Research Article

An Algorithm for the Numerical Evaluation of Certain Finite Part Integrals

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Many algorithms that have been proposed for the numerical evaluation of Cauchy principal value integrals are numerically unstable. In this work we present some formulae to evaluate the known Gaussian quadrature rules for finite part integrals $\int_a^{*b} w(x) (f(x)/(x-t)^r) dx$, $r = 1, 2, \dots$, and extend Clenshaw's algorithm to evaluate these integrals in a stable way.

1. Introduction

In the theory of the numerical approximation of Cauchy principal value integrals, basically two kinds of Gaussian quadrature formulae have been investigated. For the modified Gaussian formula, we refer to [1–7] and the literature cited therein. Results on the Gaussian quadrature formula in the strict sense can be found, for example, in [1, 3, 7–13]. In [14] Diethelm proposed a quadrature formula which is called Gaussian quadrature formula of the third kind. Many algorithms that have been proposed for the numerical evaluation of Cauchy principal value integrals are numerically unstable. In particular, this is true for the standard algorithm proposed in [3, Section 3] for the evaluation of the modified Gaussian formula [15, Section 4]. In this paper, we present some formulae to evaluate the known Gaussian quadrature rules for finite part integrals

$$\int_a^{*b} w(x) \frac{f(x)}{(x-t)^r} dx, \quad r = 1, 2, \dots, \quad (1.1)$$

and extend Clenshaw's algorithm [16] to evaluate these integrals in a stable way.

This paper is organized as follows: in Section 1, we introduce some known quadrature rules for finite part integrals and we analyze the relations among these rules. In Section 2, we extend Clenshaw's algorithm [16] to evaluate any sum of derivatives of functions. In Section 3, we apply the extended algorithm to compute Lagrange interpolation polynomial and its derivatives, to evaluate some known singular integrals exactly and quadrature rules for finite part integrals in a stable way. Finally, we present a numerical example.

Some hypersingular integrals can be found by successive differentiation; in some cases evaluation of these integrals becomes tedious and the formulas are lengthily. The evaluation of the such singular integrals is a necessary step for the numerical approach to the integral equations in crack problems. One of the main tasks of the proposed algorithm is evaluate these integrals. The appendix, with closed form solutions for class of these integrals, supplements the paper.

1.1. Gaussian Quadrature Rule

Let $w(x)$ be an integrable weight function on the interval $[a, b]$, and let $\{\pi_n(x)\}$ be orthogonal system of polynomials associated with $w(x)$ which satisfies a three-term recurrence relation:

$$\pi_{n+1}(x) = (A_n x + B_n)\pi_n(x) - l_n \pi_{n-1}(x), \quad \pi_{-1}(x) = 0, \quad \pi_1(x) = 0, \quad (1.2)$$

and hence $\{\pi_n^{(r)}(x)\}$ satisfy a recurrence relation:

$$\pi_{n+1}^{(r)}(x) = (A_n x + B_n)\pi_n^{(r)}(x) - l_n \pi_{n-1}^{(r)}(x) + r A_n \pi_n^{(r-1)}(x). \quad (1.3)$$

The known Gaussian quadrature rule $Q_n(f)$ for $I(f) := \int_a^b w(x)f(x)dx$ is defined as

$$Q_n(f) = \sum_{i=1}^n w_i f(x_i), \quad w_i = \frac{1}{\pi_n'(x_i)} \int_a^b w(x) \frac{\pi_n(x)}{x - x_i} dx, \quad \pi_n(x_i) = 0, \quad i = \overline{1, n}. \quad (1.4)$$

1.2. Modified Gaussian Quadrature Rules for Singular Integrals

Definition 1.1.

$$\begin{aligned} f[x, y] &= \frac{f(x) - f(y)}{x - y}, & f[x, x] &= f'(x), \\ f\left[x, \underbrace{t, \dots, t}_r\right] &= \frac{1}{(r-1)!} \frac{d^{r-1}}{dt^{r-1}} f[x, t]. \end{aligned} \quad (1.5)$$

The finite part integral is defined as follows.

Definition 1.2.

$$\begin{aligned} I(f, t, r) &:= \int_a^{*b} w(x) \frac{f(x)}{(x-t)^r} dx \\ &= \int_a^b w(x) f \left[x, \underbrace{t, \dots, t}_{r \text{ times}} \right] dx + \sum_{k=0}^{r-1} \frac{f^{(k)}(t)}{k!} I(1, t, r-k). \end{aligned} \quad (1.6)$$

The modified Gaussian quadrature rule for $I(f, t, r)$ can be defined as follows.

Definition 1.3.

$$Q_n^M(f, t, r) := \sum_{i=1}^n w_i f \left[x_i, \underbrace{t, \dots, t}_{r \text{ times}} \right] + \sum_{k=0}^{r-1} \frac{f^{(k)}(t)}{k!} I(1, t, r-k). \quad (1.7)$$

Since $Q_n^M(f, t, 1) = \sum_{i=1}^n w_i f[x_i, t] + f(t)I(1, t, 1)$, then by differentiating both sides $(r-1)$ -times, using Leibniz formula and Definition 1.1, we have the following the lemma.

