Research Article

Convergence Theorems for Finite Family of Multivalued Maps in Uniformly Convex Banach Spaces

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We introduce a new iterative process to approximate a common fixed point of a finite family of multivalued maps in a uniformly convex real Banach space and establish strong convergence theorems for the proposed process. Furthermore, strong convergence theorems for finite family of quasi-nonexpansive multivalued maps are obtained. Our results extend important recent results.

1. Introduction

Let D be a nonempty, closed, and convex subset of a real Hilbert space H . The set D is called *proximal* if for each $x \in H$, there exists $y \in D$ such that $\|x - y\| = d(x, D)$, where $d(x, D) = \inf\{\|x - z\| : z \in D\}$. Let $CB(D)$, $K(D)$, and $P(D)$ denote the families of nonempty, closed and bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of D , respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}, \quad (1.1)$$

for $A, B \in CB(D)$. A single-valued map $T : D \rightarrow D$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. A multivalued map $T : D \rightarrow CB(D)$ is said to be *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of $T : D \rightarrow D$ (resp., $T : D \rightarrow CB(D)$) if $p = Tp$ (resp., $p \in Tp$). The set of fixed points of T is denoted by $F(T)$. A multivalued map $T : D \rightarrow CB(D)$ is said to be *quasi-nonexpansive* if $H(Tx, Tp) \leq \|x - p\|$ for all $x \in D$ and for all $p \in F(T)$.

A $T : D \rightarrow CB(D)$ is said to satisfy *Condition (I)* if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that

$$d(x, Tx) \geq f(d(x, F(T))), \quad (1.2)$$

for all $x \in D$.

The fixed point theory of multivalued nonexpansive mappings is much more complicated and difficult than the corresponding theory of single-valued nonexpansive mappings. However, some classical fixed point theorems for single-valued nonexpansive mappings have already been extended to multivalued mappings. The first results in this direction were established by Markin [1] in Hilbert spaces and by Browder [2] for spaces having weakly continuous duality mapping. Dozo [3] generalized these results to a Banach space satisfying Opial's condition.

In 1974, by using Edelstein's method of asymptotic centers, Lim [4] obtained a fixed point theorem for a multivalued nonexpansive self-mapping in a uniformly convex Banach space.

Theorem 1.1 (Lim [4]). *Let D be a nonempty, closed convex, and bounded subset of a uniformly convex Banach space E and $T : D \rightarrow C(E)$ a multivalued nonexpansive mapping. Then, T has a fixed point.*

In 1990, Kirk and Massa [5] gave an extension of Lim's theorem proving the existence of a fixed point in a Banach space for which the asymptotic center of a bounded sequence in a closed bounded convex subset is nonempty and compact.

Theorem 1.2 (Kirk and Massa [5]). *Let D be a nonempty, closed convex, and bounded subset of a Banach space E and $T : D \rightarrow CB(E)$ a multivalued nonexpansive mapping. Suppose that the asymptotic center in E of each bounded sequence of E is nonempty and compact. Then, T has a fixed point.*

Banach contraction mapping principle was extended nicely multivalued mappings by Nadler [6] in 1969. (Below is stated in a Banach space setting).

Theorem 1.3 (Nadler [6]). *Let D be a nonempty closed subset of a Banach space E and $T : D \rightarrow CB(D)$ a multivalued contraction. Then, T has a fixed point.*

In 1953, Mann [7] introduced the following iterative scheme to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H :

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \quad (1.3)$$

where the initial point x_0 is taken arbitrarily in D and $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$. However, we note that Mann's iteration has only weak convergence; see, for example, [8].

In 2005, Sastry and Babu [9] proved that the Mann and Ishikawa iteration schemes for a multivalued map T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point p may be different from q .

In 2007, Panyanak [10] extended the results of Sastry and Babu to uniformly convex Banach spaces and proved the following theorems.

Theorem 1.4 (Panyanak [10]). *Let E be a uniformly convex Banach space, D a nonempty closed bounded convex subset of E , and $T : D \rightarrow P(D)$ a multivalued nonexpansive mapping that satisfies condition (I). Assume that (i) $0 \leq \alpha_n < 1$ and (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $F(T)$ a nonempty proximinal subset of D . Then, the Mann iterates $\{x_n\}$ defined by $x_0 \in D$,*

$$x_{n+1} = \alpha_n y_n + (1 - \alpha_n)x_n, \quad \alpha_n \in [a, b], \quad 0 < a < b < 1, \quad n \geq 0, \quad (1.4)$$

where $y_n \in Tx_n$ such that $\|y_n - u_n\| = d(u_n, Tx_n)$ and $u_n \in F(T)$ such that $\|x_n - u_n\| = d(x_n, F(T))$, converges strongly to a fixed point of T .

