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Research Article

Bounded Solutions to Nonlinear Parabolic Equations

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We deal with existence results for nonlinear parabolic equations with general quadratic gradient terms and with absorption term which depend on the solution. We note that no boundedness is assumed on the data of the problem. We prove an existence result of distributional solution via test-function method. A priori estimates and compactness arguments are our main ingredient; the method of sub-supersolution does not apply her.

1. Introduction

This work is devoted to deal with existence results concerning nonlinear parabolic equations with both first-order terms having quadratic growth with respect to the gradients and superlinear absorption terms which depend on the solution. Let us consider the following Cauchy-Dirichlet problem:

$$\begin{gathered} u_t + \Lambda u + A(x,u) = \beta(u)|\nabla u|^2 + f(x,t) \quad \text{in}\Omega \times]0,T[, \\ u = 0 \quad \text{on } \partial\Omega \times]0,T[, \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $\Lambda u = -\operatorname{div}(a(x,t,u,\nabla u))$ is a pseudomonotone, coercive, uniformly elliptic operator acting from $L^2(0,T;H_0^1(\Omega))$ to its dual. Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 1$ and $T > 0$. The operator $A(x,u)$ grows like $|u|^r$. The unknown real function u depends on $x \in \Omega$ and $t \in]0,T[$. Let $a : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function (i.e., $a(\ldots,\sigma,\zeta)$ is measurable in Q for every (x,t) in Q).

(A_1) There exists a constant $\beta > 0$ such that

$$|a(x,t,\sigma,\zeta)| \leq \beta[k(x,t) + |\sigma| + |\zeta|], \tag{1.2}$$

for almost every (x,t) in Q and every (σ,ζ) in $\mathbb{R} \times \mathbb{R}^N$, where $k(\cdot\cdot)$ is a nonnegative function in $L^2(Q)$.

(A_2) There exists a constant α such that

$$a(x,t,\sigma,\zeta) \geq \alpha|\zeta|^2, \tag{1.3}$$

for almost every (x,t) in Q and every (σ,ζ) in $\mathbb{R} \times \mathbb{R}^N$.

(A_3)

$$\left[a(x,t,\sigma,\zeta) - a\left(x,t,\sigma,\eta\right)\right] \cdot \left(\zeta - \eta\right) > 0, \tag{1.4}$$

for almost every (x,t) in Q, for every σ in $\mathbb{R}$, for every ζ,η in $\mathbb{R}^N$, $\zeta \neq \eta$.

Let us define the differential operator $A(x,u)$ as follows:

$$A(x,u) = a(x)u|u|^{r-1}. \tag{1.5}$$

The function f satisfies

$$f(x,t) \in L^r(0,T;L^q(\Omega)), \quad \text{with} \quad r > 1, \qquad \frac{N}{q} + \frac{2}{r} < 2. \tag{1.6}$$

The initial data satisfy the following hypothesis:

$$\psi(u_0) \in L^1(\Omega), \tag{1.7}$$

where

$$\psi(s) = \int_0^s e^{\gamma(r)}dr, \quad \text{with } \gamma(s) = \int_0^s \beta(\sigma)d\sigma. \tag{1.8}$$

The source term f satisfies the same condition considered by Aronson and Serrin in [1] to show the existence of a solution for the classical problem $(\partial u/\partial t) - \Delta u = f(x,t)$. The condition on the source term f is optimal. Indeed, if $f \in L^\infty(0,T;L^q(\Omega))$ or if $f(x,t) = f(x) \in L^q(\Omega)$, with $q > (N/2)$, the condition (1.6) is still satisfied.

Let us note that we studied the elliptic problem associated to (1.1) in [2]. This kind of problems has been extensively studied in the last years by many authors (see, e.g., [3–13] and the references therein). In these works, the hypothesis on the function β implies grosso modo that β is bounded, with some restrictions as in [4, 10, 14]. A special condition is assumed in [10], where β is supposed to tend to $+\infty$ for s tending to ∞.

There is many results for particular situations of our problem (see, e.g., [4, 7, 12, 13]). All this work have studied the question of existence of distributional solutions for this problem in the case where $a \equiv 0$, $u_0 \in L^\infty(\Omega)$, and $f(x,t) \in L^r(0,T;L^q(\Omega))$ with $(q(r-1)/r) > (N/2)$ The existence has also been studied in the case where $u_0 \in L^\infty(\Omega)$ and $f(x,t) = f(x) \in L^m(\Omega\times]0,T[)$, $m > 1 + (N/2)$, which are special cases of our conditions.

