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Research Article

Logistic Heat Integral Methods for the One-Phase Stefan Problem

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Logistic versions of the heat balance integral and refined integral methods are introduced. A benchmark with a one-phase Stefan problem under constant and time-dependent boundary conditions shows remarkable accuracy at estimating temperature distribution and position of the moving front.

1. Introduction

In this paper, we introduce a logistic variant of the heat balance integral method (HBIM) and its refinement, the refined integral method (RIM) for the solution of a (1 + 1)-dimensional one-phase Stefan melting problem. The heat balance integral method (HBIM), was introduced and developed by Goodman to address transport problems in [1, 2]. Suppose the approximate solutions to an initial boundary value partial differential equation are sought. The method consists, roughly, of integrating the differential equation with respect to a spatial variable to create a *heat integral*, the assumption of a *functional* profile with certain undetermined coefficients (which satisfies the appropriate boundary and initial conditions), and calculation of both the undetermined coefficients and its solution.

The pliancy and accuracy of the method have encouraged various applications in literature; thermistor models [3, 4], approximation of temperature perturbation front in heat transfer [5, 6], approximate solutions to heat transfer problems under convective boundary conditions [7–9], and application in the modeling of thermal protection systems for space vehicles [10, 11], to mention a few.

The areas usually addressed in the refinement of the method are the *structural* (which involves the framework of the integral method, see, for instance, [12, 13]) and the *prescriptive*

(which concerns suggesting effective approximate profile, for instance as in [14, 15]). This paper concerns the latter. Reviews of current trends and approaches regarding both the structural and prescriptive aspects of the integral method are given in the works by Hristov [15], Wood [16], Mitchell and Myers [17], and Sadoun [18] and references within.

It is known that the Gaussian variant by Mosally et al. [14] and Mitchell and Myers [17] is prescriptively one of the most accurate variants in current literature for obtaining approximate solutions of the (1 + 1)-dimensional one-phase Stefan melting problem. The choice of a Gaussian profile was largely motivated by (exponential) approximations of the error function [19, 20], which dominates the form of the solution

$$\Theta(x,t) = 1 - \frac{\operatorname{erf}\left(x/\left(2\sqrt{t}\right)\right)}{\operatorname{erf}(\alpha)}, \qquad \sqrt{\pi}\beta\alpha \exp \alpha^2 \operatorname{erf}(\alpha) \ = 1 \tag{1.1}$$

of the classical (1 + 1)-dimensional one-phase Stefan problem

$$\frac{\partial \Theta}{\partial t} = \frac{\partial^2 \Theta}{\partial x^2} \quad \text{in } \Omega \equiv (0, \delta(t)) \times [0, T] \tag{1.2}$$

with boundary and initial conditions

$$\begin{gathered} \frac{\partial \Theta}{\partial x}(x = \delta(t), t) = -\beta \frac{d\delta(t)}{dt}, \\ \Theta(\delta(t), t) = 0, \\ \Theta(0, t) = f(t) \in C^1[0, T], \\ \Theta(x, 0) = 0, \qquad x > 0, \end{gathered} \tag{1.3}$$

where Θ is the dimensionless temperature, $\delta(t)$ is the position of the moving front, and β is the reciprocal of the Stefan number.

Our motivation differs slightly. It relies a bit on the geometry of the error function, which is sigmoid, and more on the fact that given limited resources, growth for many dynamical systems is better modeled by logistic functions rather than the exponential ones [21]. In this case, it should be noted that phase-change problems could be considered as dynamical systems wherein the phases are the competing species.

This paper is arranged as follows. Section 2 details the proposed logistic variants of the HBIM and RIM, their application to some (1 + 1)-dimensional Stefan problems, and comparison with the Gaussian HBIM, RIM and the Enthalpy methods as deemed necessary. Section 3 concludes the paper.

2. Logistic Integral Methods

The proposed logistic profile is as follows.

Logistic Heat Profile

$\Theta(x,t;a) \approx \Theta(x,t)$ such that

$$\Theta(x,t;a) = f(t)\left[a - \frac{2(a-1)}{\left(1+(1-2/a)^{x/\delta(t)}\right)}\right], \tag{2.1}$$

$a \in (-\infty, 0) \cup (2, \infty)$.