Theorem 1.5 (Panyanak [10]). *Let E be a uniformly convex Banach space, D a nonempty compact convex subset of E , and $T : D \rightarrow P(D)$ a multivalued nonexpansive mapping with a fixed point p . Assume that (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \rightarrow 0$ and (iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$. Then, the Ishikawa iterates $\{x_n\}$ defined by $x_0 \in D$,*

$$y_n = \beta_n z_n + (1 - \beta_n)x_n, \quad \beta_n \in [0, 1], \quad n \geq 0, \quad (1.5)$$

$z_n \in Tx_n$ such that $\|z_n - p\| = d(p, Tx_n)$, and

$$x_{n+1} = \alpha_n z'_n + (1 - \alpha_n)x_n, \quad \alpha_n \in [0, 1], \quad n \neq 0, \quad (1.6)$$

$z'_n \in Ty_n$ such that $\|z'_n - p\| = d(p, Ty_n)$ converges strongly to a fixed point of T .

Later, Song and Wang [11] noted there was a gap in the proofs of Theorem 1.5 above and of [9, Theorem 5]. They further solved/revised the gap and also gave the affirmative answer Panyanak [10] question using the Ishikawa iterative scheme. In the main results, the domain of T is still compact, which is a strong condition (see [11, Theorem 1]) and T satisfies condition (I) (see [11, Theorem 1]).

Recently, Shahzad and Zegeye [12] proved the following theorems for quasi-nonexpansive multivalued map and multivalued map in uniformly convex Banach space.

Theorem 1.6 (Shahzad and Zegeye [12]). *Let E be a uniformly convex real Banach space and D a nonempty, closed and convex subset of E . Let $T : D \rightarrow CB(D)$ be a quasi-nonexpansive multivalued map with $F(T) \neq \emptyset$ for which $Tp = \{p\}$, for all $p \in F(T)$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by $x_0 \in D$,*

$$y_n = \beta_n z_n + (1 - \beta_n)x_n, \quad \beta_n \in [0, 1], \quad n \geq 0, \quad (1.7)$$

$z_n \in Tx_n$, and

$$x_{n+1} = \alpha_n z'_n + (1 - \alpha_n)x_n, \quad \alpha_n \in [0, 1], \quad n \neq 0, \quad (1.8)$$

$z'_n \in Ty_n$. Assume that T satisfies condition (I) and $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$. Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T .

Theorem 1.7 (Shahzad and Zegeye [12]). *Let E be a uniformly convex real Banach space and D a nonempty, closed, and convex subset of E . Let $T : D \rightarrow P(D)$ be a multivalued map with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by $x_0 \in D$,*

$$y_n = \beta_n z_n + (1 - \beta_n)x_n, \quad \beta_n \in [0, 1], \quad n \geq 0, \quad (1.9)$$

$z_n \in P_T(x_n)$, and

$$x_{n+1} = \alpha_n z'_n + (1 - \alpha_n)x_n, \quad \alpha_n \in [0, 1], \quad n \neq 0, \quad (1.10)$$

$z'_n \in P_T(y_n)$. Assume that T satisfies condition (I) and $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$. Then, $\{x_n\}_{n=1}^\infty$ converges strongly to a fixed point of T .

More recently, Abbas et al. [13] introduced the following one-step iterative process to compute common fixed points of two multivalued nonexpansive mappings.

$$\begin{aligned} x_1 &\in D, \\ x_{n+1} &= a_n x_n + b_n y_n + c_n z_n, \quad \forall n \geq 1. \end{aligned} \quad (1.11)$$

Using (1.11), Abbas et al. [13] proved weak and strong convergence theorems for approximation of common fixed point of two multivalued nonexpansive mappings in Banach spaces.

Motivated by the ongoing research and the above mentioned results, we introduce a new iterative scheme for approximation of common fixed points of finite family of multivalued maps in a real Banach space. Furthermore, we prove strong convergence theorems for approximation of common fixed points of finite family of multivalued maps in a uniformly convex real Banach space. Next, we prove a necessary and sufficient condition for strong convergence of our new iterative process to a common fixed point of finite family of multivalued maps. Finally, we introduce a new iterative scheme and prove strong convergence theorems for finite family of quasi-nonexpansive multivalued maps in a uniformly convex real Banach space. Our results extend the results of Sastry and Babu [9], Panyanak [10], Shahzad and Zegeye [12], and Song and Wang [11].