In the case where $a \equiv 0$, $\psi(u_0) \in L^2(\Omega)$ and for more general condition on β, existence results of a solution for parabolic convection diffusion problems

$$\frac{\partial u}{\partial t} - \Delta u = \beta(u)|\nabla u|^2 + f(x,t) \tag{1.9}$$

have been given in [15], [8].

We can reduce this problem with the change of u with the Col-Hopf change of variable

$$v = \psi(u) = \int_0^u \exp\left(\int_0^r \beta(s)ds\right)dr. \tag{1.10}$$

to the following

$$\frac{\partial v}{\partial t} - \Delta v = f(x,t)\left(1 + |v|\left|\log|v|\right|^\alpha\right), \tag{1.11}$$

In this stage, we have an existence result of distributional solutions via test function method. That gives the a priori estimates for the approximate problem associated with (1.11) which also provide a priori estimates for the approximate problem associated with (1.9) and, therefore, an existence result of distributional solution for problem (1.9).

One cannot perform such a change of variable, when trying to extend the previous results to our more general situation, where one has a general first-order term which grows quadratically with respect to the gradient and with superlinear reaction terms which grow like $|u|^r$. Therefore, we shall use some convenient test functions to prove the a priori estimates and use compactness arguments to prove an existence result of distributional solutions of (P). We point out that for this class of problems, the regularity assumed on the data f and u_0, can not expect bounded solutions.

We also point out that we are interested in solutions having finite norms in $L^2(0,T;H_0^1(\Omega))$. The techniques used in this paper are mainly based on a linear operator and on the concept of distributional solutions. These approachs allow to have, in the case of both subcritical growth and a reaction terms with u, existence results. The first ingredient of our proof consists in obtaining certain a priori bounds on the solutions of approximate problems and some suitable L^1-norm of diffusion terms. A convenient use of Young's inequality will give a uniform estimate of the $L^2(0,T;H_0^1(\Omega))$-norms and, therefore, the weak convergence up to a subsequence. We will prove that there exists u such that, up to a subsequence the solution u_n of the approximate problems converges to u, all everywhere convergence of gradient of u_n to gradient of u, up to a subsequence, which is important in the study of the limiting process. Next, we will prove the convergence of the superlinear reaction term and the quadratic gradient term in $L^1(\Omega\times]0,T[)$. Another interesting approach is in some sense the combination of the previous, in studies of the behavior of sequences of approximating solutions. Likewise, we will see that the solutions of the approximates problems converge to the solution of the model problem in $C([0,T];L^1(\Omega))$, which gives meaning to the initial condition.

2. Basic Results

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$. We denote by Q_T for $T > 0$ the set $\Omega\times]0,T[$ and by Γ the set $\partial\Omega\times]0,T[$.

We consider the following nonlinear problem that we denote by (P)

$$\begin{gathered} u_t + \Lambda u + A(x,u) = B(u,\nabla u) + f(x,t) \quad \text{in } \Omega\times]0,T[, \\ u = 0 \quad \text{on } \partial\Omega\times]0,T[, \\ u(x,0) = u_0(x) \quad \text{in}\Omega, \end{gathered} \tag{2.1}$$

where the unknown function $u = u(x,t)$ is a real function depending on $x \in \Omega$ and $t \in]0,T[$. Λ, A and B are differential operators such that

$$\begin{gathered} \Lambda u := -\operatorname{div}(a(x,t,u,\nabla u)), \\ B(u,\nabla u) = \beta(u)|\nabla u|^2. \end{gathered} \tag{2.2}$$

Let us consider the following assumptions.

(H_1) The real function β is such that

$$\beta \text{ is continuous nonincreasing,} \quad \text{with } \beta(0) = 0. \tag{2.3}$$

(H_2) The real functions u_0 and f are satisfying (1.7) and (1.6), respectively.