It is clear that $\Theta(x,t;a)$ satisfies the initial and boundary condition of (1.2) for given $f(t)$. It is also observed that

$$\begin{gathered}
\frac{\partial\Theta(x,t;a)}{\partial x}(x=l,t) = f(t) \times \begin{cases} -\dfrac{(-1+a)\log(a/(-2+a))}{2\delta(t)}, & l=0, \\ -\dfrac{(-2+a)a\log(a/(-2+a))}{2(-1+a)\delta(t)}, & l=\delta(t), \end{cases} \\
\int_0^\delta \Theta(x,t;a)dx = \delta(t)\left[a - \frac{2(-1+a)\log(1+1/(-2+a))}{\log(a/(-2+a))}\right], \\
\int_0^\delta x\Theta(x,t;a)dx = \delta(t)^2\left[\frac{-(-1+a)\left(\pi^2+12\log(2+2/(-2+a))\left(\log(a/(a-2))\right)\right)}{6\left(\log(a/(-2+a))\right)^2}\right. \\
\left. + \frac{3a\left(\log(a/(-2+a))\right)^2 - 12(-1+a)\mathrm{Li}_2(-(a/(-2+a)))}{6\left(\log(a/(-2+a))\right)^2}\right],
\end{gathered} \tag{2.2}$$

where $x \mapsto \mathrm{Li}_n(x)$ is the polylogarithm function defined by

$$\mathrm{Li}_n(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^n}, \qquad n \in \mathbb{Z}. \tag{2.3}$$

In the following, we give the logistic variants of the HBIM and RIM which, respectively, employ the framework of either the HBIM or RIM but adopt the logistic heat profile given by (2.1).

2.1. Logistic Heat Balance Integral Method (LHBIM)

The path taken here is that prescribed by Wood [16] as the most accurate and effective. It entails integrating (1.2) with respect to x over the moving region 0 to $\delta(t)$:

$$\frac{\partial\Theta(x,t;a)}{\partial x}(x=\delta(t),t) - \frac{\partial\Theta(x,t;a)}{\partial x}(x=0,t) = \frac{d}{dt}\int_0^{\delta(t)}\Theta(x,t;a)dx. \tag{2.4}$$

In evaluating the time differential of the heat integral (the right hand side of (2.4)), the Neumann boundary condition $(1.3)_1$ is applied.

Evaluating (2.4) while employing the definition of $\Theta(x,t;a)$ yields

$$\begin{aligned}\frac{f(t)}{2\delta(t)(a-1)}\log\left(\frac{a}{-2+a}\right) &= \frac{d}{dt}\int_0^{\delta(t)} f(t)\left[a-\frac{2(a-1)}{\left(1+(1-2/a)^{x/\delta}\right)}\right]dx;\\ &= \frac{\left(-2(-1+a)\log((-1+a)/(-2+a))+a\log(a/(-2+a))\right)}{\log(a/(-2+a))}\\ &\quad\times\left(\delta(t)\frac{df(t)}{dt}+f(t)\frac{d\delta(t)}{dt}\right),\end{aligned} \tag{2.5}$$

from which

$$\begin{aligned}\delta(t)^2 &= \frac{2}{f(t)^2}\times\left(\int_1^0 \frac{f(z)^2\left(\log(a/(-2+a))\right)^2}{2(-1+a)\left(2(-1+a)\log((-1+a)/(-2+a))-a\log(a/(-2+a))\right)}dz\right)\\ &\quad-\frac{2}{f(t)^2}\times\left(\int_1^t \frac{f(z)^2\left(\log(a/(-2+a))\right)^2}{2(-1+a)\left(2(-1+a)\log((-1+a)/(-2+a))-a\log(a/(-2+a))\right)}dz\right).\end{aligned} \tag{2.6}$$

But $(\partial\Theta(x,t;a)/\partial x)(x=\delta(t),t) = -\beta d\delta/dt$. Hence,

$$\begin{aligned}-\frac{(-2+a)a\log(a/(-2+a))f(t)}{2(-1+a)\delta(t)} &= -\beta\frac{d\delta(t)}{dt},\\ \frac{(-2+a)a\log(a/(-2+a))f(t)}{2(-1+a)} &= \frac{\beta}{2}\frac{d\delta(t)^2}{dt}.\end{aligned} \tag{2.7}$$

Substituting the expression for $\delta(t)^2$ in (2.6) establishes a relationship between the parameters a and β in time. The next sub-subsection considers the case wherein $f(t)=1$.