Lemma 1.4.

$$Q_n^M(f, t, r) = \frac{1}{(r-1)!} \frac{d^{r-1}}{dt^{r-1}} Q_n^M(f, t, 1). \quad (1.8)$$

Lemma 1.5. Let $q_n(t) = I(\pi_n, t, 1)$; then

$$I(1, t, m) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dt^{m-1}} \left(\frac{q_n(t)}{\pi_n(t)} \right) + \sum_{j=1}^n \frac{w_j}{(x_j - t)^m}, \quad m = 1, 2, \dots \quad (1.9)$$

Proof. Since the quadrature rule (1.7) is Gaussian:

$$I(\pi_n, t, 1) = Q_n(\pi_n, t, 1) = \sum_{i=1}^n w_i \pi_n[x_i, t] + \pi_n(t)I(1, t, 1), \quad (1.10)$$

then

$$q_n(t) = -\pi_n(t) \sum_{i=1}^n \frac{w_i}{x_i - t} + \pi_n(t)I(1, t, 1). \quad (1.11)$$

This implies that

$$I(1, t, 1) = \frac{q_n(t)}{\pi_n(t)} + \sum_{j=1}^n \frac{w_j}{(x_j - t)}. \quad (1.12)$$

Differentiate both sides $(m-1)$ -times to have (1.9). □

Thus the modified Gaussian quadrature rule for $I(f, t, r)$ can be written as

$$Q_n^M(f, t, r) := \sum_{i=1}^n w_i f \left[x_i, \underbrace{t, \dots, t}_{r \text{ times}} \right] + \sum_{k=0}^{r-1} \frac{f^{(k)}(t)}{k!} \left\{ \frac{1}{(r-k-1)!} \frac{d^{r-k-1}}{dt^{r-k-1}} \left(\frac{q_n(t)}{\pi_n(t)} \right) + \sum_{j=1}^n \frac{w_j}{(x_j - t)^{r-k}} \right\}. \quad (1.13)$$

Special Cases

The first case:

$$Q_n^M(f, t, 1) = \sum_{i=1}^n w_i \frac{f(x_i)}{x_i - t} + f(t) \frac{q_n(t)}{\pi_n(t)}, \quad (1.14)$$

The second case:

$$Q_n^M(f, t, 2) = \sum_{i=1}^n w_i \frac{f(x_i)}{(x_i - t)^2} + \frac{d}{dt} \left(f(t) \frac{q_n(t)}{\pi_n(t)} \right), \quad (1.15)$$

which are the known formulae

1.3. The Gaussian Quadrature Formula in the Strict Sense for Singular Integrals

In the integral $I(f, t, r)$, if we interpolate f by a polynomial $L_{n-1}f$ of degree $n-1$ using the zeros of $\pi_n(x)$, that is, the nodes of the classical Gauss formula, as interpolation nodes and use $I(L_{n-1}f, t, r)$ as an approximation for $I(f, t, r)$, we obtain the Gaussian quadrature formula in the strict sense for singular integrals:

$$Q_n^{\text{strict}}(f, t, 1) := I(L_{n-1}f, t, 1) = \sum_{i=1}^n \frac{w_i}{q_n(x_i)} q_n[x_i, t] f(x_i), \quad (1.16)$$

$$Q_n^{\text{strict}}(f, t, r) := \frac{1}{(r-1)!} \frac{d^{r-1}}{dt^{r-1}} Q_n^{\text{strict}}(f, t, 1), \quad (1.17)$$

so due to (1.16), one obtains the following lemma.

Lemma 1.6.

$$Q_n^{\text{strict}}(f, t, r) := \sum_{i=1}^n \frac{w_i}{q_n(x_i)} q_n \left[x_i, \underbrace{t, \dots, t}_{r \text{ times}} \right] f(x_i). \quad (1.18)$$

1.4. Gaussian Quadrature Rule of the Third Kind

Let $f \in C^{r-1}[a, b]$. According to [14], the integral $I(f, t, r)$ can be approximated by $Q_{n+1}^{L3}(f, t, r)$:

$$Q_{n+1}^{L3}(f, t, r) := \int_a^b w(x) L_{n-1} f \left[\underbrace{x, t, \dots, t}_{r \text{ times}} \right] dx + \sum_{k=0}^{r-1} \frac{f^{(k)}(t)}{k!} I(1, t, r-k), \quad (1.19)$$

which is called *Gaussian quadrature rule of the third kind*:

for $r = 1$,

$$Q_{n+1}^{L3}(f, t, 1) := \int_a^b w(x) L_{n-1} f[x, t] dx + f(t) q_0(t). \quad (1.20)$$

Lemma 1.7.

$$Q_{n+1}^{L3}(f, t, r) := \frac{1}{(r-1)!} \frac{d^{r-1}}{dt^{r-1}} Q_{n+1}^{L3}(f, t, 1). \quad (1.21)$$

1.5. The Connection between the Gaussian Quadrature Rules

Lemma 1.8.

$$Q_n^M(f, t, 1) = Q_n^{\text{strict}}(f, t, 1) + \frac{q_n(t)}{\pi_n(t)} (f(t) - L_n f(t)). \quad (1.22)$$

Proof. From (1.16),

$$\begin{aligned} Q_n^{\text{strict}}(f, t, 1) &= \sum_{i=1}^n \frac{w_i}{q_n(x_i)} q_n[x_i, t] f(x_i), \quad w_i = \frac{q_n(x_i)}{\pi_n'(x_i)} \\ &= \sum_{i=1}^n w_i \frac{f(x_i)}{x_i - t} + \frac{q_n(t)}{\pi_n(t)} \sum_{i=1}^n \frac{f(x_i) \pi_n(t)}{\pi_n'(x_i)(t - x_i)} \\ &= \sum_{i=1}^n w_i \frac{f(x_i)}{x_i - t} + \frac{q_n(t)}{\pi_n(t)} L_n f(t). \end{aligned} \quad (1.23)$$

Since $Q_n^M(f, t, 1) = \sum_{i=1}^n w_i (f(x_i)/x_i - t) + f(t)(q_n(t)/\pi_n(t))$, one obtains (1.22). $\square$

Corollary 1.9. If $\pi_n(t) = 0$, then

$$Q_n^M(f, t, 1) = Q_n^{\text{strict}}(f, t, 1) + \frac{q_n(t)}{\pi_n'(t)} (f(t) - L_n f(t))'. \quad (1.24)$$