The operator $A(x,u)$ is such that

$$A(x,u) = a(x)u|u|^{r-1}, \tag{2.4}$$

where

(H_3) $r > 1$ and $a(\cdots)$ is a real function such that

$$a \in L^\infty(\Omega), \quad a(x) \geq a_0 > 0 \quad a.e. \;\; x \in \Omega. \tag{2.5}$$

By a weak solution of the problem (1.1), we mean a function $u \in L^2(0,T;H_0^1(\Omega))$ such that

$$a(x)u|u| \in L^r(Q_T), \quad \beta(u)|\nabla u|^2 \in L^1(Q_T) \tag{2.6}$$

and satisfying

$$\int_{Q_T} u_t\phi + \int_{Q_T} a(x,t,u,\nabla u)\nabla\phi + \int_{Q_T} a(x)u|u|^{r-1}\phi = \int_{Q_T} \beta(u)|\nabla u|^2\phi + \int_{Q_T} f\phi, \tag{2.7}$$

for any test function ϕ in $C_c^1(Q_T)$ (the C^1 functions with compact support).

In the sequel we denote by θ a truncation function satisfying $\theta \in C^\infty(\mathbb{R})$, $0 \leq \theta \leq 1$, $\theta(\eta) = 1$ for $|\eta| \leq \rho$, $\theta(\eta) = 0$ for $|\eta| \geqslant 2\rho$ and $|\nabla\theta| \leq (2/\rho)$, where ρ is a positive real. By c we denote different constants in $\mathbb{R}$ which may vary from line to line. The main result in this paper is the following.

Theorem 2.1. *If the hypotheses (A_1)–(A_3) and (H_1)–(H_3) are satisfied, then the problem* (1.1) *admits at least one solution u, such that*

$$u \in L^2\left(0,T;H_0^1(\Omega)\right) \cap C^0\left([0,T];L^2(\Omega)\right),$$
$$a(x)u|u| \in L^r(Q_T), \quad \beta(u)|\nabla u|^2 \in L^1(Q_T). \tag{2.8}$$

To prove the main result, we approximate our problem by a sequence of regular problems and show a priori estimates of solutions. Next, we shall prove the convergence of approximating solutions to some function that solves our problem.

Let us recall the classical inequality of Poincaré and Sobolev (see [16, Chapters 7.7 and 7.8]).

Lemma 2.2. *Let Ω be an open subset of $\mathbb{R}^N$ with finite Lebesgue measure. Then, for every p such that $1 \leq p < \infty$, one has the following inequality:*

$$\|u\|_{L^p(\Omega)} \leq \left(\frac{|\Omega|}{\omega_N}\right)^{1/N} \|\nabla u\|_{L^p(\Omega)}, \quad \textit{for every } u \in W_0^{1,P}(\Omega), \tag{2.9}$$

where ω_N is the measure of the unit ball in $\mathbb{R}^N$. Furthermore, there exists a constant $c = c(N,p)$ such that, for all $u \in W_0^{1,P}(\Omega)$,

$$\|u\|_{L^{p^*}(\Omega)} \leq c\|\nabla u\|_{L^p(\Omega)}, \quad \textit{for any } p < N,$$
$$\|u\|_{L^\infty(\Omega)} \leq |\Omega|^{(1/(N-1))/P}\|\nabla u\|_{L^p(\Omega)}, \quad \textit{for } p > N, \tag{2.10}$$

where $p^* = Np/(N-p)$.

Remark 2.3. By an approximation argument, the same inequality holds true if we replace $|\Omega|$ by $|\{x \in \Omega : u(x) \neq 0\}|$.

Let us recall next the Gagliardo-Nirenberg's inequality for evolution spaces.

Lemma 2.4 (see, e.g., [17]). *Let Ω be a bounded open set of $\mathbb{R}^N$ and T a real positive number. Let $v(x,t)$ be a function such that $v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. Then*

$$v \in L^\rho(0,T;L^\sigma(\Omega)), \quad \textit{where, } 2 \leq \sigma \leq \frac{2N}{N-2},\ 2 \leq \rho \leq \infty,\ \frac{N}{\sigma} + \frac{2}{\rho} = \frac{N}{2}, \tag{2.11}$$

and the following estimate holds

$$\int_0^T \|v(t)\|_{L^\sigma(\Omega)}^\rho dt \leq c(N)\|v\|_{L^\infty(0,T;L^2(\Omega))}^{\rho-2} \int_0^T \|\nabla v(t)\|_{L^2(\Omega,\mathbb{R}^N)}^2 dt. \tag{2.12}$$

We are interested in studding a sequence of regular problems approximating the model problem. We prove the existence of bounded solutions for the approximating problems, and this bound does not depend on *n*. We shall prove some a priori estimates on the solutions of this sequence of problems which serves in the limiting process.