2.1.1. Boundary Condition $f(t)=1$

In this case, on fixing $f(t)=1$ in (2.6) and (2.7), we have that

$$\begin{aligned}\delta(t) &= \sqrt{\frac{(-2+a)a\,\log(a/(-2+a))}{(-1+a)\beta}}\;t^{1/2},\\ \frac{(2-a)a}{2(-1+a)\beta} &= \frac{\log(a/(-2+a))}{2(-1+a)\left(2(-1+a)\log(1+1/(-2+a))-a\log(a/(-2+a))\right)}.\end{aligned} \tag{2.8}$$

The approximate temperature profile and moving front position to (1.1) and (1.2) are, respectively, given by (2.1) and $(2.8)_1$ evaluated at (β,a) coordinates satisfying $(2.8)_2$ for prescribed period $t=T$. A contour plot of $(2.8)_2$ that is, Figure 1, shows it has two solution

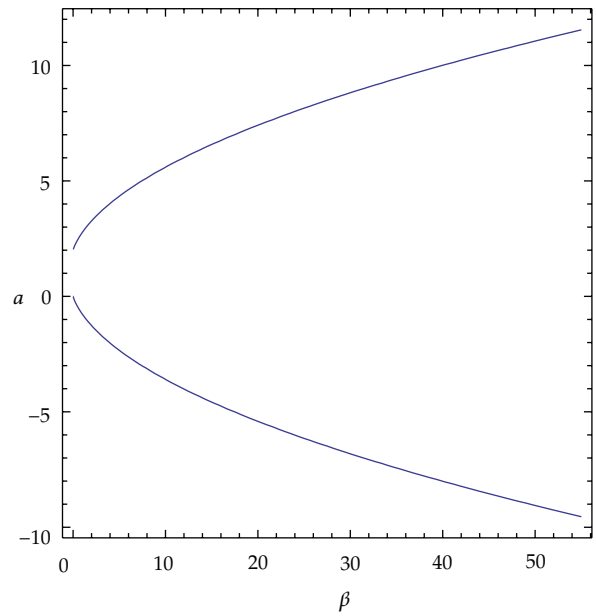

Figure 1: (β, a) contour plot for LHBIM.

branches. These are symmetrical about line $a = 1$ and are essentially equivalent vis-a-vis the Stefan problem under consideration, see, for instance, Figure 2. Selected (β, a) values at $\beta \in \mathbb{K} = \{1, 1.25, 1.67, 2.5, 5, 10\}$ (for further computational use) are given in Table 1.

In order to have a fairly global sense of the strength of the LHBIM approximation, it is natural to gauge the error in the temperature approximations in the L^2 norm sense over $(0, \delta(t)) \times [0, T]$ (i.e., over the exact instantaneous moving front length by a time period $[0, T]$, e.g.,) as

$$\sqrt{\int_0^T \int_0^{\delta(t)} \left|\Theta - \Theta_{\text{app}}\right|^2 dx\, dt}, \tag{2.9}$$

where Θ_{app} is an approximation. However, for computational convenience, we employ a L^2-norm-type error estimate

$$\left| \sqrt{\int_0^T \int_0^{\delta(t)} |\Theta|^2 dx\, dt} - \sqrt{\int_0^T \int_0^{\delta(t)} \left|\Theta_{\text{app}}\right|^2 dx\, dt} \right|, \tag{2.10}$$

which, from the reverse triangle inequality, is less or equal to (2.9). Based on (2.10), error comparison (with the Gaussian HBIM) of temperature and moving front estimates obtained through the LHBIM for $\beta \in \mathbb{K}$ values are given, respectively, in Tables 2 and 3.

A graphical comparison, which gives a perspective of the temperature estimate l^∞ errors relative to the Gaussian HBIM, is also given in Figures 3 and 4.