Proof. From Definition 1.3,

$$\begin{aligned} Q_n^M(f, t, 1) &= \sum_{i=1}^n \frac{w_i}{q_n(x_i)} q_n[x_i, t] f(x_i) + q_n(t) \sum_{i=1}^n \frac{w_i}{q_n(x_i)} f[x_i, t] \\ &= Q_n^{\text{strict}}(f, t, 1) + q_n(t) \sum_{i=1}^n \frac{f(x_i) - f(t)}{\pi'_n(x_i)(x_i - t)}, \end{aligned} \quad (1.25)$$

and since

$$\sum_{i=1}^n \frac{f(x_i) - f(t)}{\pi'_n(x_i)(x_i - t)} = \frac{1}{\pi'_n(t)} (f'(t) - L_{n-1}[f]'(t)), \quad (1.26)$$

then Lemma has been proved. $\square$

Lemma 1.10.

$$Q_{n+1}^{L3}(f, t, 1) = Q_n^{\text{strict}}(f, t, 1) + (f(t) - L_{n-1}f(t))q_0(t). \quad (1.27)$$

2. Sum Series Algorithm

Theorem 2.1. The sum $\sum_{k=m}^n a_k \Phi_k(x)$ ($m \geq 0$) of any series of functions which satisfy a linear recurrence relation

$$\Phi_{k+1}(x) + \delta_k(x)\Phi_k(x) + \eta_k(x)\Phi_{k-1}(x) = s_k f_k(x) \quad (2.1)$$

is given by

$$\begin{aligned} \sum_{k=m}^n a_k \Phi_k(x) &= b_m(x)\Phi_m(x) + b_{m+1}(x)\{\Phi_{m+1}(x) + \delta_m(x)\Phi_m(x)\} \\ &\quad + \sum_{k=m+1}^{n-1} b_{k+1} s_k f_k(x), \end{aligned} \quad (2.2)$$

where

$$\begin{aligned} b_{n+1} &= b_{n+2} = 0, \\ b_k + \delta_k b_{k+1} + \eta_{k+1} b_{k+2} &= a_k, \quad k = n, n-1, \dots, m. \end{aligned} \quad (2.3)$$

Moreover, if $f_k(x)$ satisfies the linear recurrence relation

$$f_{k+1}(x) + \lambda_k(x)f_k(x) + \mu_k(x)f_{k-1}(x) = 0, \quad (2.4)$$

then

$$\begin{aligned} \sum_{k=m}^n a_k \Phi_k(x) &= b_m(x) \Phi_m(x) + b_{m+1}(x) \{ \Phi_{m+1}(x) + \delta_m(x) \Phi_m(x) \} \\ &\quad + c_{m+1}(x) f_{m+1}(x) + c_{m+2}(x) \{ f_{m+2}(x) + \lambda_{m+1}(x) f_{m+1}(x) \}, \end{aligned} \quad (2.5)$$

where

$$\begin{aligned} c_n &= c_{n+1} = 0, \\ c_k + \lambda_k c_{k+1} + \mu_{k+1} c_{k+2} &= s_k b_{k+1}, \quad k = n-1, \dots, m+1. \end{aligned} \quad (2.6)$$

Proof.

$$\begin{aligned} \sum_{k=m}^n a_k \Phi_k &= \sum_{k=m}^n (b_k + \delta_k b_{k+1} + \eta_{k+1} b_{k+2}) \Phi_k(x) \\ &= (b_m + \delta_m b_{m+1}) \Phi_m(x) + b_{m+1} \Phi_{m+1}(x) \\ &\quad + \sum_{k=m+2}^n b_k (\Phi_k(x) + \delta_{k-1} \Phi_{k-1}(x) + \eta_{k-1} \Phi_{k-2}(x)). \end{aligned} \quad (2.7)$$

Using (2.1), we get (2.2). Moreover, if $f_k(x)$ satisfies (2.4), we can similarly deduce (2.5). $\square$

In a similar way, we can prove the following theorem.

Theorem 2.2.

$$\begin{aligned} \sum_{k=m}^n a_k \Phi_k(x) &= \eta_{n+1}(x) b_n \Phi_n(x) + b_{n-1}(x) \{ \eta_n \Phi_{n-1}(x) + \delta_n(x) \Phi_n(x) \} \\ &\quad + \sum_{k=m+1}^{n-1} b_{k-1} s_k f_k(x), \end{aligned} \quad (2.8)$$

where

$$\begin{aligned} b_{m-2} &= b_{m-1} = 0, \\ \eta_{k+1} b_k + \delta_k b_{k-1} + b_{k-2} &= a_k, \quad k = m, m+1, \dots, n. \end{aligned} \quad (2.9)$$

Moreover, if $f_k(x)$ satisfies the linear recurrence relation (2.4), then

$$\begin{aligned} \sum_{k=m}^n a_k \Phi_k(x) &= \eta_{n+1} b_n \Phi_n(x) + b_{n-1} \{ \eta_n \Phi_{n-1}(x) + \delta_n \Phi_n(x) \} \\ &\quad + \mu_n c_{n-1} f_{n-1}(x) + c_{n-2} \{ \mu_{n-1} f_{n-2}(x) + \lambda_{n-1} f_{n-1}(x) \}, \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} c_{m-1} &= c_m = 0, \\ \mu_{k+1}c_k + \lambda_k c_{k-1} + c_{k-2} &= s_k b_{k-1}, \quad k = m+1, \dots, n-1. \end{aligned} \quad (2.11)$$

Now we extend the previous theorems to evaluate $\sum_{k=0}^n a_k \Phi_k^{(r-1)}(x)$.