3. Approximating Problems

We regularize the problem (1.1) by considering the following sequence of problems:

$$\begin{gathered} \frac{\partial u_n}{\partial t} - \Lambda u_n + A_n(x,u_n) = B_n(u_n,\nabla u_n) + f_n(x,t) \quad \text{in } Q_n, \\ u_n(x,t) = 0 \quad \text{on } \Gamma_n, \\ u_n(x,0) = u_{0n}(x) \quad \text{in } \Omega_n, \end{gathered} \tag{3.1}$$

where

$$\begin{gathered} B_n(u_n,\nabla u_n) = \beta_n(u_n)|\nabla u_n|^2, \\ \beta_n(\eta) = \begin{cases} n & \text{if } \beta(\eta) > n, \\ \beta(\eta) & \text{if } -n \le \beta(\eta) \le n, \\ -n & \text{if } \beta(\eta) < -n. \end{cases} \end{gathered} \tag{3.2}$$

Let us consider

$$A_n(x,u_n) = a(x)u_n|u_n|^{r-1}. \tag{3.3}$$

Next, we consider the truncated function

$$f_n = \inf(|f|,n)\operatorname{sign}(f). \tag{3.4}$$

We denote by Ω_n a strictly increasing sequence of bounded sets $\Omega \cap B_n$ invading Ω. Next we denote

$$\begin{gathered} Q_n = \Omega_n \times]0,T[, \\ \Gamma_n = \partial\Omega_n \times]0,T[. \end{gathered} \tag{3.5}$$

From standard results (see, e.g., [14]), the following problem:

$$\begin{gathered} \frac{\partial u_n}{\partial t} - \Lambda u_n + \Upsilon_n(x,u_n,\nabla u_n) = f_n(x,t) \quad \text{in} Q_n, \\ u_n(x,t) = 0 \quad \text{on } \Gamma_n, \\ u_n(x,0) = u_{0n}(x) \quad \text{in } \Omega_n, \end{gathered} \tag{3.6}$$

where

$$\Upsilon_n(x,u_n,\nabla u_n) = A_n(x,u_n) - B_n(u_n,\nabla u_n), \tag{3.7}$$

admits at least one solution u_n satisfying

$$u_n \in L^2\left(0,T;H_0^1(\Omega_n)\right) \cap C\left(0,T;L^2(\Omega_n)\right),$$
$$u_{n_t} \in L^2\left(0,T;H^{-1}(\Omega_n)\right), \quad |u_n|^r \in L^1(Q_n). \tag{3.8}$$

Then one has the following estimates:

$$\int_0^T \int_\Omega |\nabla u_n|^2 + \int_0^T \int_\Omega a(x) u_n |u_n|^{r-1} \leq c(\rho,\lambda,q,T). \tag{3.9}$$

Indeed, let us define the following function:

$$\phi(s) = \int_0^s H(\sigma)d\sigma, \quad \text{for } s > 0,$$
$$\phi(s) = -\phi(-s), \quad \text{for } s < 0, \tag{3.10}$$

wehere

$$H(s) = \frac{1}{(1+s)^{q+1}}, \quad 0 < q < 1. \tag{3.11}$$

We introduce the function ψ defined by

$$\psi(s) = \int_0^s \phi(\sigma)d\sigma. \tag{3.12}$$

Let us consider the following sequences:

$$I_{1,n} = \int_{\Omega_n} \psi(u_n(x,T))\theta^{\lambda+1},$$
$$I_{2,n} = \alpha \int_{Q_n} |\nabla u_n|^2 \phi'(u_n)\theta^{\lambda+1},$$
$$I_{3,n} = \beta(\lambda+1) \int_{Q_n} |\nabla u_n| \left|\nabla\theta\phi(u_n)\right| \theta^{\lambda},$$
$$I_{4,n} = \int_{Q_n} a(x) u_n |u_n|^{r-1} \xi. \tag{3.13}$$

Taking $\xi = \phi\theta^{\lambda+1}$ with $\lambda > 0$, we obtain

$$I_{1,n} + I_{2,n} + I_{3,n} + I_{4,n} \leq \int_{B_{2\rho}} f(x,t)\xi + \int_{B_{2\rho}} \psi(u_0(x))\theta^{\lambda+1}. \tag{3.14}$$