2. Preliminaries

Let E be Banach space and $\dim E \geq 2$. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \rightarrow [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\| = \|y\| = 1; \epsilon = \|x - y\| \right\}. \quad (2.1)$$

E is *uniformly convex* if for any $\epsilon \in (0, 2]$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, then $\|(1/2)(x + y)\| \leq 1 - \delta$. Equivalently, E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

A family $\{T_i : D \rightarrow CB(D), i = 1, 2, \dots, m\}$ is said to satisfy *Condition (II)* if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, \infty)$ such that

$$d(x, T_i x) \geq f(d(x, \cap_{i=1}^m F(T_i))), \quad (2.2)$$

for all $i = 1, 2, \dots, m$ and $x \in D$.

The mapping $T : D \rightarrow CB(D)$ is called *hemicompact* if for any sequence $\{x_n\}$ in D such that $d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p \in D$. We note that if D is compact, then every multivalued mapping $T : D \rightarrow CB(D)$ is hemicompact.

Let D be a nonempty, closed, and convex subset of a real Banach space E . Let $T : D \rightarrow P(D)$ be a multimap and

$$P_T(x) := \{u_x \in Tx : \|x - u_x\| = d(x, Tx)\}. \quad (2.3)$$

Then, $P_T(x) : D \rightarrow P(D)$ is nonempty and compact for every $x \in D$. Furthermore, we observe that $P_T(y) = \{y\}$ if y is a fixed point of T .

A mapping $T : D \rightarrow P(D)$ is **-nonexpansive* ([14]) if for all $x, y \in D$ and $u_x \in Tx$ with $d(x, u_x) = \inf\{d(x, z) : z \in Tx\}$, there exists $u_y \in Ty$ with $d(y, u_y) = \inf\{d(y, w) : w \in Ty\}$ such that

$$d(u_x, u_y) \leq d(x, y). \quad (2.4)$$

It is known that *-nonexpansiveness is different from nonexpansiveness for multimaps. There are some *-nonexpansive multimaps which are not nonexpansive and some nonexpansive multimaps which are not *-nonexpansive ([15, 16]).

By the definition of Hausdorff metric, we obtain that if a multimap $T : D \rightarrow P(D)$ is *-nonexpansive, then P_T is nonexpansive.

Throughout this paper, we write $x_n \rightarrow x$ to indicate that the sequence $\{x_n\}$ converges strongly to x .

Also, this following lemma will be used in the sequel.

Lemma 2.1 (Schu [17]). *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n . Also, suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq r, \limsup_{n \rightarrow \infty} \|y_n\| \leq r$ and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r > 0$. Then, $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

3. Main Results

We now introduce the following iteration scheme. Let E be a real normed space and D a nonempty subset of E . Let $T_1, T_2, \dots, T_m$ be multivalued maps of D into $P(D)$ with $F := \cap_{i=1}^m F(T_i) \neq \emptyset$ such that $P_{T_1}, P_{T_2}, \dots, P_{T_m}$ are nonexpansive and $\{\alpha_{ni}\}_{n=1}^\infty, i = 0, 1, \dots, m$

a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ such that $\sum_{i=0}^m \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by

$$\begin{aligned} x_1 &\in D, \\ x_{n+1} &= \alpha_{n0}x_n + \alpha_{n1}y_n^{(1)} + \cdots + \alpha_{nm}y_n^{(m)}, \end{aligned} \quad (3.1)$$

where $y_n^{(i)} \in P_{T_i}x_n$, $i = 1, 2, \dots, m$.

Lemma 3.1. *Let E be a real normed space and D a nonempty subset of E . Let $T_1, T_2, \dots, T_m$ be multivalued maps of D into $P(D)$ with $F := \cap_{i=1}^m F(T_i) \neq \emptyset$ such that $P_{T_1}, P_{T_2}, \dots, P_{T_m}$ are nonexpansive. Let $\{\alpha_{ni}\}_{n=1}^\infty$, $i = 0, 1, \dots, m$ a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ such that $\sum_{i=0}^m \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by (3.1). Then,*

$$\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0, \quad \forall i = 1, 2, \dots, m. \quad (3.2)$$

Proof. Let $x^* \in \cap_{i=1}^m F(T_i)$. Then, from (3.1), we have the following estimates:

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \alpha_{n0}\|x_n - x^*\| + \alpha_{n1}\|y_n^{(1)} - x^*\| + \cdots + \alpha_{nm}\|y_n^{(m)} - x^*\| \\ &\leq \alpha_{n0}\|x_n - x^*\| + \alpha_{n1}d(y_n^{(1)}, P_{T_1}x^*) + \cdots + \alpha_{nm}d(y_n^{(m)}, P_{T_m}x^*) \\ &\leq \alpha_{n0}\|x_n - x^*\| + \alpha_{n1}H(P_{T_1}x_n, P_{T_1}x^*) + \cdots + \alpha_{nm}H(P_{T_m}x_n, P_{T_m}x^*) \\ &\leq \alpha_{n0}\|x_n - x^*\| + \alpha_{n1}\|x_n - x^*\| + \cdots + \alpha_{nm}\|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned} \quad (3.3)$$

Thus, $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists. Let

$$\lim_{n \rightarrow \infty} \|x_n - x^*\| = c, \quad (3.4)$$

for some $c \geq 0$. Then,

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} \|x_{n+1} - x^*\| \\ &= \lim_{n \rightarrow \infty} \left\| \alpha_{n0}(x_n - x^*) + \alpha_{n1}(y_n^{(1)} - x^*) + \cdots + \alpha_{nm}(y_n^{(m)} - x^*) \right\| \\ &= \lim_{n \rightarrow \infty} \left\| (1 - \alpha_{n0}) \left[\frac{\alpha_{n1}}{1 - \alpha_{n0}}(y_n^{(1)} - x^*) + \cdots + \frac{\alpha_{nm}}{1 - \alpha_{n0}}(y_n^{(m)} - x^*) \right] + \alpha_{n0}(x_n - x^*) \right\|. \end{aligned} \quad (3.5)$$

Since P_{T_i} , $i = 1, 2, \dots, m$ is nonexpansive mapping and $\cap_{i=1}^m F(T_i) \neq \emptyset$, we have

$$\|y_n^{(i)} - x^*\| \leq d(y_n^{(i)}, P_{T_i}x^*) \leq H(P_{T_i}x_n, P_{T_i}x^*) \leq \|x_n - x^*\|, \quad (3.6)$$

for each $x^* \in \cap_{i=1}^m F(T_i)$. Taking $\limsup$ on both sides, we obtain

$$\limsup_{n \rightarrow \infty} \|y_n^{(i)} - x^*\| \leq c, \quad i = 1, 2, \dots, m. \quad (3.7)$$

Next,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left\| \frac{\alpha_{n1}}{1 - \alpha_{n0}} (y_n^{(1)} - x^*) + \frac{\alpha_{n2}}{1 - \alpha_{n0}} (y_n^{(2)} - x^*) + \dots + \frac{\alpha_{nm}}{1 - \alpha_{n0}} (y_n^{(m)} - x^*) \right\| \\ & \leq \limsup_{n \rightarrow \infty} \left[\frac{\alpha_{n1}}{1 - \alpha_{n0}} \|y_n^{(1)} - x^*\| + \frac{\alpha_{n2}}{1 - \alpha_{n0}} \|y_n^{(2)} - x^*\| + \dots + \frac{\alpha_{nm}}{1 - \alpha_{n0}} \|y_n^{(m)} - x^*\| \right] \\ & \leq \limsup_{n \rightarrow \infty} \frac{\alpha_{n1} + \alpha_{n2} + \dots + \alpha_{nm}}{1 - \alpha_{n0}} \|x_n - x^*\| = c. \end{aligned} \quad (3.8)$$

Using (3.5), (3.8), and Lemma 2.1, we obtain

$$\lim_{n \rightarrow \infty} \left\| \frac{\alpha_{n1}}{1 - \alpha_{n0}} (y_n^{(1)} - x^*) + \frac{\alpha_{n2}}{1 - \alpha_{n0}} (y_n^{(2)} - x^*) + \dots + \frac{\alpha_{nm}}{1 - \alpha_{n0}} (y_n^{(m)} - x^*) - (x_n - x^*) \right\| = 0. \quad (3.9)$$

This yields

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \left\| \frac{\alpha_{n1}}{1 - \alpha_{n0}} y_n^{(1)} + \frac{\alpha_{n2}}{1 - \alpha_{n0}} y_n^{(2)} + \dots + \frac{\alpha_{nm}}{1 - \alpha_{n0}} y_n^{(m)} - x_n \right\| \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{1 - \alpha_{n0}} \right) \left\| \alpha_{n1} y_n^{(1)} + \alpha_{n2} y_n^{(2)} + \dots + \alpha_{nm} y_n^{(m)} - (1 - \alpha_{n0}) x_n \right\| \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{1 - \alpha_{n0}} \right) \|x_{n+1} - x_n\|. \end{aligned} \quad (3.10)$$