Suppose that $\Phi_k(x)$ satisfies (2.1) with $f_k \equiv 0$ and that $\Phi_k^{(i)}(x)$ satisfies the following recurrence relation:

$$\Phi_{k+1}^{(i)}(x) + \delta_k(x)\Phi_k^{(i)}(x) + \eta_k(x)\Phi_{k-1}^{(i)}(x) = s_k^i(x)\Phi_k^{(i-1)}(x). \quad (2.12)$$

Theorem 2.3. Let $M = \max\{n, 2r-1\}$; then

$$\sum_{i=0}^{n-1} a_i \Phi_i^{(r-1)} = \sum_{i=1}^r \left\{ c_{i-1}^{r-i} \Phi_{i-1}^{(r-i)} + c_i^{r-i} [\delta_{i-1} \Phi_{i-1}^{(r-i)} + \Phi_i^{(r-i)}] \right\}, \quad (2.13)$$

where

$$\begin{aligned} c_{M-i+1}^{r-i} &= c_{M-i+2}^{r-i} = 0, \\ c_k^{r-i} + \delta_k c_{k+1}^{r-i} + \eta_{k+1} c_{k+2}^{r-i} &= a_k^{r-i}, \quad k = M-i, \dots, i-1, \quad 1 \leq i \leq r, \\ a_k^{r-1} &= \begin{cases} a_k, & 0 \leq k \leq n-1, \\ 0, & n \leq k \leq M, \end{cases} \\ a_k^{r-i} &= s_k^{r-i+1} c_{k+1}^{r-i+1}, \quad i \geq 2. \end{aligned} \quad (2.14)$$

Proof. Let

$$Y_m = \sum_{i=m-1}^{M-m} a_i^{r-m} \Phi_i^{(r-m)} \quad (1 \leq m \leq r). \quad (2.15)$$

Due to Theorem 2.1, we obtain $Y_m = X_m + Y_{m+1}$,

$$\begin{aligned} X_m &= c_{m-1}^{r-m} \Phi_{m-1}^{(r-m)} + c_m^{r-m} [\delta_{m-1} \Phi_{m-1}^{(r-m)} + \Phi_m^{(r-m)}], \\ c_{M-m+1}^{r-m} &= c_{M-m+2}^{r-m} = 0, \\ c_k^{r-m} + \delta_k c_{k+1}^{r-m} + \eta_{k+1} c_{k+2}^{r-m} &= a_k^{r-m}, \quad k = M-m, \dots, m-1. \end{aligned} \quad (2.16)$$

By induction, we have $Y_1 = \sum_{i=1}^{r-1} X_i + Y_r$, and by using Theorem 2.1, we obtain

$$Y_r = \sum_{i=r-1}^{M-r} a_i^0 \Phi_i = c_{r-1}^0 \Phi_{r-1} + c_r^0 [\delta_{r-1} \Phi_{r-1} + \Phi_r], \quad (2.17)$$

where

$$\begin{aligned} c_{M-r+1}^0 &= c_{M-r+2}^0 = 0, \\ c_k^0 + \delta_k c_{k+1}^0 + \eta_{k+1} c_{k+2}^0 &= a_k^0, \quad k = M-r, \dots, r-1. \end{aligned} \quad (2.18)$$

Finally

$$Y_1 = \sum_{i=1}^r \left\{ c_{i-1}^{r-i} \Phi_{i-1}^{r-i} + c_i^{r-i} \left[\delta_{i-1} \Phi_{i-1}^{(r-i)} + \Phi_i^{(r-i)} \right] \right\}, \quad (2.19)$$

so

$$\sum_{i=0}^{n-1} a_i \Phi_i^{(r-1)} = Y_1 = \sum_{i=1}^r \left\{ c_{i-1}^{r-i} \Phi_{i-1}^{(r-i)} + c_i^{r-i} \left[\delta_{i-1} \Phi_{i-1}^{(r-i)} + \Phi_i^{(r-i)} \right] \right\}. \quad (2.20)$$

□

The alternative algorithm that incorporate a_i 's in an upward direction can be deduced as follows.

Theorem 2.4. Let $M = \max\{n, 2r-1\}$; then

$$\sum_{i=0}^{n-1} a_i \Phi_i^{(r-1)} = \sum_{j=1}^r \left\{ \eta_{M-j+1} c_{M-j}^{j-1} \Phi_{M-j}^{(r-j)} + c_{M-j-1}^{j-1} \left[\eta_{M-j} \Phi_{M-j-1}^{(r-j)} + \delta_{M-j} \Phi_{M-j}^{(r-j)} \right] \right\}, \quad (2.21)$$

where

$$\begin{aligned} c_{j-3}^{j-1} &= c_{j-2}^{j-1} = 0, \\ \eta_{k+1} c_k^{j-1} + \delta_k c_{k-1}^{j-1} + c_{k-2}^{j-1} &= a_k^{j-1}, \quad k = j-1, \dots, M-j, \quad 1 \leq j \leq r, \\ a_k^0 &= \begin{cases} a_k, & 0 \leq k \leq n-1, \\ 0, & n \leq k \leq M, \end{cases} \\ a_k^{j-1} &= s_k^{j-2} c_{k-1}^{j-2}, \quad k = j-1, \dots, M-j, \quad j \geq 2. \end{aligned} \quad (2.22)$$

Proof. Let

$$Y_m = \sum_{i=m-1}^{M-m} a_i^{(m-1)} \Phi_i^{(r-m)} \quad (1 \leq m \leq r). \quad (2.23)$$

Due to Theorem 2.2, we obtain $Y_m = X_m + Y_{m+1}$,

$$X_m = \eta_{M-m+1} c_{M-m}^{m-1} \Phi_{M-m}^{(r-m)} + c_{M-m-1}^{m-1} \left[\eta_{M-m} \Phi_{M-m-1}^{(r-m)} + \delta_{M-m} \Phi_{M-m}^{(r-m)} \right],$$

$$c_{m-3}^{m-1} = c_{m-2}^{m-1} = 0, \quad (2.24)$$

$$\eta_{k+1} c_k^{m-1} + \delta_k c_{k-1}^{m-1} + c_{k-2}^{m-1} = a_k^{m-1}, \quad k = m-1, \dots, M-m, \quad 1 \leq m \leq r-1.$$