Applying Young's inequality, one has the following inequality:

$$|\nabla u|\theta^{\lambda} \leq \frac{\epsilon}{2}\frac{|\nabla u|^2\theta^{\lambda+1}}{(1+|u|)^q(\lambda+1)} + \frac{c(\lambda)}{2\epsilon}\theta^{\lambda-1}(1+|u|)^{q+1}. \tag{3.15}$$

We now choose q such that $q < r$, which is possible, since $r > 1$ and $q > 0$. We use again Young's inequality twice and we obtain

$$\begin{aligned} |\nabla u|\theta^{\lambda} &\leq \epsilon\phi'(u)\frac{|\nabla u|^2\theta^{\lambda+1}}{2(\lambda+1)} + \frac{\epsilon\ a_0}{\lambda+1}|u|^{r-1}u\phi(u)\theta^{\lambda+1} + c(\rho,\lambda,q,a_0) \\ &\leq \frac{1}{2}\frac{|\nabla u|^2\theta^{\lambda+1}}{(1+|u|)^q} + \frac{a(x)}{2}|u|^{r-1}u\phi(u)\theta^{\lambda+1} + c(\rho,\lambda,q,a_0). \end{aligned} \tag{3.16}$$

Then

$$\frac{1}{\beta}I_{3,n} \leq \frac{1}{2}\int_{Q_n}|\nabla u_n|^2\phi'(u_n)\theta^{\lambda+1} + \int_{Q_n}a(x)u_n|u_n|^{r-1}\phi(u)\theta^{\lambda+1} + c(\rho,\lambda,q). \tag{3.17}$$

From (3.14), we obtain

$$\int_{\Omega_n}\psi(u_n(x,T))\theta^{\lambda+1} + \frac{1}{2}\int_{Q_n}|\nabla u_n|^2\phi'(u_n)\theta^{\lambda+1} + \frac{1}{2}\int_{Q_n}a(x)u_n|u_n|^{r-1}\phi(u)\theta^{\lambda+1} \leq c(\rho,\lambda,q,T). \tag{3.18}$$

In consequence,

$$\int_0^T\int_\Omega|\nabla u_n|^2 + \int_0^T\int_\Omega|u_n|^r \leq c(\rho,\lambda,q,T). \tag{3.19}$$

Next, we substitute T by t for any t, $0 \leq t \leq T$ in (3.18), which is possible. We obtain

$$\int_{B_n}|u_n(x,t)|^r \leq c, \tag{3.20}$$

where c is a constant that does not depend on n. then, the approximate problem admits at least one solution which is bounded independently on n in $L^\infty(0,T;L^r(\Omega))$.

Let us now prove that the sequence $\beta_n(u_n)|\nabla u_n|^2$ is bounded in $L^1(Q_T)$. We denote

$$\psi(r) = \int_0^r e^{|\gamma(\sigma)|}d\sigma, \tag{3.21}$$

where

$$\gamma(r) = \int_0^r \beta(\sigma)d\sigma. \tag{3.22}$$

We define, for any $k > 0$ fixed, the functions $h_k(r)$ defined as follows:

$$h_k(r) = \chi_{[|r|>k]}(s) \int_k^r \beta(\sigma) e^{(|\gamma(\sigma)|-\gamma(k))} d\sigma,$$
$$\phi_k(r) = \int_0^r h_k(\sigma) d\sigma. \tag{3.23}$$

Let us consider the following sequences:

$$T_{1,n} = \int_{\Omega\cap[|u_n(T)|>k]} \phi_k(u_n(T,x))dx,$$
$$T_{2,n} = \alpha \int_{[|u_n|>k]} |\nabla u_n|^2 \beta_n(u_n) e^{(|\gamma(u_n)|-\gamma(k))},$$
$$T_{3,n} = \int_{\Omega\cap[|u_{0n}|>k]} \phi_k(u_{0n}(x))dx, \tag{3.24}$$
$$T_{4,n} = \int_{[|u_n|>k]} |\nabla u_n|^2 \beta_n(u_n)\left(e^{(|\gamma(u_n)|-\gamma(k))} - 1\right).$$

We can choose $h_k(u_n)$ as test function, and we obtain

$$T_{1,n} + T_{2,n} - T_{3,n} \le \int_{[|u_n|>k]} f(x,t)\operatorname{sign}(u_n)\left(e^{(|\gamma(u_n)|-\gamma(k))} - 1\right) + T_{4,n}. \tag{3.25}$$

