Research Article

On Generalized Rotation Matrices

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Received 11 March 2011; Accepted 17 April 2011

Academic Editors: D. Kuhl and K. Takaba

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A general class of matrices, covering, for instance, an important set of proper rotations, is considered. Several characteristics of the class are established, which deal with such notions and properties as determinant, eigenspaces, eigenvalues, idempotency, Moore-Penrose inverse, or orthogonality.

1. Introduction and Basic Properties

Let $\mathbb{C}_{m,n}$ denote the set of $m \times n$ complex matrices. The symbols $\mathbf{L}'$, $\mathbf{L}^*$, $\mathcal{R}(\mathbf{L})$, and $\text{rk}(\mathbf{L})$ will stand for the transpose, conjugate transpose, column space, and rank, respectively, of $\mathbf{L} \in \mathbb{C}_{m,n}$. Further, $\mathbf{L}^\dagger \in \mathbb{C}_{n,m}$ will be the Moore-Penrose inverse of $\mathbf{L} \in \mathbb{C}_{m,n}$, that is, the unique matrix satisfying the equations

$$\mathbf{L}\mathbf{L}^\dagger\mathbf{L} = \mathbf{L}, \quad \mathbf{L}^\dagger\mathbf{L}\mathbf{L}^\dagger = \mathbf{L}^\dagger, \quad (\mathbf{L}\mathbf{L}^\dagger)^* = \mathbf{L}\mathbf{L}^\dagger, \quad (\mathbf{L}^\dagger\mathbf{L})^* = \mathbf{L}^\dagger\mathbf{L}, \quad (1.1)$$

and $\mathbf{I}_n$ will mean the identity matrix of order n . The Moore-Penrose inverse $\mathbf{L}^\dagger$ is useful in representing the orthogonal (in the sense of the standard inner product) projectors onto $\mathcal{R}(\mathbf{L})$ and $\mathcal{R}(\mathbf{L}^*)$, denoted by $\mathbf{P}_\mathbf{L}$ and $\mathbf{P}_{\mathbf{L}^*}$, as well as the orthogonal projectors onto the orthogonal complements of these subspaces, denoted by $\mathbf{Q}_\mathbf{L}$ and $\mathbf{Q}_{\mathbf{L}^*}$. To be precise, for $\mathbf{L} \in \mathbb{C}_{m,n}$,

$$\mathbf{P}_\mathbf{L} = \mathbf{L}\mathbf{L}^\dagger, \quad \mathbf{P}_{\mathbf{L}^*} = \mathbf{L}^\dagger\mathbf{L}, \quad \mathbf{Q}_\mathbf{L} = \mathbf{I}_m - \mathbf{L}\mathbf{L}^\dagger, \quad \mathbf{Q}_{\mathbf{L}^*} = \mathbf{I}_n - \mathbf{L}^\dagger\mathbf{L}. \quad (1.2)$$

With respect to a scalar, say $\alpha \in \mathbb{C}$, the inverse $\alpha^\dagger$ is defined as: $\alpha^\dagger = 0$ when $\alpha = 0$ and $\alpha^\dagger = \alpha^{-1}$ when $\alpha \neq 0$.

The considerations of the present paper concern 3×3 matrices and 3×1 vectors having either complex or real entries; in the latter case their subsets will be denoted by $\mathbb{R}_{3,3}$ and $\mathbb{R}_{3,1}$, respectively. Customarily, the symbol $\|\mathbf{l}\|$ will stand for the Euclidean norm of $\mathbf{l} \in \mathbb{C}_{3,1}$, that is, $\|\mathbf{l}\| = \sqrt{\mathbf{l}^* \mathbf{l}}$. Let

$$\mathbf{T}_{\mathbf{a}} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \quad (1.3)$$

be generated by $\mathbf{a} = (a_1, a_2, a_3)' \in \mathbb{R}_{3,1}$. The matrix $\mathbf{T}_{\mathbf{a}}$ can be used to define the vector cross product in $\mathbb{R}_{3,1}$, with $\mathbf{T}_{\mathbf{a}} \mathbf{l} = \mathbf{a} \times \mathbf{l}$ for any $\mathbf{l} \in \mathbb{R}_{3,1}$; see [1]. Further properties of $\mathbf{T}_{\mathbf{a}}$ are listed in the following lemma, whose proof is easy and thus omitted.

Lemma 1.1. *Let $\lambda, \mu \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}_{3,1}$. Moreover, let $\mathbf{T}_{\mathbf{l}}$ for some $\mathbf{l} \in \mathbb{R}_{3,1}$ be of the form (1.3). Then,*

- (i) $\mathbf{T}_{\lambda \mathbf{a} + \mu \mathbf{b}} = \lambda \mathbf{T}_{\mathbf{a}} + \mu \mathbf{T}_{\mathbf{b}}$,
- (ii) $\mathbf{T}_{\mathbf{a}} \mathbf{b} = -\mathbf{T}_{\mathbf{b}} \mathbf{a}$,
- (iii) $\mathbf{T}_{\mathbf{a}} = -\mathbf{T}'_{\mathbf{a}}$,
- (iv) $\mathbf{T}_{\mathbf{a}} \mathbf{a} = \mathbf{0}$,
- (v) $\mathbf{T}_{\mathbf{a}} \mathbf{T}_{\mathbf{b}} = \mathbf{b} \mathbf{a}' - (\mathbf{a}' \mathbf{b}) \mathbf{I}_3$,
- (vi) $(\mathbf{T}_{\mathbf{a}} \mathbf{T}_{\mathbf{b}})' = \mathbf{T}_{\mathbf{b}} \mathbf{T}_{\mathbf{a}}$,
- (vii) $\mathbf{T}_{\mathbf{a}} \mathbf{T}_{\mathbf{b}} \mathbf{T}_{\mathbf{c}} = \mathbf{T}_{\mathbf{a}} \mathbf{c} \mathbf{b}' - (\mathbf{b}' \mathbf{c}) \mathbf{T}_{\mathbf{a}}$,
- (viii) $\mathbf{T}_{\mathbf{a}} \mathbf{T}_{\mathbf{b}} \mathbf{T}_{\mathbf{a}} = -(\mathbf{a}' \mathbf{b}) \mathbf{T}_{\mathbf{a}}$,
- (ix) $\mathbf{T}_{\mathbf{a}}^3 = -\|\mathbf{a}\|^2 \mathbf{T}_{\mathbf{a}}$,
- (x) $\mathbf{c}' \mathbf{T}_{\mathbf{a}} \mathbf{b} = \mathbf{a}' \mathbf{T}_{\mathbf{b}} \mathbf{c} = \mathbf{b}' \mathbf{T}_{\mathbf{c}} \mathbf{a}$,
- (xi) $\mathbf{T}_{\mathbf{a}} \mathbf{T}_{\mathbf{b}} \mathbf{c} = (\mathbf{a}' \mathbf{c}) \mathbf{b} - (\mathbf{a}' \mathbf{b}) \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$,
- (xii) $(\mathbf{T}_{\mathbf{a}} \mathbf{b})' \mathbf{T}_{\mathbf{c}} \mathbf{d} = (\mathbf{a}' \mathbf{c})(\mathbf{b}' \mathbf{d}) - (\mathbf{a}' \mathbf{d})(\mathbf{b}' \mathbf{c}) = (\mathbf{a} \times \mathbf{b})'(\mathbf{c} \times \mathbf{d})$,
- (xiii) $\mathbf{T}_{\mathbf{T}_{\mathbf{a}} \mathbf{b}} = \mathbf{b} \mathbf{a}' - \mathbf{a} \mathbf{b}' = \mathbf{T}_{\mathbf{a}} \mathbf{T}_{\mathbf{b}} - \mathbf{T}_{\mathbf{b}} \mathbf{T}_{\mathbf{a}}$.