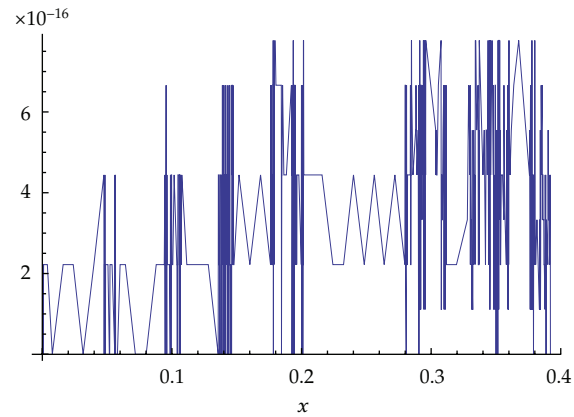

Figure 2: Absolute difference in LHBIM solutions, to (1.2) and (1.3), corresponding to both branches at $\beta = 1$ and $t = 0.1$.

Table 1: Generated (β, a) coordinates for lower solution branches of $(2.8)_2$ (LHBIM) and LRIM.

β	1	1.25	1.67	2.5	5	10
a (LHBIM)	−0.7705	−0.9078	−1.1178	−1.4801	−2.3404	−3.6003
a (LRIM)	−0.7628	−0.9004	−1.1109	−1.4741	−2.3357	−3.5967

2.2. *Logistic Refined Integral Method (LRIM)*

The refined approach given by Sadoun et al. [12, 13] to the heat balance integral method is employed. Here and after, at the expense of abuse of notation, we shall drop $\Theta(x, t; a)$ and use $\Theta(x, t)$ instead. We integrate (1.2) twice to have

$$\int_0^{\delta(t)} \left(\int_0^x \frac{\partial \Theta(\xi, t)}{\partial t} d\xi \right) dx = \Theta(x = \delta(t), t) - \Theta(x = 0, t) - \delta(t) \frac{\partial \Theta}{\partial x}(x = 0, t), \tag{2.11}$$

leading to

$$\delta(t) \frac{d}{dt} \int_0^{\delta(t)} \Theta(x, t) dx - \frac{d}{dt} \int_0^{\delta(t)} x\Theta(x, t) dx = -\Theta(x = 0, t) - \delta(t) \frac{\partial \Theta}{\partial x}(x = \delta(t), t). \tag{2.12}$$

Applying $(1.3)_1$ and $(1.3)_3$ to (2.12), we obtain

$$\begin{gathered} \frac{d}{dt} \int_0^{\delta(t)} x\Theta(x, t) dx = \Theta(x = 0, t) - \beta\delta(t) \frac{d\delta(t)}{dt}, \\ \frac{d}{dt} \int_0^{\delta(t)} x\Theta(x, t) dx = 1 - \beta\delta(t) \frac{d\delta(t)}{dt}. \end{gathered} \tag{2.13}$$

Table 2: L^2 norm error comparison of Gaussian and logistic HBIM and RIM moving front estimates for (1.1) and (1.2) over domain $t \in [0,1]$.

	$\delta(t)$ errors			
β	Gaussian HBIM	Gaussian RIM	LHBIM	LRIM
1	2.1082×10^{-3}	3.1648×10^{-4}	1.4426×10^{-3}	2.1460×10^{-4}
1.25	1.3280×10^{-3}	1.7065×10^{-4}	9.2234×10^{-4}	1.1768×10^{-4}
1.67	7.1373×10^{-4}	7.3970×10^{-5}	5.0393×10^{-4}	5.1957×10^{-5}
2.5	2.9065×10^{-4}	2.1833×10^{-5}	2.0892×10^{-4}	1.5642×10^{-5}
5	5.7986×10^{-5}	2.3876×10^{-6}	4.2524×10^{-5}	1.7483×10^{-6}
10	1.0938×10^{-5}	2.3685×10^{-7}	8.1094×10^{-6}	1.7547×10^{-7}

Table 3: L^2-norm-type error comparison of Gaussian and logistic HBIM and RIM temperature estimates for (1.1) and (1.2) over domain $(x,t) \in (0,\delta(t)) \times [0,1]$.