By induction, we have $Y_1 = \sum_{i=1}^{r-1} X_i + Y_r$, and by using Theorem 2.2, we obtain

$$Y_r = \sum_{i=r-1}^{M-r} a_i^{r-1} \Phi_i = \eta_{M-r+1} c_{M-r}^{r-1} \Phi_{M-r} + c_{M-r-1}^{r-1} \left[\eta_{M-r} \Phi_{M-r-1} + \delta_{M-r} \Phi_{M-r} \right], \quad (2.25)$$

where

$$c_{r-3}^{r-1} = c_{r-2}^{r-1} = 0,$$

$$\eta_{k+1} c_k^{r-1} + \delta_k c_{k-1}^{r-1} + c_{k-2}^{r-1} = a_k^{r-1}, \quad k = r-1, \dots, M-r. \quad (2.26)$$

Finally

$$Y_1 = \sum_{j=1}^r \eta_{M-j+1} c_{M-j}^{j-1} \Phi_{M-j}^{(r-j)} + c_{M-j-1}^{j-1} \left[\eta_{M-j} \Phi_{M-j-1}^{(r-j)} + \delta_{M-j} \Phi_{M-j}^{(r-j)} \right], \quad (2.27)$$

so

$$\sum_{i=0}^{n-1} a_i \Phi_i^{(r-1)} = Y_1 = \sum_{j=1}^r \eta_{M-j+1} c_{M-j}^{j-1} \Phi_{M-j}^{(r-j)} + c_{M-j-1}^{j-1} \left[\eta_{M-j} \Phi_{M-j-1}^{(r-j)} + \delta_{M-j} \Phi_{M-j}^{(r-j)} \right]. \quad (2.28)$$

□

3. Computational Aspects and Numerical Examples

In this section, we use the previous algorithm to evaluate the known Gaussian quadrature rules for the integrals in (1.6).

Lemma 3.1. Let $L_{n-1}(f) = \sum_{i=0}^{n-1} a_i[f] \pi_i$; then

$$a_i[f] = \frac{1}{h_i} \sum_{j=1}^n w_j \pi_i(x_j) f(x_j), \quad (3.1)$$

where w_i is the weight of the classical Gaussian quadrature formula corresponding to the node x_i .

Proof. Since

$$\sum_{i=0}^{n-1} a_i [f] \pi_i = \sum_{i=1}^n \frac{f(x_i) \pi_n(x)}{\pi'_n(x_i)(x_i - x)}, \quad (3.2)$$

then multiply both sides by $w(x)\pi_j(x)$ ($0 \leq j \leq n-1$), and integrate over $[a, b]$, we derive

$$a_j [f] = \frac{1}{h_j} \sum_{i=1}^n \frac{f(x_i)}{\pi'_n(x_i)} I(\pi_n \pi_j, x_i, 1), \quad (3.3)$$

by using the orthogonality of π_n , we get $I(\pi_n \pi_j, x_i, 1) = \pi_j(x_i) q_n(x_i)$ and by knowing that $w_i = q_n(x_i) / \pi'_n(x_i)$, we obtain (3.1). $\square$

Now, since $\pi_n(x)$ satisfy (2.1) which $f_r = 0$ and $\eta_n = l_n$, the following statement follows from Theorem 2.3.

Lemma 3.2.

$$L_{n-1}[f] = c_0 + c_1(\alpha_0 + \pi_1), \quad (3.4)$$

where

$$\begin{aligned} c_{n+1} &= c_{n+2} = 0, \\ c_k + \alpha_k c_{k+1} + l_{k+1} c_{k+2} &= a_k, \quad k = n-1, \dots, 0, \end{aligned} \quad (3.5)$$

$$\alpha_k = -(A_k t + B_k). \quad (3.6)$$

Since π'_n satisfies recurrence relation (2.12), so we can apply Theorem 2.3 to have the following lemma.

Lemma 3.3.

$$L'_{n-1}[f] = c_1^1 \pi_1' + c_1^0 \pi_1 + c_2^0 (\alpha_1 \pi_1 + \pi_2), \quad (3.7)$$

where c_1^1, c_1^0, c_2^0 are given in Theorem 2.3.

Since $q_i^{(r-1)}(t)$ obeys recurrence relation (2.12), so we can evaluate the pervious Gaussian quadrature rules introduced in Sections 1.2–1.4 by the following theorems.

Theorem 3.4.

$$Q_n^{strict}(f, t, r) = \frac{1}{(r-1)!} \sum_{i=0}^{n-1} a_i [f] q_i^{(r-1)}(t), \quad (3.8)$$

where

$$\sum_{i=0}^{n-1} a_i [f] q_i^{(r-1)}(t) = \sum_{i=1}^r \left\{ c_{i-1}^{r-i} q_{i-1}^{(r-i)}(t) + c_i^{r-i} \left[\alpha_{i-1} q_{i-1}^{(r-i)}(t) + q_i^{(r-i)} \right] \right\}, \quad (3.9)$$

where c_i^k 's are given by Theorem 2.3.

Theorem 3.5. If $\pi_n(t) \neq 0$, then

$$Q_n^M(f, t, r) = \frac{1}{(r-1)!} \sum_{i=0}^{n-1} a_i [f] \rho_i^{(r-1)}(t) + \frac{1}{(r-1)!} \frac{d^{r-1}}{dt^{r-1}} \left\{ \frac{q_n(t)}{\pi_n(t)} f(t) \right\}, \quad (3.10)$$

where (using Theorem 2.3)

$$\sum_{i=0}^{n-1} a_i [f] \rho_i^{(r-1)}(t) = \sum_{i=1}^r \left\{ c_{i-1}^{r-i} \rho_{i-1}^{(r-i)}(t) + c_i^{r-i} \left[\alpha_{i-1} \rho_{i-1}^{(r-i)}(t) + \rho_i^{(r-i)} \right] \right\}, \quad (3.11)$$

where $\rho_i(t) = q_i(t) - q_n(t)\pi_i(t)/\pi_n(t)$, which fulfill a three-term recurrence (1.2), and c_i^k are given by Theorem 2.3.