Relationships listed in Lemma 1.1 are available in the literature; see Trenkler [2, 3], Groß et al. [4], Bernstein [5, Ch. 3], and G. Trenkler and D. Trenkler [6]. It is noteworthy that the three scalars involved in point (x) represent the scalar triple products, for example, $\mathbf{c}' \mathbf{T}_{\mathbf{a}} \mathbf{b} = (\mathbf{a} \times \mathbf{b})' \mathbf{c}$. Moreover, the right-hand sides equalities in points (xi) and (xii) are known as the Grassmann and Lagrange identities, respectively. From the point of view of the present paper, conditions (iii)–(v) of the lemma are of particular importance and will be extensively utilized in the subsequent derivations.

It is known that if $\mathbf{a} \in \mathbb{R}_{3,1}$ generating $\mathbf{T}_{\mathbf{a}}$ given in (1.3) is such that $\|\mathbf{a}\| = 1$, then the matrices of the form

$$R = \sin \theta \mathbf{T}_{\mathbf{a}} + \mathbf{I}_3 + (1 - \cos \theta) \mathbf{T}_{\mathbf{a}}^2 \quad (1.4)$$

describe proper rotations in $\mathbb{R}_{3,1}$ with θ being the angle of rotation about the axis given by $\mathbf{a}$; see Noble [7, Ch. 12] or Murray et al. [8, Ch. 2]. In what follows we consider a more general class of matrices than the one spanned by matrices of the form (1.4), namely,

$$\Upsilon = \{\mathbf{T} \in \mathbb{C}_{3,3} : \mathbf{T} = \alpha \mathbf{T}_{\mathbf{a}} + \beta \mathbf{I}_3 + \gamma \mathbf{a} \mathbf{a}', \alpha, \beta, \gamma \in \mathbb{C}, \mathbf{a} \in \mathbb{R}_{3,1}, \mathbf{a} \neq 0\}. \quad (1.5)$$

It can be verified that Υ covers all proper rotations of the form (1.4). Moreover, it comprises also improper rotations, that is, orthogonal matrices with determinant equal to -1 [9, Ch. VIII], symmetric elementary matrices [10, Sec. 1], and all matrices commuting with $\mathbf{T}_{\mathbf{a}}$ [11].

The purpose of the present paper is to identify various properties of the class of matrices specified in (1.5). As a result, several characteristics of the class Υ are established, dealing with such notions and properties as idempotency, determinant, eigenvalues, Moore-Penrose inverse, orthogonality, or eigenspaces.

2. Results

Subsequently, the symbol i is interpreted as $i = \sqrt{-1}$. The theorem below states that Υ is closed against multiplication.

Theorem 2.1. *Let $\mathbf{T}_1, \mathbf{T}_2 \in \Upsilon$ with $\mathbf{T}_k = \alpha_k \mathbf{T}_{\mathbf{a}} + \beta_k \mathbf{I}_3 + \gamma_k \mathbf{a} \mathbf{a}'$, $k = 1, 2$. Then $\mathbf{T}_1 \mathbf{T}_2 \in \Upsilon$.*

Proof. Direct calculations show that

$$\mathbf{T}_1 \mathbf{T}_2 = (\alpha_1 \beta_2 + \beta_1 \alpha_2) \mathbf{T}_{\mathbf{a}} + (\beta_1 \beta_2 - \alpha_1 \alpha_2 \|\mathbf{a}\|^2) \mathbf{I}_3 + (\alpha_1 \alpha_2 + \beta_1 \gamma_2 + \beta_2 \gamma_1 + \gamma_1 \gamma_2 \|\mathbf{a}\|^2) \mathbf{a} \mathbf{a}', \quad (2.1)$$

establishing the assertion. □

It is clear that, besides ensuring the closure property, multiplication in Υ is also associative. Furthermore, for each $\mathbf{T} \in \Upsilon$ there exists the identity element, namely, $\mathbf{I}_3$. On the other hand, since Υ includes also singular matrices, not every $\mathbf{T} \in \Upsilon$ has an inverse element in Υ . Thus, the set Υ is a semigroup under (matrix) multiplication. (As will be seen subsequently, the Moore-Penrose inverse of every $\mathbf{T} \in \Upsilon$ belongs to Υ , in particular, $\mathbf{T} \in \Upsilon \Leftrightarrow \mathbf{T}^{-1} \in \Upsilon$ for each nonsingular $\mathbf{T} \in \Upsilon$.)

The next theorem provides necessary and sufficient conditions for $\mathbf{T}^2 = \mathbf{T}$.

Theorem 2.2. *Let $\mathbf{T} \in \Upsilon$. Then $\mathbf{T}$ is idempotent if and only if*

$$2\alpha\beta = \alpha, \quad \beta^2 - \alpha^2 \|\mathbf{a}\|^2 = \beta, \quad \alpha^2 + 2\beta\gamma + \gamma^2 \|\mathbf{a}\|^2 = \gamma. \quad (2.2)$$

Proof. The equivalence established in the theorem follows straightforwardly from Theorem 2.1. □

In view of Theorem 2.2, we can distinguish two subsets of idempotent matrices belonging to Υ corresponding to $\alpha \neq 0$ and $\alpha = 0$. In the former of them, $\alpha = \pm(i/2\|\mathbf{a}\|)$,