	$\Theta(x,t)$ errors			
β	Gaussian HBIM	Gaussian RIM	LHBIM	LRIM
1	1.4421×10^{-5}	8.0325×10^{-4}	2.7660×10^{-5}	5.6510×10^{-4}
1.25	1.2284×10^{-5}	5.5010×10^{-4}	1.9031×10^{-5}	3.9086×10^{-4}
1.67	8.8848×10^{-6}	3.2795×10^{-4}	1.1350×10^{-5}	2.3563×10^{-4}
2.5	4.9245×10^{-6}	1.5296×10^{-4}	5.2625×10^{-6}	1.1128×10^{-4}
5	1.4269×10^{-6}	3.7774×10^{-5}	1.2800×10^{-6}	2.7874×10^{-5}
10	3.5314×10^{-7}	8.6657×10^{-6}	2.8998×10^{-7}	6.4450×10^{-6}

Assuming the logistic profile for $\Theta(x,t;a)$, (2.13) and the Stefan condition $(1.3)_1$, respectively, yield

$$\begin{aligned}
&\frac{d}{dt}\left[\left[\frac{\delta(t)^2(1-a)\left(\pi^2+12\log(2(-1+a)/(-2+a))\left(\log(a/(-2+a))\right)\right)}{6\log\left(a/(-2+a)\right)^2}\right.\right.\\
&\qquad\left.\left.+\frac{3a\log\left(a/(-2+a)\right)^2-12(-1+a)\mathrm{Li}_2(a/(2-a))}{6\log\left(a/(-2+a)\right)^2}\right]f(t)\right]\\
&\qquad=1-\beta\delta(t)\frac{d\delta(t)}{dt},\\
&\frac{(-2+a)a\log(a/(2-a))f(t)}{2(1-a)\beta}=\delta(t)\frac{d\delta(t)}{dt}.
\end{aligned} \tag{2.14}$$

Parameter a (corresponding to specific βs and $f(t)$), $\delta(t)$ and consequently approximate $\Theta(x,t;a)$ to (1.2) and (1.3) are evaluated from $(2.14)_1$ and $(2.14)_2$

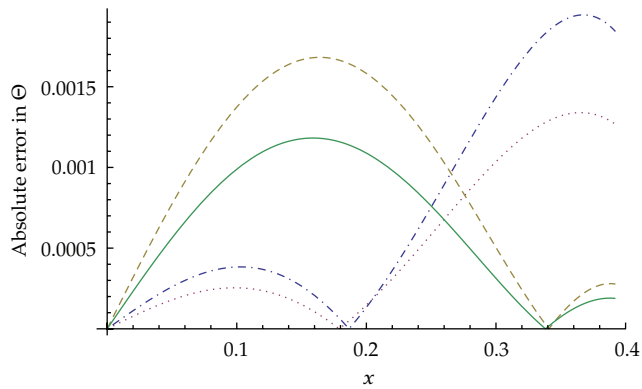

Figure 3: Θ absolute error plots at $\beta = 1$, $t = 0.1$; dotted line: LHBIM, dot-dashed line: Gaussian HBIM, dashed line: Gaussian RIM, plain line: LRIM.

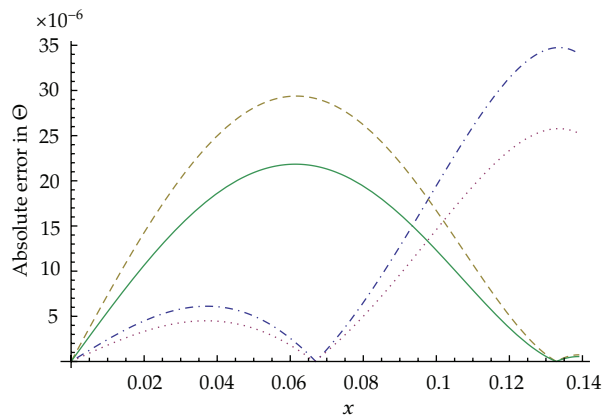

Figure 4: Θ absolute error plots at $\beta = 10$, $t = 0.1$; dotted line: LHBIM, dot-dashed line: Gaussian HBIM, dashed line: Gaussian RIM, plain line: LRIM.