Proof. Using Lemma 1.8, we have

$$\begin{aligned} Q_n^M(f, t, 1) &= \sum_{i=0}^{n-1} a_i [f] I(\pi_i, t, 1) + \frac{q_n(t)}{\pi_n(t)} (f(t) - L_{n-1}[f](t)) \\ &= \sum_{i=0}^{n-1} a_i [f] q_i(t) + \frac{q_n(t)}{\pi_n(t)} (f(t) - L_{n-1}[f](t)) \\ &= \sum_{i=0}^{n-1} a_i [f] \rho_i(t) + \frac{q_n(t)}{\pi_n(t)} f(t). \end{aligned} \quad (3.12)$$

Then differentiating both sides $(r-1)$ -times, we obtain relation (3.10). □

Theorem 3.6. If $\pi_n(t) = 0$, then

$$\begin{aligned} Q_n^M(f, t, r) &= \frac{1}{(r-1)!} \sum_{i=0}^{n-1} a_i [f] q_i^{(r-1)}(t) \\ &\quad + \frac{1}{(r-1)!} \frac{d^{r-1}}{dt^{r-1}} \left\{ \frac{q_n(t)}{\pi_n(t)} (f'(t) - L'_{n-1}[f](t)) \right\}. \end{aligned} \quad (3.13)$$

Proof. Using Corollary 1.9, we have

$$\begin{aligned} Q_n^M(f, t, 1) &= \sum_{i=0}^{n-1} a_i[f] I(\pi_i, t, 1) + \frac{q_n(t)}{\pi_n(t)} (f'(t) - L'_{n-1}[f](t)) \\ &= \sum_{i=0}^{n-1} a_i[f] q_i(t) + \frac{q_n(t)}{\pi_n(t)} (f'(t) - L'_{n-1}[f](t)), \end{aligned} \quad (3.14)$$

and by differentiating both sides $(r-1)$ -times, we obtain relation (3.13). $\square$

A direct result of Lemma 1.10 and Theorem 2.3 is the following theorem.

Theorem 3.7.

$$\begin{aligned} Q_{n+1}^{L3}(f, t, r) &= \frac{1}{(r-1)!} \sum_{i=0}^{n-1} a_i[f] \sigma_i^{(r-1)}(t) + \frac{1}{(r-1)!} \frac{d^{r-1}}{dt^{r-1}} (f(t) q_0(t)), \\ Q_{n+1}^{L3}(f, t, r) &= \frac{1}{(r-1)!} \sum_{i=0}^{r-1} \left\{ c_i^{r-i-1} \sigma_i^{(r-i-1)} + c_{i+1}^{r-i-1} [\alpha_i \sigma_i^{(r-i-1)} + \sigma_i^{(r-i-1)}] \right\} \\ &\quad + \frac{1}{(r-1)!} \frac{d^{r-1}}{dt^{r-1}} (f(t) q_0(t)), \end{aligned} \quad (3.15)$$

where $\sigma_i = q_i(t) - q_0(t)\pi_i(t)$ and $c_i^{(k)}$ are given by Theorem 2.3.

4. A Numerical Example

Hui* and Shia [17] consider the modified Gaussian quadrature

$$\int_{-1}^{*1} \frac{\sqrt{1-x^2} f(x)}{(x-t)^2} dx \cong -\frac{\pi f'(t)}{U_{n-1}(t)} T_n(t) - \frac{\pi f(t) W(U_{n-1}, T_n)}{U_{n-1}(t)^2} + \sum_{i=1}^n w_i \frac{f(x_i)}{(x_i-t)^2}, \quad (4.1)$$

where $W(U_{n-1}, T_n) = U_{n-1} T'_n - U'_{n-1} T_n$.

In Table 1, numerical results are obtained for

$$I(f, 0, 2) = \int_{-1}^{*1} \frac{\sqrt{1-x^2} e^x}{x^2} dx \cong -2.3395562533389707, \quad (4.2)$$

by using the quadrature rule $Q_n^M(f, 0, 2)$; in this case, $w(x) = \sqrt{1-x^2}$; and $x_{i,n}$ are the zeros of Chebyshev polynomial of the second kind U_n ; then we have

$$x_{i,n} = \cos\left(\frac{i+1}{n+1}\pi\right), \quad w_{i,n} = \frac{\pi}{n+1} (1-x_{i,n}^2), \quad q_n(t) = -\pi T_{n+1}(x). \quad (4.3)$$

Table 1

n	Equation (4.1)	Equation (3.13)
4	-2.33956	-2.33956
5	NA	-2.33942
6	-2.33956	-2.33956
7	NA	-2.33956
8	-2.33956	-2.33956

Then applying the algorithm in Theorem 3.6 for $r = 2$, $l_i = 1$, $\alpha_i = -2t$, we obtain the following results comparing with these of Hui* and Shia [17].

The results indicate that the algorithm is indeed stable even when t coincides with one of the nodes of the quadrature formula (in this case, this node is located at the origin). However, in deriving the modified method [17] there are no numerical values presented for $n = 5$ and 7 that is why it is assumed that the roots of the orthogonal polynomials are distinct from t .