Thus, $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. Furthermore,

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} \|x_{n+1} - x^*\| \\ &= \lim_{n \rightarrow \infty} \left\| \alpha_{n0} (x_n - x^*) + \alpha_{n1} (y_n^{(1)} - x^*) + \dots + \alpha_{nm} (y_n^{(m)} - x^*) \right\| \\ &= \lim_{n \rightarrow \infty} \left\| (1 - \alpha_{n1}) \left[\frac{\alpha_{n0}}{1 - \alpha_{n1}} (x_n - x^*) + \frac{\alpha_{n2}}{1 - \alpha_{n1}} (y_n^{(2)} - x^*) + \dots + \frac{\alpha_{nm}}{1 - \alpha_{n1}} (y_n^{(m)} - x^*) \right] \right. \\ & \quad \left. + \alpha_{n1} (y_n^{(1)} - x^*) \right\|, \end{aligned}$$

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \left\| \frac{\alpha_{n0}}{1 - \alpha_{n1}} (x_n - x^*) + \frac{\alpha_{n2}}{1 - \alpha_{n1}} (y_n^{(2)} - x^*) + \cdots + \frac{\alpha_{nm}}{1 - \alpha_{n1}} (y_n^{(m)} - x^*) \right\| \\
& \leq \limsup_{n \rightarrow \infty} \left[\frac{\alpha_{n0}}{1 - \alpha_{n1}} \|x_n - x^*\| + \frac{\alpha_{n2}}{1 - \alpha_{n1}} \|y_n^{(2)} - x^*\| + \cdots + \frac{\alpha_{nm}}{1 - \alpha_{n1}} \|y_n^{(m)} - x^*\| \right] \\
& \leq \limsup_{n \rightarrow \infty} \frac{\alpha_{n0} + \alpha_{n2} + \cdots + \alpha_{nm}}{1 - \alpha_{n1}} \|x_n - x^*\| = c.
\end{aligned} \tag{3.11}$$

Using (3.11) and Lemma 2.1, we obtain

$$\lim_{n \rightarrow \infty} \left\| \frac{\alpha_{n0}}{1 - \alpha_{n1}} (x_n - x^*) + \frac{\alpha_{n2}}{1 - \alpha_{n1}} (y_n^{(2)} - x^*) + \cdots + \frac{\alpha_{nm}}{1 - \alpha_{n1}} (y_n^{(m)} - x^*) - (y_n^{(1)} - x^*) \right\| = 0. \tag{3.12}$$

This yields

$$\begin{aligned}
0 &= \lim_{n \rightarrow \infty} \left\| \frac{\alpha_{n0}}{1 - \alpha_{n1}} x_n + \frac{\alpha_{n2}}{1 - \alpha_{n1}} y_n^{(2)} + \cdots + \frac{\alpha_{nm}}{1 - \alpha_{n1}} y_n^{(m)} - y_n^{(1)} \right\| \\
&= \lim_{n \rightarrow \infty} \left(\frac{1}{1 - \alpha_{n1}} \right) \left\| \alpha_{n0} x_n + \alpha_{n2} y_n^{(2)} + \cdots + \alpha_{nm} y_n^{(m)} - (1 - \alpha_{n1}) y_n^{(1)} \right\| \\
&= \lim_{n \rightarrow \infty} \left(\frac{1}{1 - \alpha_{n1}} \right) \|x_{n+1} - y_n^{(1)}\|.
\end{aligned} \tag{3.13}$$

Thus, $\lim_{n \rightarrow \infty} \|x_{n+1} - y_n^{(1)}\| = 0$. So,

$$\|x_n - y_n^{(1)}\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - y_n^{(1)}\| \rightarrow 0, \quad n \rightarrow \infty. \tag{3.14}$$

Then,

$$d(x_n, T_1 x_n) \leq d(x_n, P_{T_1} x_n) \leq \|x_n - y_n^{(1)}\| \rightarrow 0, \quad n \rightarrow \infty. \tag{3.15}$$

In a similar way, we can show that

$$\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0, \quad i = 2, 3, \dots, m. \tag{3.16}$$

This completes the proof. $\square$

Theorem 3.2. Let E be a uniformly convex real Banach space and D a nonempty, closed, and convex subset of E . Let $T_1, T_2, \dots, T_m$ be multivalued maps of D into $P(D)$ with $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ such that $P_{T_1}, P_{T_2}, \dots, P_{T_m}$ are nonexpansive and $\{T_i\}_{i=1}^m$ satisfying condition (II). Let $\{\alpha_{ni}\}_{n=1}^\infty$, $i = 0, 1, \dots, m$

a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ such that $\sum_{i=0}^m \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by (3.1). Then, $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^m$.