2.2.1. Boundary Condition $f(t) = 1$

In this case, $(2.14)_1$ and $(2.14)_2$ give

$$\delta(t) = \sqrt{\frac{(-2+a)a\,\log(\mathcal{A})}{(-1+a)\beta}}\; t^{1/2},$$

$$\frac{(-2+a)a}{(-1+a)\beta} = \frac{6\,\log(\mathcal{A})}{3\beta\log(\mathcal{A})^2+(1-a)\left(\pi^2+12\log(\mathcal{B})\left(\log(\mathcal{A})\right)\right)+3a\left(\log(\mathcal{A})\right)^2-12(-1+a)\mathrm{Li}_2(-\mathcal{A})}. \tag{2.15}$$

where $\mathcal{A}$ denotes $a/(-2+a)$, $\mathcal{B}$ denotes $2+(2/(-2+a))$.

The contour plot for $(2.15)_2$ takes approximately the same shape as of the LHBIM: at the resolution of Figure 1, no difference in both is noticeable. However, selected (β, a) values satisfying $(2.15)_2$ for $\beta \in \mathbb{K}$ are given in Table 1. Plots and tables of error estimates in moving

front and temperature estimates/approximations due to the LRIM (as well as comparison with other methods) are furnished in Tables 2 and 3 and Figures 3 and 4.

It is observed from these that the LRIM offers the best compromise in approximating the temperature profile and locating the moving front of the considered Stefan problem. In point of fact, the LRIM has the lowest l^∞ $((0,\delta(t)) \times [0,1])$ temperature error and the lowest L^2 error at estimating the position of the moving front. Likewise worth noting is the fact that both logistic methods overpredict the location of the moving front (just like the Gaussian HBIM) in the constant boundary condition case.

2.3. Boundary Condition $f(t) = \exp(t) - 1$

In this subsection we will study the Stefan problem Equations (1.2) and (1.3), when the boundary condition is time-dependent in the form $f(t) = \exp(t) - 1$, for approximate location of its moving front. The LRIM will be employed as it is the most accurate in this regard (as observed from the constant boundary condition considerations of the previous subsection). The approach is similar to the constant boundary condition case.

The analytic solution to the one-phase Stefan problem with boundary condition $f(t) = \exp(t) - 1$ and in the specific case wherein $\beta = 1$ [22] is

$$\Theta(x,t) = \exp(t-x) - 1, \qquad \delta(t) = t. \tag{2.16}$$

Setting $f(t) = \exp(t) - 1$ in (2.14) and solving the resultant ordinary differential equation yield

$$\begin{gathered}\delta(t) = \sqrt{6\frac{(-1+a)(-1+e^t-t)\log((a/(-2+a)))^2}{\mathscr{H}(t;\beta,a)}} \\ \Re\left(\delta(t)\frac{d\delta(t)}{dt}\right) = \Re\left(\frac{((-2+a)a(-1+\exp(t))\log(a/(-2+a)))}{2\beta(-1+a)}\right),\end{gathered} \tag{2.17}$$

where $\mathscr{H}(t;\beta,a)$ is as defined in the appendix and $\Re(\cdot)$ connotes the real part of its argument. Further limiting equation (2.17) to the case $\beta = 1$ yields

$$\begin{gathered}\delta(t) = \sqrt{6\frac{(-1+a)(-1+e^t-t)\log((a/(-2+a)))^2}{\mathscr{H}(t;1,a)}}, \\ \Re\left(\delta(t)\frac{d\delta(t)}{dt}\right) = \Re\left(\frac{((-2+a)a(-1+\exp(t))\log(a/(-2+a)))}{2(-1+a)}\right).\end{gathered} \tag{2.18}$$

A glimpse into the dependency in time $t \in (0,1]$, given by contour plot Figure 5, shows $(2.17)_2$ has two solution branches which are symmetric about line $a = 1$. Selected lower branch (t,a) coordinates, generated with the *FindRoot* command of the symbolic software *Mathematica*, are given in Table 4.