$\beta = 1/2$, and $\gamma = \pm(1/2\|\mathbf{a}\|^2)$. Hence, since $\mathbf{a}^\dagger = (1/\|\mathbf{a}\|^2)\mathbf{a}'$, it follows that $\mathbf{T} \in \{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4\}$, with

$$\begin{aligned} \mathbf{P}_1 &= \frac{1}{2} \left(\frac{i}{\|\mathbf{a}\|} \mathbf{T}_a + \mathbf{I}_3 + \mathbf{P}_a \right), & \mathbf{P}_2 &= \frac{1}{2} \left(\frac{i}{\|\mathbf{a}\|} \mathbf{T}_a + \mathbf{Q}_a \right), \\ \mathbf{P}_3 &= \frac{1}{2} \left(-\frac{i}{\|\mathbf{a}\|} \mathbf{T}_a + \mathbf{I}_3 + \mathbf{P}_a \right), & \mathbf{P}_4 &= \frac{1}{2} \left(-\frac{i}{\|\mathbf{a}\|} \mathbf{T}_a + \mathbf{Q}_a \right). \end{aligned} \quad (2.3)$$

(Note that matrices $\mathbf{P}_k$, $k = 1, \dots, 4$, were considered by Trenkler [3] to characterize certain eigenspaces.) On the other hand, if $\alpha = 0$, then Theorem 2.2 entails four cases, namely, $\beta = 0$, $\gamma = 0$; $\beta = 0$, $\gamma = 1/\|\mathbf{a}\|^2$; $\beta = 1$, $\gamma = 0$; and $\beta = 1$, $\gamma = -1/\|\mathbf{a}\|^2$, leading to $\mathbf{P}_5 = 0$, $\mathbf{P}_6 = \mathbf{P}_a$, $\mathbf{P}_7 = \mathbf{I}_3$, and $\mathbf{P}_8 = \mathbf{Q}_a$, respectively.

The next task is to characterize the eigenvalues of matrices belonging to Υ . The subsequent theorem expresses the determinant of $\mathbf{T} \in \Upsilon$ in terms of the scalars α , β , and γ . Its proof is based on the so-called Leverrier-Sourian-Frame algorithm, which provides a useful tool to calculate the coefficients in a characteristic polynomial. Since the algorithm is not widely known, it is restated in the following lemma; see for example, Meyer [12, page 504]. Customarily, $\text{tr}(\cdot)$ denotes the trace of a matrix argument.

Lemma 2.3. *Let $\mathbf{L} \in \mathbb{C}_{n,n}$, and let $\mu^n + c_1\mu^{n-1} + c_2\mu^{n-2} + \dots + c_n = 0$ be the characteristic equation for $\mathbf{L}$. Then*

$$c_1 = -\text{tr}(\mathbf{L}), \quad c_k = -\frac{1}{k} \text{tr}(\mathbf{L}\mathbf{B}_{k-1}), \quad k = 2, 3, \dots, n, \quad (2.4)$$

where $\mathbf{B}_1 = c_1\mathbf{I}_n + \mathbf{L}$ and $\mathbf{B}_k = c_k\mathbf{I}_n + \mathbf{L}\mathbf{B}_{k-1}$, $k = 2, 3, \dots, n-1$.

Theorem 2.4. *Let $\mathbf{T} \in \Upsilon$, and let $\tau = \alpha^2\|\mathbf{a}\|^2 + \beta^2$. Then $\det(\mathbf{T}) = \tau(\beta + \gamma\|\mathbf{a}\|^2)$.*

Proof. From Lemma 2.3 it follows that $c_1 = -\text{tr}(\mathbf{T})$ is given by $c_1 = -(3\beta + \gamma\|\mathbf{a}\|^2)$, whence it is seen that $\mathbf{B}_1 = c_1\mathbf{I}_3 + \mathbf{T}$ takes the form

$$\mathbf{B}_1 = \alpha\mathbf{T}_a - (2\beta + \gamma\|\mathbf{a}\|^2)\mathbf{I}_3 + \gamma\mathbf{a}\mathbf{a}'. \quad (2.5)$$

Further, if $k = 2$, then $c_2 = -(1/2) \text{tr}(\mathbf{T}\mathbf{B}_1)$, where

$$\mathbf{T}\mathbf{B}_1 = -(\alpha\beta + \alpha\gamma\|\mathbf{a}\|^2)\mathbf{T}_a - (\alpha^2\|\mathbf{a}\|^2 + 2\beta^2 + \beta\gamma\|\mathbf{a}\|^2)\mathbf{I}_3 + (\alpha^2 - \beta\gamma)\mathbf{a}\mathbf{a}'. \quad (2.6)$$

Since $\text{tr}(\mathbf{T}_a) = 0$, in consequence we get $c_2 = \alpha^2\|\mathbf{a}\|^2 + 3\beta^2 + 2\beta\gamma\|\mathbf{a}\|^2$, leading to

$$\mathbf{B}_2 = c_2\mathbf{I}_3 + \mathbf{T}\mathbf{B}_1 = -(\alpha\beta + \alpha\gamma\|\mathbf{a}\|^2)\mathbf{T}_a + (\beta^2 + \beta\gamma\|\mathbf{a}\|^2)\mathbf{I}_3 + (\alpha^2 - \beta\gamma)\mathbf{a}\mathbf{a}'. \quad (2.7)$$

Finally, if $k = 3$, then $c_3 = -(1/3) \text{tr}(\mathbf{T}\mathbf{B}_2)$, where

$$\mathbf{T}\mathbf{B}_2 = (\alpha^2\beta\|\mathbf{a}\|^2 + \alpha^2\gamma\|\mathbf{a}\|^4 + \beta^3 + \beta^2\gamma\|\mathbf{a}\|^2)\mathbf{I}_3. \quad (2.8)$$

Hence, straightforward calculations yield $c_3 = -\tau(\beta + \gamma\|\mathbf{a}\|^2)$. Combining this result with the property $\det(\mathbf{T}) = (-1)^3 c_3$ completes the proof. $\square$

By virtue of Theorem 2.4, it is easy to determine the spectrum of matrices belonging to Υ .

Theorem 2.5. *Let $\mathbf{T} \in \Upsilon$. Then the eigenvalues of $\mathbf{T}$ are solutions μ to the equation $[\alpha^2\|\mathbf{a}\|^2 + (\beta - \mu)^2][\beta - \mu + \gamma\|\mathbf{a}\|^2] = 0$.*

Proof. It is clear that

$$\det(\mathbf{T} - \mu\mathbf{I}_3) = \det[\alpha\mathbf{T}_a + (\beta - \mu)\mathbf{I}_3 + \gamma\mathbf{a}\mathbf{a}']. \quad (2.9)$$

Hence, on account of Theorem 2.4, we get

$$\det(\mathbf{T} - \mu\mathbf{I}_3) = \det\left\{\left[\alpha^2\|\mathbf{a}\|^2 + (\beta - \mu)^2\right]\left[\beta - \mu + \gamma\|\mathbf{a}\|^2\right]\right\}, \quad (2.10)$$

establishing the assertion. $\square$

Theorem 2.5 leads to what follows.