Then, we deduce

$$T_{2,n} \le \int_{[|u_n|>k]} f(x,t)\operatorname{sign}(u_n)\left(e^{(|\gamma(u_n)|-\gamma(k))} - 1\right) + \int_{\Omega\cap[|u_{0n}|>k]} \phi_k(u_{0n}(x))dx$$
$$\le \int_{[|u_n|>k]} f(x,t) e^{|\gamma(u_n)|} + \int_{\Omega\cap[|u_{0n}|>k]} \phi_k(u_{0n}(x))dx. \tag{3.26}$$

In consequence,

$$\alpha \int_{[|u_n|>k]} |\nabla u_n|^2 \beta_n(u_n) \le c \left\| f\chi_{[|u_n|>k]}\right\|_{r,q} \left\| e^{|\gamma(u_n)|}\right\|_{r',q'} + \int_{\Omega\cap[|u_{0n}|>k]} \psi(u_{0n}). \tag{3.27}$$

Finally, we obtain

$$\left\| e^{|\gamma(u_n)|}\right\|_{r',q'} \le c \left\| 1 + |\psi(u_n)|^2 \right\|_{r',q'} \le c\left(1 + \|\psi(u_n)\|^2_{2r',2q'}\right). \tag{3.28}$$

Then, the sequence $\beta_n(u_n)|\nabla u_n|^2$ is bounded in $L^1(Q_T)$.

We require the all everywhere convergence of gradient u_n to the gradient of u. Let us consider

$$\psi(s) = \begin{cases} \inf(s,\epsilon) & \text{if } s \geq 0, \\ -\inf\ (s,\epsilon) & \text{if } s \leq 0. \end{cases} \tag{3.29}$$

Substituting u in the approximating problem successively with u_n and u_m, we consider the following function:

$$\phi = \psi(u_n - u_m)\theta. \tag{3.30}$$

After substraction, for $n, m \geq 4\rho$, we obtain the following inequality:

$$\begin{aligned} &\int_{Q_T} \frac{\partial(u_n - u_m)}{\partial t}\phi + \alpha\int_{Q_T} |\nabla u_n - \nabla u_m|^2\phi + \int_{Q_T} a(x)\left(u_n|u_n|^{r-1} - u_m|u_m|^{r-1}\right)\phi \\ &\qquad \leq \int_{Q_T}\left(\beta_n(u_n)|\nabla u_n|^2 - \beta_m(u_m)|\nabla u_m|^2\right)\phi + \int_{Q_T}(f_n - f_m)\phi. \end{aligned} \tag{3.31}$$

Let us now consider for $n,\ m$ sufficiently large the following sequence:

$$F_n = f_n - a(x)u_n|u_n|^{r-1} + \beta_n(u_n)|\nabla u_n|^2. \tag{3.32}$$

Then we get

$$\int_0^T\int_{[|u_n - u_m|\leq\epsilon]\cap B_\rho} |\nabla u_n - \nabla u_m|^2 \leq c\epsilon\int_0^T\int_{B_{2\rho}}(|F_n| + |F_m|)dx. \tag{3.33}$$

Since (3.20) holds, then $a(x)u_n|u_n|^{r-1}$ is in $L^1(Q_T)$. So that, F_n is bounded in $L^1(Q_T)$, then one has

$$\int_0^T\int_{[|u_n - u_m|\leq\epsilon]\cap B_\rho} |\nabla u_n - \nabla u_m|^2 \leq c\epsilon. \tag{3.34}$$

Using Holder's inequality, we obtain

$$\begin{aligned} \int_0^T\int_{B_\rho}|\nabla u_n - \nabla u_m| \leq &\left(\int_0^T\int_{[|u_n - u_m|\leq\epsilon]\cap B_\rho} |\nabla u_n - \nabla u_m|^2\right)^{1/2} c \\ &+ \int_0^T\int_{[|u_n - u_m|\geq\epsilon]\cap B_\rho} |\nabla u_n - \nabla u_m|. \end{aligned} \tag{3.35}$$

Using (3.34), one has

$$\int_0^T \int_{[|u_n - u_m| \le \epsilon] \cap B_\rho} |\nabla u_n - \nabla u_m|^2 \longrightarrow 0 \quad \text{as } n, m \longrightarrow +\infty. \tag{3.36}$$