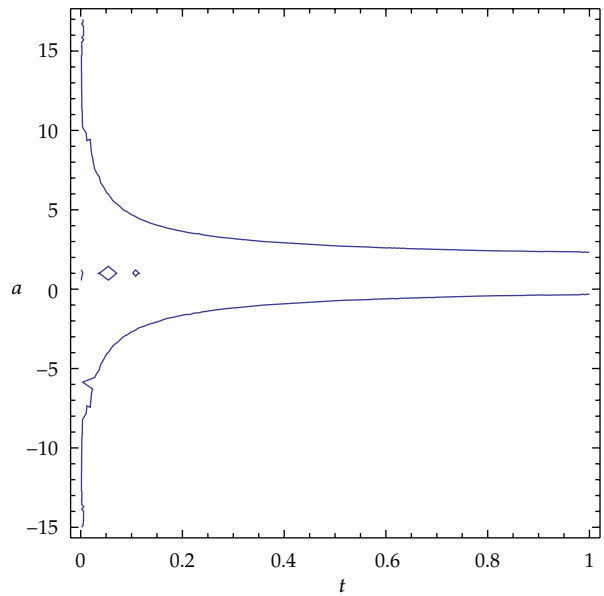

Figure 5: (t, a) contour plot for LRIM when $\beta = 1$ and $f(t) = \exp(t) - 1$.

Table 4: Selected lower branch (t, a) solutions of $(2.18)_2$.

t	0.25	0.50	0.75	1
a	−1.36937	−0.72695	−0.45862	−0.31002

A comparison of obtained $\delta(t)$ values (corresponding to lower branch (t, a) coordinates listed in Table 4) with results from the enthalpy method [22], exponent-type hybrid refined integral method (ETRIM) [23] which employs a test profile of the form

$$\Theta(x, t; b) = f(t)\left[\left(1 - \frac{x}{\delta(t)}\right) + \frac{x}{\delta(t)}\left(b - b^{x^2/\delta(t)^2}\right)\right] \tag{2.19}$$

$b \in (0, 1)$ is detailed in Table 5. It is observed from Table 5 that even without discretization as in the Enthalpy method, the LRIM offers comparable results: it underpredicts the location of the moving front with an l^∞ error of 0.0041 over $t \in [0, 1]$ when $\beta = 1$. This is expected to reduce as β increases.

3. Conclusion

In this paper, motivated by both the geometry of the error function and the potency of the logistic function at modeling growth for dynamical systems, logistic variants of the HBIM and RIM have been proposed. The utility and precision of these methods are evinced in the lowest l^∞ and L^2 temperature estimates errors and L^2 moving front location errors incurred by heat integral methods in current literature for the (1 + 1)-dimensional Stefan melting problem. Compliance with time-dependent boundary conditions is shown with a suitable illustration. The approach given here promises to be applicable to other moving boundary problems,

Table 5: Comparison of exact, ETRIM, LRIM, and Enthalpy methods moving front approximations for the one-phase Stefan problem with time-dependent boundary condition $f(t) = \exp(t) - 1$ when $\beta = 1$ over $t \in [0, 1]$.

	$\delta(t)$			
t	Exact	ETRIM	Enthalpy	LRIM
0.25	0.2500	0.2499	0.2505	0.2499
0.5	0.5000	0.4990	0.5006	0.4992
0.75	0.7500	0.7468	0.7507	0.7479
1	1.0000	0.9926	1.0007	0.9959
N	N/A	N/A	100	N/A
Δt	N/A	N/A	0.001	N/A

and it is supposed that an appropriate finite difference discretization in temporal and spatial coordinates will further improve its accuracy. Furthermore, the stability and convergence studies of such finite-difference schemes are potential areas of investigation.