Corollary 2.6. *Let $\mathbf{T} \in \Upsilon$. Then the eigenvalues of $\mathbf{T}$ are*

$$\mu_1 = \beta + \gamma\|\mathbf{a}\|^2, \quad \mu_2 = \beta + i\alpha\|\mathbf{a}\|^2, \quad \mu_3 = \beta - i\alpha\|\mathbf{a}\|^2. \quad (2.11)$$

An expected result originating from Corollary 2.6 is that $\mu_1\mu_2\mu_3 = \det(\mathbf{T})$, with $\det(\mathbf{T})$ given in Theorem 2.4. Furthermore, when $\alpha = 1$, $\beta = 0$, and $\gamma = 0$, then the eigenvalues given in (2.11) reduce to $\mu_1 = 0$, $\mu_2 = i\|\mathbf{a}\|$, and $\mu_3 = -i\|\mathbf{a}\|$, that is, to the eigenvalues of $\mathbf{T}_a$; see [3, Theorem 2].

The following theorem will be useful in the subsequent calculations of the Moore-Penrose inverses of $\mathbf{T} \in \Upsilon$.

Theorem 2.7. *Let $\mathbf{A} \in \mathbb{C}_{3,3}$ be such that $\mathbf{A} = \alpha\mathbf{T}_a + \beta\mathbf{I}_3$, with $\alpha, \beta \in \mathbb{C}$ and $\mathbf{T}_a$ of form (1.3) generated by nonzero $\mathbf{a} \in \mathbb{R}_{3,1}$. Moreover, let $\lambda = 1 + \gamma\mathbf{a}'\mathbf{A}^\dagger\mathbf{a}$, with $\gamma \in \mathbb{C}$, and $\tau = \alpha^2\|\mathbf{a}\|^2 + \beta^2$. Then,*

- (i) $\det(\mathbf{A}) = \beta\tau$,
- (ii) the eigenvalues of $\mathbf{A}$ are $\sigma_1 = \beta$, $\sigma_2 = \beta + i\alpha\|\mathbf{a}\|$, $\sigma_3 = \beta - i\alpha\|\mathbf{a}\|$,
- (iii) if $\beta = 0$, then $\mathbf{A}^\dagger = -(\alpha^\dagger/\|\mathbf{a}\|^2)\mathbf{T}_a$, $\lambda = 1$,
- (iv) if $\beta \neq 0$, $\tau = 0$, then $\mathbf{A}^\dagger = (1/4\beta)((\alpha/\beta)\mathbf{T}_a + \mathbf{I}_3 + 3\mathbf{P}_a)$, $\lambda = 1 + (\gamma/\beta)\|\mathbf{a}\|^2$,
- (v) if $\beta \neq 0$, $\tau \neq 0$, then $\mathbf{A}^{-1} = (1/\tau)(-\alpha\mathbf{T}_a + \beta\mathbf{I}_3 + (\alpha^2/\beta)\mathbf{a}\mathbf{a}')$, $\lambda = 1 + (\gamma/\beta)\|\mathbf{a}\|^2$.

Proof. Assertion (i) follows from Theorem 2.4 by setting $\gamma = 0$. Statement (ii) is a consequence of (i), whereas the validity of points (iii) and (v) can be confirmed by straightforward calculations; see [3, Theorem 1]. For the proof of statement (iv) note that $\beta \neq 0$, $\tau = 0$ imply $\alpha/\beta = \pm(i/\|\mathbf{a}\|)$, that is, α/β is purely imaginary. Taking this fact into account, in view of $\mathbf{a}^\dagger = (1/\|\mathbf{a}\|^2)\mathbf{a}'$, the validity of the formula for $\mathbf{A}^\dagger$ given in point (iv) is seen by direct

verification of conditions (1.1). Similarly, the formula for $\mathbf{A}^{-1}$ provided in point (v) can be confirmed by examining the condition $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_3$. The proof is thus complete, for the expressions for λ given in points (iv) and (v) are easily obtainable. $\square$

Note that regardless whether β and/or τ in Theorem 2.7 are zero or not, the matrix $\mathbf{A}$ is such that $\mathbf{P}_\mathbf{A} = \mathbf{P}_{\mathbf{A}^*}$, or, in other words, $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A}^*)$, that is, $\mathbf{A}$ is an EP matrix. Another observation is that by setting $\alpha = 1, \beta = 1$ in Theorem 2.7 leads to the relationship

$$(\mathbf{T}_\mathbf{a} + \mathbf{I}_3)^{-1} = \frac{1}{\|\mathbf{a}\|^2 + 1} (-\mathbf{T}_\mathbf{a} + \mathbf{I}_3 + \mathbf{a}\mathbf{a}'). \quad (2.12)$$

Hence, the so called Cayley transform of $\mathbf{T}_\mathbf{a}$ (see [13, p. 219]), being of the form $\mathbf{S} = (-\mathbf{T}_\mathbf{a} + \mathbf{I}_3)(\mathbf{T}_\mathbf{a} + \mathbf{I}_3)^{-1}$, of which it is known that is orthogonal (see [9, Theorem 8.1.10]), takes the form

$$\mathbf{S} = \frac{1}{\|\mathbf{a}\|^2 + 1} [-2\mathbf{T}_\mathbf{a} + (1 - \|\mathbf{a}\|^2)\mathbf{I}_3 + 2\mathbf{a}\mathbf{a}']. \quad (2.13)$$

Since $\det(\mathbf{S}) = 1$, the matrix $\mathbf{S}$ represents in fact a proper rotation.

We now have the tools necessary to establish formulae for the Moore-Penrose inverses of $\mathbf{T} \in \mathcal{Y}$.