Proof. Since $\{T_i\}_{i=1}^m$ satisfies condition (II), we have that $d(x_n, \cap_{i=1}^m F(T_i)) \rightarrow 0$ as $n \rightarrow \infty$. Thus, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\} \subset \cap_{i=1}^m F(T_i)$ such that

$$\|x_{n_k} - p_k\| < \frac{1}{2^k}, \quad (3.17)$$

for all k . By Lemma 3.1, we obtain

$$\|x_{n_{k+1}} - p_k\| \leq \|x_{n_k} - p_k\| < \frac{1}{2^k}. \quad (3.18)$$

We now show that $\{p_k\}$ is a Cauchy sequence in D . Observe that

$$\begin{aligned} \|p_{k+1} - p_k\| &\leq \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned} \quad (3.19)$$

This shows that $\{p_k\}$ is a Cauchy sequence in D , and thus converges to $p \in D$. Since

$$\begin{aligned} d(p_k, T_i p) &\leq d(p_k, P_{T_i} p) \leq H(P_{T_i} p, P_{T_i} p_k) \\ &\leq \|p - p_k\|, \end{aligned} \quad (3.20)$$

and $p_k \rightarrow p$ as $k \rightarrow \infty$, it follows that $d(p, T_i p) = 0$, and thus $p \in \cap_{i=1}^m F(T_i)$, and $\{x_{n_k}\}$ converges strongly to p . Since $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, it follows that $\{x_n\}$ converges strongly to p . This completes the proof. $\square$

Corollary 3.3. Let E be a uniformly convex real Banach space and D a nonempty, closed, and convex subset of E . Let $T_1, T_2, \dots, T_m$ be $*$ -nonexpansive multimaps of D into $P(D)$ with $F := \cap_{i=1}^m F(T_i) \neq \emptyset$ and $\{T_i\}_{i=1}^m$ satisfying condition (II). Let $\{\alpha_{ni}\}_{n=1}^\infty$, $i = 0, 1, \dots, m$ a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ such that $\sum_{i=0}^m \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by (3.1). Then, $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^m$.

Theorem 3.4. Let E be a uniformly convex real Banach space and D a nonempty, closed and convex subset of E . Let $T_1, T_2, \dots, T_m$ be multivalued maps of D into $P(D)$ with $F := \cap_{i=1}^m F(T_i) \neq \emptyset$ such that $P_{T_1}, P_{T_2}, \dots, P_{T_m}$ are nonexpansive and T_i is hemicompact and continuous for each $i = 1, 2, \dots, m$. Let $\{\alpha_{ni}\}_{n=1}^\infty$, $i = 0, 1, \dots, m$ a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ such that $\sum_{i=0}^m \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by (3.1). Then, $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^m$.

Proof. Since $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$, for all $i = 1, 2, \dots, m$ and T_i is hemicompact for each $i = 1, 2, \dots, m$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p$ as $k \rightarrow \infty$ for some $p \in D$.

Since T_i is continuous for each $i = 1, 2, \dots, m$, we have $d(x_{n_k}, T_i x_{n_k}) \rightarrow d(p, T_i p)$. As a result, we have that $d(p, T_i p) = 0$, for all $i = 1, 2, \dots, m$, and so, $p \in \cap_{i=1}^m F(T_i)$. Since $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, it follows that $\{x_n\}$ converges strongly to p . This completes the proof. $\square$

Corollary 3.5. *Let E be a uniformly convex real Banach space and D a nonempty, closed, and convex subset of E . Let $T_1, T_2, \dots, T_m$ be $*$ -nonexpansive multimaps of D into $P(D)$ with $F := \cap_{i=1}^m F(T_i) \neq \emptyset$ and T_i is hemicompact and continuous for each $i = 1, 2, \dots, m$. Let $\{\alpha_{ni}\}_{n=1}^\infty$, $i = 0, 1, \dots, m$ a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ such that $\sum_{i=0}^m \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by (3.1). Then, $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^m$.*

Theorem 3.6. *Let E be a uniformly convex real Banach space and D a nonempty compact convex subset of E . Let $T_1, T_2, \dots, T_m$ be multivalued maps of D into $P(D)$ with $F := \cap_{i=1}^m F(T_i) \neq \emptyset$ such that $P_{T_1}, P_{T_2}, \dots, P_{T_m}$ are nonexpansive. Let $\{\alpha_{ni}\}_{n=1}^\infty$, $i = 0, 1, \dots, m$ a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ such that $\sum_{i=0}^m \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by (3.1). Then, $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^m$.*