Appendix

A. Evaluation for Some Finite Part Integrals Using the Proposed Algorithm

The goal of this appendix is to provide the evaluations for some integrals using the proposed algorithm with suitable n points

A.1. $w(x) = \sqrt{1-x^2}$, $r = 1$

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_0(x)}{x-t} dx &= -t, \\ \frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_1(x)}{x-t} dx &= \frac{1}{2} - t^2, \\ \frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_2(x)}{x-t} dx &= 2t - 2t^3, \\ \frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_3(x)}{x-t} dx &= -1 + 5t^2 - 4t^4, \\ \frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_4(x)}{x-t} dx &= -4t + 12t^3 - 8t^5, \\ \frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_5(x)}{x-t} dx &= 1 - 13t^2 + 28t^4 - 16t^6, \\ \frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_6(x)}{x-t} dx &= 6t - 38t^3 + 64t^5 - 32t^7, \end{aligned}$$

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_7(x)}{x-t} dx &= -1 + 25t^2 - 104t^4 + 144t^6 - 64t^8, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_8(x)}{x-t} dx &= -8t + 88t^3 - 272t^5 + 320t^7 - 128t^9,
\end{aligned}
\tag{A.1}$$

A.2. $w(x) = (1 - x^2)^{3/2}$, $r = 1$

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_0(x)}{x-t} dx &= t^3 - \frac{3}{2}t, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_1(x)}{x-t} dx &= \frac{3}{8} - \frac{3}{2}t^2 + t^4, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_2(x)}{x-t} dx &= \frac{9}{4}t - 4t^3 + 2t^5, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_3(x)}{x-t} dx &= -\frac{7}{8} + 6t^2 - 9t^4 + 4t^6, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_4(x)}{x-t} dx &= -4t + 16t^3 - 20t^5 + 8t^7, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_5(x)}{x-t} dx &= 1 - 14t^2 + 41t^4 - 44t^6 + 16t^8, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_6(x)}{x-t} dx &= 6t - 44t^3 + 102t^5 - 96t^7 + 32t^9, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_7(x)}{x-t} dx &= -1 + 26t^2 - 129t^4 + 248t^6 - 208t^8 + 64t^{10}, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_8(x)}{x-t} dx &= -8t + 96t^3 - 360t^5 + 592t^7 - 448t^9 + 128t^{11},
\end{aligned}
\tag{A.2}$$

A.3. $w(x) = (1 - x^2)^{5/2}$, $r = 1$

(a)

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_0(x)}{x-t} dx &= -\frac{15}{8}t + \frac{5}{2}t^3 - t^5, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_1(x)}{x-t} dx &= \frac{5}{16} - \frac{15}{8}t^2 + \frac{5}{2}t^4 - t^6,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_2(x)}{x-t} dx &= \frac{5}{12}t - \frac{25}{4}t^3 + 6t^5 - 2t^7, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_3(x)}{x-t} dx &= -\frac{25}{32} + \frac{55}{8}t^2 - 15t^4 + 13t^6 - 4t^8, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_4(x)}{x-t} dx &= -\frac{65}{16}t + 20t^3 - 36t^5 + 28t^7 - 8t^9, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_5(x)}{x-t} dx &= \frac{31}{32} - 15t^2 + 55t^4 - 85t^6 + 60t^8 - 16t^{10}, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_6(x)}{x-t} dx &= 6t - 50t^3 + 146t^5 - 198t^7 + 128t^9 - 32t^{11}, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_7(x)}{x-t} dx &= -1 + 27t^2 - 155t^4 + 377t^6 - 456t^8 + 272t^{10} - 64t^{12}, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_8(x)}{x-t} dx &= -8t + 104t^3 - 456t^5 + 952t^7 - 1040t^9 + 576t^{11} - 128t^{13},
\end{aligned} \tag{A.3}$$

(b)

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_0(x)}{x-t} dx &= -\frac{15}{8}t + \frac{5}{2}t^3 - t^5, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_1(x)}{x-t} dx &= \frac{5}{8} - \frac{15}{4}t^2 + 5t^4 - 2t^6, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_2(x)}{x-t} dx &= \frac{25}{8}t - 10t^3 + 11t^5 - 4t^7, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_3(x)}{x-t} dx &= -\frac{15}{16} + 10t^2 - 25t^4 + 24t^6 - 8t^8, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_4(x)}{x-t} dx &= -5t + 30t^3 - 61t^5 + 52t^7 - 16t^9, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_5(x)}{x-t} dx &= 1 - 20t^2 + 85t^4 - 146t^6 + 112t^8 - 32t^{10}, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_6(x)}{x-t} dx &= 7t - 70t^3 + 231t^5 - 344t^7 + 240t^9 - 64t^{11},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_7(x)}{x-t} dx &= -1 + 34t^2 - 225t^4 + 608t^6 - 800t^8 + 512t^{10} - 128t^{12}, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_8(x)}{x-t} dx &= -9t + 138t^3 - 681t^5 + 1560t^7 - 1840t^9 + 1088t^{11} - 256t^{13},
\end{aligned}
\tag{A.4}$$

A.4. $w(x) = (1 - x^2)^{5/2}$, $r = 2$

(a)

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_0(x)}{(x-t)^2} dx &= -\frac{15}{8} + \frac{15}{2}t^2 - 5t^4, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_1(x)}{(x-t)^2} dx &= -\frac{15}{4}t + 10t^3 - 6t^5, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_2(x)}{(x-t)^2} dx &= \frac{5}{12} - \frac{75}{4}t^2 + 30t^4 - 14t^6, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_3(x)}{(x-t)^2} dx &= \frac{55}{4}t - 60t^3 + 78t^5 - 32t^7, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_4(x)}{(x-t)^2} dx &= -\frac{65}{16} + 60t^2 - 180t^4 + 196t^6 - 72t^8, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_5(x)}{(x-t)^2} dx &= -30t + 220t^3 - 510t^5 + 480t^7 - 160t^9, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_6(x)}{(x-t)^2} dx &= 6 - 150t^2 + 730t^4 - 1386t^6 + 1152t^8 - 352t^{10}, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_7(x)}{(x-t)^2} dx &= 54t - 620t^3 + 2262t^5 - 3648t^7 + 2720t^9 - 768t^{11}, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_8(x)}{(x-t)^2} dx &= -8 + 312t^2 - 2280t^4 + 6664t^6 - 9360t^8 + 6336t^{10} - 1664t^{12},
\end{aligned}
\tag{A.5}$$

(b)