Theorem 2.8. *Let $\mathbf{T} \in \mathcal{Y}$ be decomposed as $\mathbf{T} = \mathbf{A} + \gamma\mathbf{a}\mathbf{a}'$, where $\mathbf{A} = \alpha\mathbf{T}_\mathbf{a} + \beta\mathbf{I}_3$. Moreover, let $\lambda = 1 + \gamma\mathbf{a}'\mathbf{A}^\dagger\mathbf{a}$ and $\tau = \alpha^2\|\mathbf{a}\|^2 + \beta^2$. Then,*

- (i) if $\beta = 0$, then $\mathbf{T}^\dagger = (1/\|\mathbf{a}\|^2)(-\alpha^\dagger\mathbf{T}_\mathbf{a} + \gamma^\dagger\mathbf{P}_\mathbf{a})$,
- (ii) if $\beta \neq 0, \tau = 0, \lambda = 0$, then $\mathbf{T}^\dagger = (1/4\beta)((\alpha/\beta)\mathbf{T}_\mathbf{a} + \mathbf{Q}_\mathbf{a})$,
- (iii) if $\beta \neq 0, \tau = 0, \lambda \neq 0$, then $\mathbf{T}^\dagger = (1/4\beta)((\alpha/\beta)\mathbf{T}_\mathbf{a} + \mathbf{I}_3 + ((3\beta - \gamma\|\mathbf{a}\|^2)/(\beta + \gamma\|\mathbf{a}\|^2))\mathbf{P}_\mathbf{a})$,
- (iv) if $\beta \neq 0, \tau \neq 0, \lambda = 0$, then $\mathbf{T}^\dagger = (1/\tau)(-\alpha\mathbf{T}_\mathbf{a} + \beta\mathbf{Q}_\mathbf{a})$,
- (v) if $\beta \neq 0, \tau \neq 0, \lambda \neq 0$, then $\mathbf{T}^{-1} = (1/\tau)(-\alpha\mathbf{T}_\mathbf{a} + \beta\mathbf{I}_3 + ((\alpha^2 - \beta\gamma)/(\beta + \gamma\|\mathbf{a}\|^2))\mathbf{a}\mathbf{a}')$.

Proof. The first observation is that if $\beta = 0$, then on account of Theorem 3.1.1 in [14], we have $\mathbf{T}^\dagger = (\alpha\mathbf{T}_\mathbf{a})^\dagger + (\gamma\mathbf{a}\mathbf{a}')^\dagger$. Hence, by utilizing $\mathbf{T}_\mathbf{a}^\dagger = -(1/\|\mathbf{a}\|^2)\mathbf{T}_\mathbf{a}$ and $(\mathbf{a}\mathbf{a}')^\dagger = (1/\|\mathbf{a}\|^2)\mathbf{P}_\mathbf{a}$, the formula for $\mathbf{T}^\dagger$ given in point (i) follows.

Assume now that $\beta \neq 0$, in which case we can still have $\tau = 0$ or $\tau \neq 0$. In the former of these situations, point (iv) of Theorem 2.7 implies $\mathbf{P}_\mathbf{A} = (1/2)((\alpha/\beta)\mathbf{T}_\mathbf{a} + \mathbf{I}_3 + \mathbf{P}_\mathbf{a})$, whence $\mathbf{P}_\mathbf{A}\mathbf{a} = \mathbf{a}$, or, in other words, $\mathbf{a} \in \mathcal{R}(\mathbf{A})$. This inclusion is clearly satisfied also when $\tau \neq 0$, for then $\det(\mathbf{A}) \neq 0$.

In order to apply the results of Baksalary et al. in [15], we introduce $\mathbf{b} = \gamma\mathbf{a}, \mathbf{c} = \mathbf{a}$. Then, $\mathbf{T} = \mathbf{A} + \mathbf{b}\mathbf{c}' = \mathbf{A} + \mathbf{b}\mathbf{c}^*$, that is, $\mathbf{T}$ is a rank-one modification of $\mathbf{A}$. As in [15], we define also the vectors $\mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g} \in \mathbb{C}_{3,1}$ according to

$$\mathbf{d} = \mathbf{A}^\dagger\mathbf{b}, \quad \mathbf{e} = (\mathbf{A}^\dagger)^* \mathbf{c}, \quad \mathbf{f} = \mathbf{Q}_\mathbf{A}\mathbf{b}, \quad \mathbf{g} = \mathbf{Q}_\mathbf{A}^* \mathbf{c}, \quad (2.14)$$

and denote the squares of the norms of the first two of them by δ and η , that is, $\delta = \|\mathbf{d}\|^2, \eta = \|\mathbf{e}\|^2$. As is seen from Theorem 2.7, the scalar $\lambda \in \mathbb{C}$ specified in [15] by $\lambda = 1 + \mathbf{c}^* \mathbf{A}^\dagger \mathbf{b}$ now takes the form $\lambda = 1 + \gamma\mathbf{a}'\mathbf{A}^\dagger\mathbf{a} = 1 + (\gamma/\beta)\|\mathbf{a}\|^2$.

Let us first consider case (ii) of the theorem, characterized, in addition to $\beta \neq 0$, by $\tau = 0$, $\lambda = 0$. On account of Theorem 1.1 in [15] this case corresponds to $\text{rk}(\mathbf{T}) = \text{rk}(\mathbf{A}) - 1$. As can be directly verified with the use of $\mathbf{A}^\dagger$ given in point (iv) of Theorem 2.7, $\mathbf{d} = (\gamma/\beta)\mathbf{a}$ and $\mathbf{e} = (1/\bar{\beta})\mathbf{a}$, from where $\delta = |\gamma/\beta|^2\|\mathbf{a}\|^2$ and $\eta = (1/|\beta|^2)\|\mathbf{a}\|^2$. Moreover, we get $\mathbf{A}^\dagger\mathbf{e} = (1/|\beta|^2)\mathbf{a}$, implying $\mathbf{d}^*\mathbf{A}^\dagger\mathbf{e} = (\bar{\gamma}/|\beta|^2\bar{\beta})\|\mathbf{a}\|^2$ and $\mathbf{A}^\dagger\mathbf{e}\mathbf{e}^* = (1/|\beta|^2\beta)\mathbf{a}\mathbf{a}'$. Furthermore, $\mathbf{d}\mathbf{d}^* = (|\gamma/\beta|^2)\mathbf{a}\mathbf{a}'$ and $\mathbf{d}\mathbf{e}^* = (\gamma/\beta^2)\mathbf{a}\mathbf{a}'$. Thus,

$$\delta^{-1}\mathbf{d}\mathbf{d}^* = \mathbf{P}_a, \quad \eta^{-1}\mathbf{A}^\dagger\mathbf{e}\mathbf{e}^* = \frac{1}{\beta}\mathbf{P}_a, \quad \delta^{-1}\eta^{-1}(\mathbf{d}^*\mathbf{A}^\dagger\mathbf{e})\mathbf{d}\mathbf{e}^* = \frac{1}{\beta}\mathbf{P}_a. \quad (2.15)$$

In consequence, from formula (2.1) in [15] we get $\mathbf{T}^\dagger = \mathbf{Q}_a\mathbf{A}^\dagger$, whence the expression for $\mathbf{T}^\dagger$ claimed in point (ii) of the theorem follows.

According to Theorem 1.1 in [15], another case which corresponds to $\text{rk}(\mathbf{T}) = \text{rk}(\mathbf{A}) - 1$ is given in point (iv) of the theorem, where $\tau \neq 0$, $\lambda = 0$. Direct calculations with the use of $\mathbf{A}^{-1}$ given in point (v) of Theorem 2.7 show that formulae (2.15) remain valid also in this case. Hence, from relationship (2.1) in [15] we obtain $\mathbf{T}^\dagger = \mathbf{Q}_a\mathbf{A}^{-1}$, leading to the expression claimed in point (iv) of the theorem.