Appendix

$$
\begin{aligned}
\mathscr{H}(t;\beta,a) = {} & 5\pi^2 - 4a\pi^2 - a^2\pi^2 - 5e^t\pi^2 + 4ae^t\pi^2 + a^2e^t\pi^2 - 6\log(4)\log(2-a) \\
& + 6a\log(4)\log(2-a) + 6e^t\log(4)\log(2-a) - 6ae^t\log(4)\log(2-a) \\
& - 12\log\left(1+\frac{1}{-2+a}\right)\log(2-a) + 12a\log\left(1+\frac{1}{-2+a}\right)\log(2-a) \\
& + 12e^t\log\left(1+\frac{1}{-2+a}\right)\log(2-a) - 12ae^t\log\left(1+\frac{1}{-2+a}\right)\log(2-a) \\
& - 3a\log(2-a)^2 + 3a^2\log(2-a)^2 + 3ae^t\log(2-a)^2 \\
& - 3a^2e^t\log(2-a)^2 + 12a\log(2-a)\log\left(\frac{-1+a}{a}\right) \\
& - 12a^2\log(2-a)\log\left(\frac{-1+a}{a}\right) - 12ae^t\log(2-a)\log\left(\frac{-1+a}{a}\right) \\
& + 12a^2e^t\log(2-a)\log\left(\frac{-1+a}{a}\right) + 6a\log(2-a)\log(-4a) \\
& - 6a^2\log(2-a)\log(-4a) - 6ae^t\log(2-a)\log(-4a) \\
& + 6a^2e^t\log(2-a)\log(-4a) - 27a\log(2)\log(-a) + 15a^2\log(2)\log(-a) \\
& + 27ae^t\log(2)\log(-a) - 15a^2e^t\log(2)\log(-a) \\
& - 12a\log\left(1+\frac{1}{-2+a}\right)\log(-a) + 12ae^t\log\left(1+\frac{1}{-2+a}\right)\log(-a) \\
& + 12\log\left(2+\frac{2}{-2+a}\right)\log(-a) - 12e^t\log\left(2+\frac{2}{-2+a}\right)\log(-a)
\end{aligned}
$$

$$
\begin{aligned}
&-12a\log\left(\frac{-1+a}{a}\right)\log(-a)+12a^2\log\left(\frac{-1+a}{a}\right)\log(-a)\\
&+12ae^t\log\left(\frac{-1+a}{a}\right)\log(-a)-12a^2e^t\log\left(\frac{-1+a}{a}\right)\log(-a)\\
&-3a\log(-a)\log\left(-\frac{a}{2}\right)+3a^2\log(-a)\log\left(-\frac{a}{2}\right)+3ae^t\log(-a)\log\left(-\frac{a}{2}\right)\\
&-3a^2e^t\log(-a)\log\left(-\frac{a}{2}\right)+24\sqrt{-\frac{(-1+a)^2}{(-2+a)^2}}\pi\log\left(-\frac{a}{-2+a}\right)\\
&-12\sqrt{-\frac{(-1+a)^2}{(-2+a)^2}}a\pi\log\left(-\frac{a}{-2+a}\right)-24\sqrt{-\frac{(-1+a)^2}{(-2+a)^2}}e^t\pi\log\left(-\frac{a}{-2+a}\right)\\
&+12\sqrt{-\frac{(-1+a)^2}{(-2+a)^2}}ae^t\pi\log\left(-\frac{a}{-2+a}\right)-6\log\left(-\frac{a}{-2+a}\right)^2+6a\log\left(-\frac{a}{-2+a}\right)^2\\
&+6e^t\log\left(-\frac{a}{-2+a}\right)^2-6ae^t\log\left(-\frac{a}{-2+a}\right)^2-3\beta\log\left(\frac{a}{-2+a}\right)^2\\
&+3a\beta\log\left(\frac{a}{-2+a}\right)^2-12\mathrm{Li}_2\left(-1+\frac{2}{a}\right)+24a\mathrm{Li}_2\left(-1+\frac{2}{a}\right)-12a^2\mathrm{Li}_2\left(-1+\frac{2}{a}\right)\\
&+12e^t\mathrm{Li}_2\left(-1+\frac{2}{a}\right)-24ae^t\mathrm{Li}_2\left(-1+\frac{2}{a}\right)+12a^2e^t\mathrm{Li}_2\left(-1+\frac{2}{a}\right).
\end{aligned}
\tag{A.1}
$$

Nomenclature

t:	Nondimensional time
x:	Nondimensional spatial variable
$\Theta(x,t)$:	Dimensionless temperature
$\delta(t)$:	Position of the moving front in time t
β:	Reciprocal of the Stefan number
$a,\ b$:	Real parameters
$\mathbb{C}$:	Complex number field
$\Re(z)$:	Real part of $z \in \mathbb{C}$
$\mathrm{Li}_n(z)$:	nth polylogarithm of $z \in \mathbb{C}$
$C^1[0,T]$:	Space of continuously differentiable functions defined on $[0,T] \in \mathbb{R}^+ \cup \{0\}$.

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