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_0(x)}{(x-t)^2} dx &= -\frac{15}{8} + \frac{15}{2}t^2 - 5t^4, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_1(x)}{(x-t)^2} dx &= -\frac{15}{2}t + 20t^3 - 12t^5,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_2(x)}{(x-t)^2} dx &= \frac{25}{8} - 30t^2 + 55t^4 - 28t^6, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_3(x)}{(x-t)^2} dx &= 20t - 100t^3 + 144t^5 - 64t^7, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_4(x)}{(x-t)^2} dx &= -5 + 90t^2 - 305t^4 + 364t^6 - 144t^8, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_5(x)}{(x-t)^2} dx &= -40t + 340t^3 - 876t^5 + 896t^7 - 320t^9, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_6(x)}{(x-t)^2} dx &= 7 - 210t^2 + 1155t^4 - 2408t^6 + 2160t^8 - 704t^{10}, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_7(x)}{(x-t)^2} dx &= 68t - 900t^3 + 3648t^5 - 6400t^7 + 5120t^9 - 1536t^{11}, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_8(x)}{(x-t)^2} dx &= -9 + 414t^2 - 3405t^4 + 10920t^6 - 16560t^8 + 11968t^{10} - 3328t^{12},
\end{aligned} \tag{A.6}$$

A.5. $w(x) = (1 - x^2)^{5/2}$, $r = 3$

(a)

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_0(x)}{(x-t)^3} dx &= \frac{15}{2}t - 10t^3, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_1(x)}{(x-t)^3} dx &= -\frac{15}{8} + 15t^2 - 15t^4, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_2(x)}{(x-t)^3} dx &= -\frac{75}{4}t + 60t^3 - 42t^5, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_3(x)}{(x-t)^3} dx &= \frac{55}{8} - 90t^2 + 195t^4 - 112t^6, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_4(x)}{(x-t)^3} dx &= 60t - 360t^3 + 588t^5 - 288t^7, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_5(x)}{(x-t)^3} dx &= -15 + 330t^2 - 1275t^4 + 1680t^6 - 720t^8, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_6(x)}{(x-t)^3} dx &= 150t + 1460t^3 - 4158t^5 + 4608t^7 - 1760t^9,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_7(x)}{(x-t)^3} dx &= 27 - 930t^2 + 5655t^4 - 12768t^6 + 12240t^8 - 4224t^{10}, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_8(x)}{(x-t)^3} dx &= 312t - 4560t^3 + 19992t^5 - 37440t^7 + 31680t^9 - 9984t^{11},
\end{aligned}
\tag{A.7}$$

(b)

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_0(x)}{(x-t)^3} dx &= \frac{15}{2}t - 10t^3, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_1(x)}{(x-t)^3} dx &= -\frac{15}{4} + 30t^2 - 30t^4, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_2(x)}{(x-t)^3} dx &= -30t + 110t^3 - 84t^5, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_3(x)}{(x-t)^3} dx &= 10 - 150t^2 + 360t^4 - 224t^6, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_4(x)}{(x-t)^3} dx &= 90t - 610t^3 + 1092t^5 - 576t^7, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_5(x)}{(x-t)^3} dx &= -20 + 510t^2 - 2190t^4 + 3136t^6 - 1440t^8, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_6(x)}{(x-t)^3} dx &= -210t + 2310t^3 - 7224t^5 + 8640t^7 - 3520t^9, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_7(x)}{(x-t)^3} dx &= 34 - 1350t^2 + 9120t^4 - 22400t^6 + 23040t^8 - 8448t^{10}, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_8(x)}{(x-t)^3} dx &= 414t - 6810t^3 + 32760t^5 - 66240t^7 + 59840t^9 - 19968t^{11},
\end{aligned}
\tag{A.8}$$

A.6. $w(x) = (1 - x^2)^{5/2}$, $r = 4$

(a)

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_0(x)}{(x-t)^4} dx &= \frac{5}{2} - 10t^2, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_1(x)}{(x-t)^4} dx &= 10t - 20t^3,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_2(x)}{(x-t)^4} dx &= -\frac{25}{4} + 60t^2 - 70t^4, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_3(x)}{(x-t)^4} dx &= -60t + 260t^3 - 224t^5, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_4(x)}{(x-t)^3} dx &= 20 - 360t^2 + 980t^4 - 672t^6, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_5(x)}{(x-t)^4} dx &= 220t - 1700t^3 + 3360t^5 - 1920t^7, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_6(x)}{(x-t)^4} dx &= -50 + 1460t^2 - 6930t^4 + 10752t^6 - 5280t^8, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_7(x)}{(x-t)^4} dx &= -620t + 7540t^3 - 25536t^5 + 32640t^7 - 14080t^9, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)T_8(x)}{(x-t)^4} dx &= 104 - 4560t^2 + 33320t^4 - 87360t^6 + 95040t^8 - 36608t^{10},
\end{aligned} \tag{A.9}$$

(b)

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_0(x)}{(x-t)^4} dx &= \frac{5}{2} - 10t^2, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_1(x)}{(x-t)^4} dx &= 20t - 40t^3, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_2(x)}{(x-t)^4} dx &= -10 + 110t^2 - 140t^4, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_3(x)}{(x-t)^4} dx &= -100t + 480t^3 - 448t^5, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_4(x)}{(x-t)^4} dx &= 30 - 610t^2 + 1820t^4 - 1344t^6, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_5(x)}{(x-t)^4} dx &= 340t - 2920t^3 + 6272t^5 - 3840t^7, \\
\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_6(x)}{(x-t)^4} dx &= -70 + 2310t^2 - 12040t^4 + 20160t^6 - 10560t^8,
\end{aligned}$$

$$\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_7(x)}{(x-t)^4} dx = -900t + 12160t^3 - 44800t^5 + 61440t^7 - 28160t^9,$$

$$\frac{1}{\pi} \int_{-1}^{*1} \frac{w(x)U_8(x)}{(x-t)^4} dx = 138 - 6810t^2 + 54600t^4 - 154560t^6 + 179520t^8 - 73216t^{10}.$$

(A.10)

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