Another conclusion originating from Theorem 1.1 in [15] is that case (iii) of the theorem, in which $\tau = 0$, $\lambda \neq 0$, corresponds to $\text{rk}(\mathbf{T}) = \text{rk}(\mathbf{A})$. Direct calculations with the use of $\mathbf{A}^\dagger$ given in point (iv) of Theorem 2.7 show that

$$\lambda^{-1}\mathbf{d}\mathbf{e}^* = \frac{\gamma}{\beta(\beta + \gamma\|\mathbf{a}\|^2)}\mathbf{a}\mathbf{a}', \quad (2.16)$$

and substituting this relationship into formula (2.2) in [15] leads to the expression for $\mathbf{T}^\dagger$ given in the theorem.

Case (v), in which $\tau \neq 0$, $\lambda \neq 0$, is left to be considered. According to the remark on p. 210 in [15], in such a situation $\mathbf{T}$ is nonsingular. Hence, $\mathbf{T}^\dagger = \mathbf{T}^{-1}$, and the validity of the formula given in the theorem can be confirmed by direct verifications of the condition $\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}_3$. $\square$

A conclusion originating from Theorem 2.8 is that every $\mathbf{T} \in \Upsilon$ satisfies $\mathbf{T} \in \Upsilon \Leftrightarrow \mathbf{T}^\dagger \in \Upsilon$, the property which was already mentioned in the remark following Theorem 2.1. Further consequences of Theorem 2.8 are in what follows.

Corollary 2.9. *Let $\mathbf{T} \in \Upsilon$, and let $\lambda = 1 + \gamma\mathbf{a}'\mathbf{A}^\dagger\mathbf{a}$, $\tau = \alpha^2\|\mathbf{a}\|^2 + \beta^2$. Then $\mathbf{P}_\mathbf{T} = \mathbf{P}_{\mathbf{T}^*}$, where*

- (i) if $\beta = 0$, then $\mathbf{P}_\mathbf{T} = \alpha\alpha^\dagger\mathbf{Q}_a + \gamma\gamma^\dagger\mathbf{P}_a$,
- (ii) if $\beta \neq 0$, $\tau = 0$, $\lambda = 0$, then $\mathbf{P}_\mathbf{T} = \mathbf{P}_2$ provided that $\alpha/\beta = i/\|\mathbf{a}\|$ and $\mathbf{P}_\mathbf{T} = \mathbf{P}_4$ provided that $\alpha/\beta = -i/\|\mathbf{a}\|$,
- (iii) if $\beta \neq 0$, $\tau = 0$, $\lambda \neq 0$, then $\mathbf{P}_\mathbf{T} = \mathbf{P}_1$ provided that $\alpha/\beta = i/\|\mathbf{a}\|$ and $\mathbf{P}_\mathbf{T} = \mathbf{P}_3$ provided that $\alpha/\beta = -i/\|\mathbf{a}\|$,
- (iv) if $\beta \neq 0$, $\tau \neq 0$, $\lambda = 0$, then $\mathbf{P}_\mathbf{T} = \mathbf{Q}_a$,
- (v) if $\beta \neq 0$, $\tau \neq 0$, $\lambda \neq 0$, then $\mathbf{P}_\mathbf{T} = \mathbf{I}_3$,

with $\mathbf{P}_k$, $k = 1, \dots, 4$, as specified in (2.3).

Proof. The corollary is established by direct calculations. $\square$

Point (v) of Theorem 2.8 enables to formulate necessary and sufficient conditions for $\mathbf{T} \in \Upsilon$ to be orthogonal.

Theorem 2.10. *Let $\mathbf{T} \in \Upsilon$ with nonzero $\alpha, \beta, \gamma \in \mathbb{R}$. Moreover, let $\tau = \alpha^2 \|\mathbf{a}\|^2 + \beta^2$. Then $\mathbf{T} \in \Upsilon$ is orthogonal if and only if*

$$\tau = 1, \quad (\beta + \gamma \|\mathbf{a}\|^2)^2 = 1. \quad (2.17)$$

Proof. The matrix $\mathbf{T}$ is orthogonal if and only if it is nonsingular and $\mathbf{T}^{-1} = \mathbf{T}'$. On account of point (v) of Theorem 2.8, it is seen that $\mathbf{T}^{-1} = \mathbf{T}'$ is equivalent to

$$\alpha = \frac{\alpha}{\tau}, \quad \beta = \frac{\beta}{\tau}, \quad \gamma = \frac{\alpha^2 - \beta\gamma}{\beta + \gamma \|\mathbf{a}\|^2}, \quad (2.18)$$

or, in other words, $\tau = 1$ and $\gamma(\beta + \gamma \|\mathbf{a}\|^2) = \alpha^2 - \beta\gamma$. Taking into account that $\tau = 1$ implies $\alpha^2 = (1/\|\mathbf{a}\|^2)(1 - \beta^2)$, the assertion follows. $\square$

Observe that the right-hand side condition in (2.17) admits two possibilities, namely, either $\beta + \gamma \|\mathbf{a}\|^2 = 1$ or $\beta + \gamma \|\mathbf{a}\|^2 = -1$. Since $\tau = 1$, Theorem 2.4 ensures that in the former situation $\mathbf{T}$ is a proper rotation, whereas in the latter one $\mathbf{T}$ is an improper rotation.

The next theorem concerns eigenspaces attributed to $\mathbf{T} \in \Upsilon$.

Theorem 2.11. *Let $\mathbf{T} \in \Upsilon$, and let $\mathcal{E}(\mu_j)$, $j = 1, 2, 3$, be the eigenspaces of $\mathbf{T}$ associated with its eigenvalues μ_j given in (2.11). Then,*