Proof. From the compactness of D , there exists a subsequence $\{x_{n_k}\}_{n=k}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $\lim_{k \rightarrow \infty} \|x_{n_k} - q\| = 0$ for some $q \in D$. Thus,

$$\begin{aligned} d(q, T_i q) &\leq d(q, P_{T_i} q) \leq \|x_{n_k} - q\| + d(x_{n_k}, P_{T_i} x_{n_k}) + H(P_{T_i} x_{n_k}, P_{T_i} q) \\ &\leq 2\|x_{n_k} - q\| + d(x_{n_k}, P_{T_i} x_{n_k}) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \end{aligned} \quad (3.21)$$

Hence, $q \in \cap_{i=1}^m F(T_i)$. Now, on taking q in place of x^* , we get that $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists. This completes the proof. $\square$

The following result gives a necessary and sufficient condition for strong convergence of the sequence in (3.1) to a common fixed point of $\{T_i\}_{i=1}^m$.

Theorem 3.7. *Let D be a nonempty, closed, and convex subset of a real Banach space E . Let $T_1, T_2, \dots, T_m$ be multivalued maps of D into $P(D)$ with $F := \cap_{i=1}^m F(T_i) \neq \emptyset$ such that $P_{T_1}, P_{T_2}, \dots, P_{T_m}$ are nonexpansive. Let $\{\alpha_{ni}\}_{n=1}^\infty$, $i = 0, 1, \dots, m$ a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ such that $\sum_{i=0}^m \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by (3.1). Then, $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^m$ if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.*

Proof. The necessity is obvious. Conversely, suppose that $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$. By (3.3), we have

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|. \quad (3.22)$$

This gives

$$d(x_{n+1}, F) \leq d(x_n, F). \quad (3.23)$$

Hence, $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. By hypothesis, $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, so we must have $\lim_{n \rightarrow \infty} d(x_n, F) = 0$.

Next, we show that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in D . Let $\epsilon > 0$ be given, and since $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, there exists n_0 such that for all $n \geq n_0$, we have

$$d(x_n, F) < \frac{\epsilon}{4}. \quad (3.24)$$

In particular, $\inf\{\|x_{n_0} - p\| : p \in F\} < \epsilon/4$ so that there must exist a $p^* \in F$ such that

$$\|x_{n_0} - p^*\| < \frac{\epsilon}{2}. \quad (3.25)$$

Now, for $m, n \geq n_0$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\leq 2\|x_{n_0} - p^*\| \\ &< 2\left(\frac{\epsilon}{2}\right) = \epsilon. \end{aligned} \quad (3.26)$$

Hence, $\{x_n\}$ is a Cauchy sequence in a closed subset D of a Banach space E , and therefore, it must converge in D . Let $\lim_{n \rightarrow \infty} x_n = p$. Now, for each $i = 1, 2, \dots, m$, we obtain

$$\begin{aligned} d(p, T_i p) &\leq d(p, P_{T_i} p) \leq d(p, x_n) + d(x_n, P_{T_i} x_n) + H(P_{T_i} x_n, P_{T_i} p) \\ &\leq d(p, x_n) + d(x_n, P_{T_i} x_n) + d(x_n, p) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \end{aligned} \quad (3.27)$$

gives that $d(p, T_i p) = 0$, $i = 1, 2, \dots, m$ which implies that $p \in T_i p$. Consequently, $p \in F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. $\square$

All the results we have obtained so far can be established for finite family of quasi-nonexpansive multivalued maps. Let $T_1, T_2, \dots, T_m$ be quasi-nonexpansive multivalued maps of D into $CB(D)$ such that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ for which $T_i p = \{p\}$, for all $p \in \bigcap_{i=1}^m F(T_i)$. Let $\{\alpha_{ni}\}_{n=1}^\infty$, $i = 0, 1, \dots, m$ a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ such that $\sum_{i=0}^m \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by

$$\begin{aligned} x_1 &\in D, \\ x_{n+1} &= \alpha_{n0} x_n + \alpha_{n1} y_n^{(1)} + \dots + \alpha_{nm} y_n^{(m)}, \end{aligned} \quad (3.28)$$

where $y_n^{(i)} \in T_i x_n$, $i = 1, 2, \dots, m$. Thus, we obtain the following theorems using iterative process (3.28).