On the other hand, since the measure of $[|u_n - u_m| \ge \epsilon] \cap B_\rho$ converges to 0 as n, then, as m converges to $+\infty$, then

$$\int_0^T \int_{[|u_n - u_m| \ge \epsilon] \cap B_\rho} |\nabla u_n - \nabla u_m| \to 0. \tag{3.37}$$

Therefore

$$\int_{Q_T} |\nabla u_n - \nabla u_m| \longrightarrow 0 \quad \text{as } n, m \longrightarrow +\infty. \tag{3.38}$$

4. Limiting Process

We denote by u_n the solution of the approximate problems (P_n) on Ω_n with initial condition U_{0n}. To prove the main result, we deal with the limiting process of the approximating problems. First of all, we will prove that there exists u such that, up to a subsequence, (u_n) converges to u, for almost every $(x,t) \in Q_T$. First, we will prove the all everywhere convergence of the gradients of u_n to the gradient of u, up to a subsequence, in Q_T. Next, we will prove the convergence of the superlinear reaction term and the quadratic gradient term in $L^1(Q_T)$. Finally, we will see that (u_n) converges to u in $C([0,T]; L^1(\Omega))$, which gives meaning to the initial condition.

From (3.38) and up to a subsequence (u_n), we have

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } Q_T. \tag{4.1}$$

By consequence, since ∇u_n is bounded in $L^1(Q_T)$, Vitali's theorem implies that

$$\nabla u_n \longrightarrow \nabla u \quad \text{in } L^1(Q_T). \tag{4.2}$$

Since one has

$$u_n \in L^2\left(0, T; H_0^1(B_n)\right) \cap C\left([0,T]; L^1(B_n)\right), \tag{4.3}$$

By a diagonal process, we may select a subsequence, also denoted by $\{u_n\}$, such that

$$u_n \longrightarrow u \quad \text{weakly in } L^2\left(0, T; H_0^1(\Omega)\right), \tag{4.4}$$

and also

$$u_n \longrightarrow u \quad \text{a.e. in } \Omega. \tag{4.5}$$

From the construction of u_{0n} and f_n, we have

$$\begin{aligned} u_{0n} &\longrightarrow u_0 \quad \text{in } L^1(\Omega), \\ f_n &\longrightarrow f \quad \text{in } L^r(0,T;L^q(\Omega)). \end{aligned} \tag{4.6}$$

Since $u_{n_t} \in L^2(0,T;H_0^{-1}(\Omega_n)) + L^1(Q_n)$, Then using compactness arguments (see [18]), we have

$$u_n \longrightarrow u \quad \text{strongly in } L^1(Q_T). \tag{4.7}$$

From (4.5) and the fact that $a(x)u_n$ is bounded in $L^r(Q_T)$, the equi-integrability of $a(x)u_n|u_n|^{r-1}$ is derived and then from Vitali's theorem, we have

$$a(x)u_n|u_n|^{r-1} \longrightarrow a(x)u|u|^{r-1} \quad \text{in } L^1(Q_T). \tag{4.8}$$

By consequence,

$$A_n(x,u_n) \longrightarrow A(x,u) \quad \text{in } L^1(Q_T). \tag{4.9}$$

Since

$$\begin{aligned} u_n &\longrightarrow u \quad \text{a.e. in } Q_T, \\ \nabla u_n &\longrightarrow \nabla u \quad \text{a.e. in } Q_T, \end{aligned} \tag{4.10}$$

then

$$\begin{aligned} \beta_n(u_n)|\nabla u_n|^2 &\longrightarrow \beta(u)|\nabla u|^2 \quad \text{a.e. in } Q_T, \\ B_n(u_n,\nabla u_n) &\longrightarrow B(u,\nabla u) \quad \text{a.e. in } Q_T. \end{aligned} \tag{4.11}$$

Therefore, from (4.1) and Vitali's theorem, we conclude that

$$\begin{aligned} \beta_n(u_n)|\nabla u_n|^2 &\longrightarrow \beta(u)|\nabla u|^2 \quad \text{in } L^1(Q_T), \\ B_n(u_n,\nabla u_n) &\longrightarrow B(u,\nabla u) \quad \text{in } L^1(Q_T). \end{aligned} \tag{4.12}$$

Finally, the sequence $(u_n)_n$ belongs to $C^0(0,T;H^{-1}(\Omega))$ and $u_n(x,0) = u_{0n}(x)$; this implies that the initial condition $u(x,0) = u_0(x)$ is satisfied.

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