- (i) if $\alpha = 0$, then $\mathcal{E}(\mu_2) = \mathcal{R}(\mathbf{I}_3 - \gamma\gamma^\dagger \mathbf{P}_\mathbf{a})$, $\mathcal{E}(\mu_3) = \mathcal{R}(\mathbf{I}_3 - \gamma\gamma^\dagger \mathbf{P}_\mathbf{a})$,
- (ii) if $\alpha \neq 0$, then $\mathcal{E}(\mu_2) = \mathcal{R}(\mathbf{Q}_2)$ provided that $\gamma/\alpha = i/\|\mathbf{a}\|$ and $\mathcal{E}(\mu_2) = \mathcal{R}(\mathbf{Q}_1)$ otherwise and, simultaneously, $\mathcal{E}(\mu_3) = \mathcal{R}(\mathbf{Q}_4)$ provided that $\gamma/\alpha = -(i/\|\mathbf{a}\|)$ and $\mathcal{E}(\mu_3) = \mathcal{R}(\mathbf{Q}_3)$ otherwise
- (iii) if $\gamma = 0$, then $\mathcal{E}(\mu_1) = \mathcal{R}(\mathbf{I}_3 - \alpha\alpha^\dagger \mathbf{Q}_\mathbf{a})$,
- (iv) if $\gamma \neq 0$, then $\mathcal{E}(\mu_1) = \mathcal{R}(\mathbf{Q}_2)$ provided that $\alpha/\gamma = -i\|\mathbf{a}\|$, $\mathcal{E}(\mu_1) = \mathcal{R}(\mathbf{Q}_4)$ provided that $\alpha/\gamma = i\|\mathbf{a}\|$, and $\mathcal{E}(\mu_1) = \mathcal{R}(\mathbf{a})$ otherwise,

where $\mathbf{Q}_k = \mathbf{I}_3 - \mathbf{P}_k$, $k = 1, \dots, 4$, with $\mathbf{P}_k$ as specified in (2.3).

Proof. It is known that $\mathcal{E}(\mu_j) = \mathcal{R}[\mathbf{I}_3 - \mathbf{T}(\mu_j)\mathbf{T}(\mu_j)^\dagger]$, where $\mathbf{T}(\mu_j) = \mathbf{T} - \mu_j \mathbf{I}_3$, $j = 1, 2, 3$; see, for example, [3]. Clearly, for each μ_j , matrix $\mathbf{T}(\mu_j)$ can be written as

$$\mathbf{T}(\mu_j) = \alpha \mathbf{T}_\mathbf{a} + \tilde{\beta} \mathbf{I}_3 + \gamma \mathbf{a} \mathbf{a}', \quad (2.19)$$

where $\tilde{\beta} = \beta - \mu_j$. For $\mu_1 = \beta + \gamma \|\mathbf{a}\|^2$, we have $\tilde{\beta} = -\gamma \|\mathbf{a}\|^2$. By virtue of the equivalence $\tilde{\beta} = 0 \Leftrightarrow \gamma = 0$, statement (i) of Corollary 2.9 leads to point (iii) of the theorem. If, however, $\tilde{\beta} \neq 0$, that is, $\gamma \neq 0$, then $\lambda = 1 + (\gamma/\tilde{\beta})\|\mathbf{a}\|^2 = 0$, which means that cases (iii) and (v) of Corollary 2.9,

characterized by $\lambda \neq 0$, are not attainable in the present situation. Further observations are that $\tau = \alpha^2 \|\mathbf{a}\|^2 + \tilde{\beta}^2 = (\alpha^2 + \gamma^2 \|\mathbf{a}\|^2) \|\mathbf{a}\|^2$ and $\alpha/\tilde{\beta} = -(\alpha/\gamma)(1/\|\mathbf{a}\|^2)$. In view of these facts, it is seen that statements (ii) and (iv) of Corollary 2.9 lead to characterizations of $\mathcal{E}(\mu_1)$ given in point (iv) of the theorem.

Next we consider eigenvalue $\mu_2 = \beta + i\alpha\|\mathbf{a}\|$, which ensures that $\tilde{\beta}$ occurring in (2.19) is given by $\tilde{\beta} = -i\alpha\|\mathbf{a}\|$. Since $\tilde{\beta} = 0$ is equivalent to $\alpha = 0$, on account of statement (i) of Corollary 2.9 we arrive at characterization of $\mathcal{E}(\mu_2)$ given in point (i) of the theorem. On the other hand, if $\tilde{\beta} \neq 0$, that is, $\alpha \neq 0$, then τ is necessarily equal to zero, which means that cases (iv) and (v) of Corollary 2.9 are to be excluded from the present considerations. Furthermore, it is seen that $\lambda = 1 + i(\gamma/\alpha)\|\mathbf{a}\|$ and $\alpha/\tilde{\beta} = i/\|\mathbf{a}\|$. With these facts taken into account, we conclude that statements (ii) and (iii) of Corollary 2.9 lead to characterizations of $\mathcal{E}(\mu_2)$ provided in point (ii) of the theorem.

The last eigenvalue to be considered is $\mu_3 = \beta - i\alpha\|\mathbf{a}\|$, for which $\tilde{\beta} = i\alpha\|\mathbf{a}\|$. In this case, analogous arguments to the ones used with respect to μ_2 lead to the eigenspaces $\mathcal{E}(\mu_3)$ in points (i) and (ii) of the theorem. The proof is complete. $\square$

Observe that if $\alpha = 1$, $\beta = 0$, and $\gamma = 0$, then from Theorem 2.11 we get $\mathcal{E}(\mu_1) = \mathcal{R}(\mathbf{a})$, $\mathcal{E}(\mu_2) = \mathcal{R}(\mathbf{Q}_1)$, and $\mathcal{E}(\mu_3) = \mathcal{R}(\mathbf{Q}_3)$, that is, the eigenspaces of $\mathbf{T}_\mathbf{a}$ identified by Trenkler [3, Sec. 3].

Theorem 2.11 is supplemented with examples demonstrating its applicability. Let $\mathbf{a} = (1, 0, -1)'$. Then,

$$\mathbf{T}_\mathbf{a} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{a}\mathbf{a}' = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad (2.20)$$

leading to

$$\mathbf{T} = \begin{pmatrix} \beta + \gamma & \alpha & -\gamma \\ -\alpha & \beta & -\alpha \\ -\gamma & \alpha & \beta + \gamma \end{pmatrix} \quad (2.21)$$

of eigenvalues $\mu_1 = \beta + 2\gamma$, $\mu_2 = \beta + i\sqrt{2}\alpha$, $\mu_3 = \beta - i\sqrt{2}\alpha$. From the right-hand side formula in (2.20) we get

$$\mathbf{P}_\mathbf{a} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \quad (2.22)$$

and the projectors $\mathbf{Q}_k$, $k = 1, \dots, 4$, involved in Theorem 2.11 are of the forms

$$\begin{aligned} \mathbf{Q}_1 &= \begin{pmatrix} \frac{1}{4} & -i\frac{\sqrt{2}}{4} & \frac{1}{4} \\ i\frac{\sqrt{2}}{4} & \frac{1}{2} & i\frac{\sqrt{2}}{4} \\ \frac{1}{4} & -i\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix}, & \mathbf{Q}_2 &= \begin{pmatrix} \frac{3}{4} & -i\frac{\sqrt{2}}{4} & -\frac{1}{4} \\ i\frac{\sqrt{2}}{4} & \frac{1}{2} & i\frac{\sqrt{2}}{4} \\ -\frac{1}{4} & -i\frac{\sqrt{2}}{4} & \frac{3}{4} \end{pmatrix}, \\ \mathbf{Q}_3 &= \begin{pmatrix} \frac{1}{4} & i\frac{\sqrt{2}}{4} & \frac{1}{4} \\ -i\frac{\sqrt{2}}{4} & \frac{1}{2} & -i\frac{\sqrt{2}}{4} \\ \frac{1}{4} & i\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix}, & \mathbf{Q}_4 &= \begin{pmatrix} \frac{3}{4} & i\frac{\sqrt{2}}{4} & -\frac{1}{4} \\ -i\frac{\sqrt{2}}{4} & \frac{1}{2} & -i\frac{\sqrt{2}}{4} \\ -\frac{1}{4} & i\frac{\sqrt{2}}{4} & \frac{3}{4} \end{pmatrix}. \end{aligned} \quad (2.23)$$

Hence, from Theorem 2.11 we obtain what follows.