Theorem 3.8. *Let E be a uniformly convex real Banach space and D a nonempty, closed, and convex subset of E . Let $T_1, T_2, \dots, T_m$ be quasi-nonexpansive multivalued maps of D into $CB(D)$ such that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ for which $T_i p = \{p\}$, for all $p \in \bigcap_{i=1}^m F(T_i)$ and $\{T_i\}_{i=1}^m$ satisfying condition (II). Let $\{\alpha_{ni}\}_{n=1}^\infty$, $i = 0, 1, \dots, m$ a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ such that $\sum_{i=0}^m \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by (3.28). Then, $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^m$.*

Theorem 3.9. Let E be a uniformly convex real Banach space and D a nonempty, closed, and convex subset of E . Let $T_1, T_2, \dots, T_m$ be quasi-nonexpansive multivalued maps of D into $CB(D)$ such that $F := \cap_{i=1}^m F(T_i) \neq \emptyset$ for which $T_i p = \{p\}$, for all $p \in \cap_{i=1}^m F(T_i)$ and T_i is hemicompact and continuous for each $i = 1, 2, \dots, m$. Let $\{\alpha_{ni}\}_{n=1}^\infty$, $i = 0, 1, \dots, m$ a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ such that $\sum_{i=0}^m \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by (3.28). Then, $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^m$.

Theorem 3.10. Let E be a uniformly convex real Banach space and D a nonempty compact convex subset of E . Let $T_1, T_2, \dots, T_m$ be quasi-nonexpansive multivalued maps of D into $CB(D)$ such that $F := \cap_{i=1}^m F(T_i) \neq \emptyset$ for which $T_i p = \{p\}$, for all $p \in \cap_{i=1}^m F(T_i)$. $\{\alpha_{ni}\}_{n=1}^\infty$, $i = 0, 1, \dots, m$ a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ such that $\sum_{i=0}^m \alpha_{ni} = 1$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by (3.28). Then, $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^m$.

Corollary 3.11 (Abbas et al. [13]). Let E be a uniformly convex real Banach space satisfying Opial's condition. Let D be a nonempty, closed, and convex of E . Let T, S be multivalued nonexpansive mappings of D into $K(D)$ such that $F := F(T) \cap F(S) \neq \emptyset$. Let $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$, and $\{c_n\}_{n=1}^\infty$ be sequence in $(0, 1)$ satisfying $a_n + b_n + c_n \leq 1$. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by

$$\begin{aligned} x_1 &\in D, \\ x_{n+1} &= a_n x_n + b_n y_n + c_n z_n, \quad \forall n \geq 1, \end{aligned} \tag{3.29}$$

where $y_n \in Tx_n$, $z_n \in Sx_n$ such that $\|y_n - p\| \leq d(p, Tx_n)$ and $\|z_n - p\| \leq d(p, Sx_n)$ whenever p is a fixed point of any one of mappings T and S . Then, $\{x_n\}_{n=1}^\infty$ converges weakly to a common fixed point of $F(T) \cap F(S)$.

Corollary 3.12 (Abbas et al. [13]). Let E be a real Banach space and D a nonempty, closed, and convex subset of E . Let T, S be multivalued nonexpansive mappings of D into $K(D)$ such that $F := F(T) \cap F(S) \neq \emptyset$. Let $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ and $\{c_n\}_{n=1}^\infty$ be sequence in $(0, 1)$ satisfying $a_n + b_n + c_n \leq 1$. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by

$$\begin{aligned} x_1 &\in D, \\ x_{n+1} &= a_n x_n + b_n y_n + c_n z_n, \quad \forall n \geq 1, \end{aligned} \tag{3.30}$$

where $y_n \in Tx_n$, $z_n \in Sx_n$ such that $\|y_n - p\| \leq d(p, Tx_n)$ and $\|z_n - p\| \leq d(p, Sx_n)$ whenever p is a fixed point of any one of mappings T and S . Then, $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point of $F(T) \cap F(S)$ if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.

Remark 3.13. Our results extend the results of Sastry and Babu [9], Panyanak [10], and Song and Wang [11] from approximation of a fixed point of a single multivalued nonexpansive mapping to approximation of common fixed point of a finite family of quasi-nonexpansive multivalued mappings.

Remark 3.14. Our results extend the results of Shahzad and Zegeye [12] from approximation of a fixed point of a single quasi-nonexpansive multivalued mapping and single multivalued map to approximation of common fixed point of a finite family of quasi-nonexpansive multivalued mappings and a finite family of multivalued maps.

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