the problem (5.1)-(5.2) has an S -asymptotically ω -periodic mild solution.*

Proof. We have the following estimates:

$$\|g(t, u)\|_{H_0^1(\Omega) \times L^2(\Omega)} \leq \|a\|_\infty \left(L_b \|u\|_{H_0^1(\Omega) \times L^2(\Omega)}^\tau + \|b(0)\|_{L^2(\Omega)} \right), \quad (5.12)$$

$$\|g(t + \omega, u) - g(t, u)\|_{H_0^1(\Omega) \times L^2(\Omega)} \leq |a(t + \omega) - a(t)| \left(L_b \|u\|_{H_0^1(\Omega) \times L^2(\Omega)}^\tau + \|b(0)\|_{L^2(\Omega)} \right), \quad (5.13)$$

for all $t \geq 0$ and $u \in H_0^1(\Omega) \times L^2(\Omega)$. It follows from (5.12)-(5.13) that the function $g : [0, \infty) \times H_0^1(\Omega) \times L^2(\Omega) \rightarrow H_0^1(\Omega) \times L^2(\Omega)$ is uniformly S -asymptotically ω -periodic on bounded sets. In addition, from the estimate

$$\|g(t, u) - g(t, v)\|_{H_0^1(\Omega) \times L^2(\Omega)} \leq |a(t)| L_b \|u - v\|_{H_0^1(\Omega) \times L^2(\Omega)}^\tau, \quad (5.14)$$

for all $t \geq 0$ and $u, v \in H_0^1(\Omega) \times L^2(\Omega)$, we obtain that g is asymptotically uniformly continuous on bounded sets. By (5.12) we are led to define $W(\xi) = \|a\|_\infty (L_b \xi^\tau + \|b(0)\|_{L^2(\Omega)})$. Consequently,

the function g satisfies the assumption (H-2). From (H-4), for $u, v \in C_h(H_0^1(\Omega) \times L^2(\Omega))$ we can infer that

$$\begin{aligned} & \frac{1}{h(t)} \int_0^t e^{-(\gamma/2)(t-s)} W(vh(s)) ds \\ & \leq \|a\|_\infty L_b v^\tau \left(\frac{1}{h(t)} \int_0^t e^{-(\gamma/2)(t-s)} h(s)^\tau ds \right) + 2 \frac{\|b(0)\|_{L^2(\Omega)}}{\gamma h(t)} \rightarrow 0, \quad t \rightarrow \infty, \\ & \sup_{t \geq 0} \int_0^t e^{-(\gamma/2)(t-s)} \|g(s, u(s)) - g(s, v(s))\|_{H_0^1(\Omega) \times L^2(\Omega)} ds \\ & \leq \left(\sup_{t \geq 0} \int_0^t e^{-(\gamma/2)(t-s)} |a(s)| h(s)^\tau ds \right) \|u - v\|_h^\tau. \end{aligned} \quad (5.15)$$

Therefore, conditions (a) and (b) of Theorem 3.5 are satisfied. A straightforward computation shows that $\liminf_{\xi \rightarrow \infty} (\xi / \rho(\xi)) > 1$, where

$$\rho(\xi) = \sup_{t \geq 0} \frac{1}{h(t)} \left(\left\| R(\cdot) \begin{pmatrix} \theta_0 \\ \eta_0 \end{pmatrix} \right\| + \widetilde{M} \int_0^t e^{-(\gamma/2)(t-s)} W(\xi h(s)) ds \right). \quad (5.16)$$

Finally, since Ω is bounded set with C^∞ boundary, Rellich-Kondrachov's Theorem [25, Theorem IX.16] leads to the conclusion that $\{a(s)b(\theta) : 0 \leq s \leq s_0, \theta \in H_0^1(\Omega), \|\theta\|_{H_0^1(\Omega)} \leq r\}$, $s_0 > 0$, is relatively compact in $L^2(\Omega)$. Hence, condition (c) holds. Using the Theorem 3.5, we conclude that the problem (5.1)-(5.2) has an S -asymptotically ω -periodic mild solution. $\square$

To complete these applications we consider (5.1) with a heat source depending on the past of the temperature. This is a usual situation in control systems. To simplify our exposition, we consider only a system which presents a finite transmission delay time $r > 0$. In this case the equation is

$$\theta''(t) + \beta(0)\theta'(t) = \alpha(0)\Delta\theta(t) - \int_0^t \beta'(t-s)\theta'(s)ds + \int_0^t \alpha'(t-s)\Delta\theta(s)ds + a(t)b(\theta(t-r)), \quad (5.17)$$

for $t \geq 0$. Using the previous development, we model this problem in the space $X = H_0^1(\Omega) \times L^2(\Omega)$, and we consider $u_t \in C([-r, 0]; X)$ for $t \geq 0$. To be consistent with our model, we study the equation (5.17) with initial condition

$$u_0 = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \quad (5.18)$$

where $\varphi \in C([-r, 0]; H_0^1(\Omega))$ and $\psi \in C([-r, 0]; L^2(\Omega))$. The function $g : [0, \infty) \times C([-r, 0]; X) \rightarrow X$ is given by

$$g\left(t, \begin{pmatrix} \varphi^1 \\ \psi^1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ a(t)b(\varphi^1(-r)) \end{pmatrix}. \quad (5.19)$$

Applying now the Corollary 4.3, we obtain the following result.

Proposition 5.3. *Under the above conditions, if $(2\widetilde{M}/\gamma)\|a\|_{\infty}L_b < 1$, then the problem (5.17)-(5.18) has a unique S -asymptotically ω -periodic mild solution.*

Acknowledgments

Claudio Cuevas is partially supported by CNPQ/Brazil under Grant 300365/2008-0. Hernán R. Henríquez is supported in part by CONICYT under Grant FONDECYT no. 1090009.

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