If $\alpha = 0$, then $\mathcal{E}(\mu_2) = \mathcal{E}(\mu_3) = \mathbb{C}_{3,1}$ provided that $\gamma = 0$ and

$$\mathcal{E}(\mu_2) = \mathcal{E}(\mu_3) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \quad (2.24)$$

otherwise.

Next, if $\alpha \neq 0$, then

$$\mathcal{E}(\mu_2) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -i\sqrt{2} \end{pmatrix} \right\} \quad (2.25)$$

provided that $\gamma/\alpha = i(\sqrt{2}/2)$ and

$$\mathcal{E}(\mu_2) = \text{Span} \left\{ \begin{pmatrix} 1 \\ i\sqrt{2} \\ 1 \end{pmatrix} \right\} \quad (2.26)$$

otherwise and, simultaneously,

$$\mathcal{E}(\mu_3) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ i\sqrt{2} \end{pmatrix} \right\} \quad (2.27)$$

provided that $\gamma/\alpha = -i(\sqrt{2}/2)$ and

$$\mathcal{E}(\mu_3) = \text{Span} \left\{ \begin{pmatrix} 1 \\ -i\sqrt{2} \\ 1 \end{pmatrix} \right\} \quad (2.28)$$

otherwise.

Further, if $\gamma = 0$, then $\mathcal{E}(\mu_1) = \mathbb{C}_{3,1}$ provided that $\alpha = 0$ and $\mathcal{E}(\mu_1) = \mathcal{R}(\mathbf{a})$ otherwise.

Finally, if $\gamma \neq 0$, then

$$\mathcal{E}(\mu_1) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -i\sqrt{2} \end{pmatrix} \right\} \quad (2.29)$$

provided that $\alpha/\gamma = -i\sqrt{2}$,

$$\mathcal{E}(\mu_1) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ i\sqrt{2} \end{pmatrix} \right\} \quad (2.30)$$

provided that $\alpha/\gamma = i\sqrt{2}$, and $\mathcal{E}(\mu_1) = \mathcal{R}(\mathbf{a})$ otherwise.

Further consequences of Theorem 2.11 deal with the proper rotation matrices; for more detailed discussion see [16]. G. Trenkler and D. Trenkler [6, Sec. 3] identified three types of proper rotations, namely, Type I, covering matrices of the form $\mathbf{R} = \mathbf{I}_3$; Type II, covering matrices of the form $\mathbf{R} = 2\mathbf{a}\mathbf{a}' - \mathbf{I}_3$, where $\|\mathbf{a}\| = 1$; Type III, covering matrices of the form $\mathbf{R} = \mathbf{I}_3 + f_{\mathbf{a}}(\mathbf{T}_{\mathbf{a}} + \mathbf{T}_{\mathbf{a}}^2)$, where $f_{\mathbf{a}} = 2/(\|\mathbf{a}\|^2 + 1)$. Moreover, it was pointed out in [6] that each proper rotation can be attributed to one of these types. Direct calculations show that rotations of Type I are obtained from the representation (1.5) by taking $\alpha = 0$, $\beta = 1$, $\gamma = 0$; Type II are obtained from the representation (1.5) by taking $\alpha = 0$, $\beta = -1$, $\gamma = 2$, and $\|\mathbf{a}\| = 1$; Type III are obtained from the representation (1.5) by taking $\alpha = f_{\mathbf{a}}$, $\beta = 1 - f_{\mathbf{a}}\|\mathbf{a}\|^2$, $\gamma = f_{\mathbf{a}}$, where $f_{\mathbf{a}} = 2/(\|\mathbf{a}\|^2 + 1)$. Combining these observations with Corollary 2.6 leads to the conclusion that rotations of Type I have eigenvalues $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = 1$; Type II have eigenvalues $\mu_1 = 1$, $\mu_2 = -1$, $\mu_3 = -1$; Type III have eigenvalues $\mu_1 = 1$, $\mu_2 = 1 - f_{\mathbf{a}}\|\mathbf{a}\|(\|\mathbf{a}\| - i)$, $\mu_3 = 1 - f_{\mathbf{a}}\|\mathbf{a}\|(\|\mathbf{a}\| + i)$. Furthermore, from Theorem 2.11 we obtain the following characterizations of the eigenspaces.

Corollary 2.12. *Let $\mathbf{R} \in \mathbf{Y}$ be a proper rotation, and let $\mathcal{E}(\mu_j)$, $j = 1, 2, 3$, be the eigenspaces of $\mathbf{R}$ associated with its eigenvalues μ_j . Then,*

- (i) *if $\mathbf{R}$ is of Type I, then $\mathcal{E}(\mu_1) = \mathbb{C}_{3,1}$, $\mathcal{E}(\mu_2) = \mathbb{C}_{3,1}$, $\mathcal{E}(\mu_3) = \mathbb{C}_{3,1}$,*
- (ii) *if $\mathbf{R}$ is of Type II, then $\mathcal{E}(\mu_1) = \mathcal{R}(\mathbf{a})$, $\mathcal{E}(\mu_2) = \mathcal{R}(\mathbf{Q}_a)$, $\mathcal{E}(\mu_3) = \mathcal{R}(\mathbf{Q}_a)$,*
- (iii) *if $\mathbf{R}$ is of Type III, then $\mathcal{E}(\mu_1) = \mathcal{R}(\mathbf{a})$, $\mathcal{E}(\mu_2) = \mathcal{R}(\mathbf{Q}_1)$, $\mathcal{E}(\mu_3) = \mathcal{R}(\mathbf{Q}_3)$, where $\mathbf{Q}_k = \mathbf{I}_3 - \mathbf{P}_k$, $k = 1, 3$, with $\mathbf{P}_k$ as specified in (2.3).*

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