Research Article

A Family of Mappings Associated with Hadamard's Inequality on a Hypercube

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This paper provides a method to generalize and refine the Hadamard's inequality on a hypercube in n -dimensional Euclidean space by establishing a family of mappings.

1. Introduction

For a convex function f defined on an open interval I of real numbers and $a, b \in I$ with $a < b$, then the following inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} \quad (1.1)$$

is known as the classical Hadamard's inequality for the convex function.

In [1], Dragomir established the following mapping:

$$H : [0, 1] \longrightarrow \mathbf{R}, \quad H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx, \quad (1.2)$$

then the mapping H is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have

$$f\left(\frac{a+b}{2}\right) = H(0) \leq H(t) \leq H(1) = \frac{1}{b-a} \int_a^b f(x) dx, \quad (1.3)$$

which is a refinement to the left side of (1.1).

In [2], Yang and Hong established the following mapping:

$$G : [0, 1] \longrightarrow \mathbf{R}, \quad G(t) = \frac{1}{2(b-a)} \int_a^b \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right) \right] dx, \quad (1.4)$$

then the mapping G is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have

$$\frac{1}{b-a} \int_a^b f(x) dx = G(0) \leq G(t) \leq G(1) = \frac{f(a) + f(b)}{2}, \quad (1.5)$$

which is a refinement to the right side of (1.1).

In this paper, we first establish a mapping in Section 2, defined by

$$A_1 : [0, 1] \longrightarrow \mathbf{R}, \quad A_1(t) = \frac{1}{2} \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) + f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \right], \quad (1.6)$$

then the mapping A_1 is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$ we have

$$f\left(\frac{a+b}{2}\right) = A_1(0) \leq A_1(t) \leq A_1(1) = \frac{f(a) + f(b)}{2}. \quad (1.7)$$

Furthermore, there is a point $\xi \in (0, 1)$ so that $A_1(\xi) = (1/(b-a)) \int_a^b f(x) dx$. Thus, (1.1) has been proved based on the properties of A_1 , and the inequality (1.7) is a refinement to the both sides of (1.1).

In Section 3, we also use an analogue of (1.6) to obtain the following Hadamard's type inequality on a square:

$$f(a, b) \leq \frac{1}{h^2} \iint_{\substack{|x-a| \leq h/2 \\ |y-b| \leq h/2}} f(x, y) dx dy \leq \frac{1}{4h} \oint_{\substack{|x-a|=h/2 \\ |y-b|=h/2}} f(x, y) ds, \quad (1.8)$$

which is a generalization of (1.1) to two-dimensional Euclidean space.

In Section 4, by establishing another analogue of (1.6) on a hypercube, the following Hadamard's inequality is also pointed out:

$$f(c) \leq \frac{1}{h^n} \int_{\|x-c\|_\infty \leq h/2} f(x) dx \leq \frac{1}{2nh^{n-1}} \oint_{\{x: |x_i-c_i|=h/2\}} f(x) ds, \quad (1.9)$$

which is a generalization of (1.1) to n -dimensional Euclidean space.

All the above Hadamard's inequalities (1.1), (1.8), and (1.9) have a uniform format

$$f(c) \leq \frac{1}{\text{Vol}(\Omega)} \int_{\Omega} f(x) dx \leq \frac{1}{\text{Area}(\Omega)} \oint_{\partial\Omega} f(x) ds. \quad (1.10)$$

Thus, by establishing a family of the mappings, we have provided a method to get Hadamard's inequality of the convex function on a hypercube in n -dimensional Euclidean space. The motivation for the present work is from many generalizations and refinements of the classical inequality (1.1) in [1–8].

2. Properties of the Mapping

The main properties of the mapping A_1 defined by (1.6) are embodied in the following theorem.

Theorem 2.1. *If f is a convex function defined on an open interval I of real numbers, $a, b \in I$ with $a < b$, and the mapping A_1 is defined by (1.6), then, one has*

(i) *the mapping A_1 is convex on $[0, 1]$,*

(ii) *one has the bounds:*

$$\begin{aligned} \inf_{t \in [0,1]} A_1(t) &= A_1(0) = f\left(\frac{a+b}{2}\right), \\ \sup_{t \in [0,1]} A_1(t) &= A_1(1) = \frac{f(a) + f(b)}{2}, \end{aligned} \quad (2.1)$$

(iii) *the mapping A_1 is increasing on $[0, 1]$,*

(iv) *there is a point $\xi \in (0, 1)$ so that*

$$A_1(\xi) = \frac{1}{b-a} \int_a^b f(x) dx, \quad (2.2)$$

(v) *one obtains, the classical inequality (1.1) and has a refinement of (1.1):*

$$A_1(0) \leq A_1(\eta) \leq A_1(\xi) \leq A_1(\mu) \leq A_1(1), \quad \forall \eta \in (0, \xi), \mu \in (\xi, 1). \quad (2.3)$$

Proof. (i) Let $t_1, t_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, then we have

$$\begin{aligned} &A_1(\alpha t_1 + \beta t_2) \\ &= \frac{1}{2} \left[f\left(\frac{1+\alpha t_1+\beta t_2}{2}a + \frac{1-\alpha t_1-\beta t_2}{2}b\right) + f\left(\frac{1-\alpha t_1-\beta t_2}{2}a + \frac{1+\alpha t_1+\beta t_2}{2}b\right) \right] \\ &= \frac{1}{2} \left\{ f\left[\alpha\left(\frac{1+t_1}{2}a + \frac{1-t_1}{2}b\right) + \beta\left(\frac{1+t_2}{2}a + \frac{1-t_2}{2}b\right)\right] \right. \\ &\quad \left. + f\left[\alpha\left(\frac{1-t_1}{2}a + \frac{1+t_1}{2}b\right) + \beta\left(\frac{1-t_2}{2}a + \frac{1+t_2}{2}b\right)\right] \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2} \left[\alpha f \left(\frac{1+t_1}{2}a + \frac{1-t_1}{2}b \right) + \beta f \left(\frac{1+t_2}{2}a + \frac{1-t_2}{2}b \right) \right. \\
&\quad \left. + \alpha f \left(\frac{1-t_1}{2}a + \frac{1+t_1}{2}b \right) + \beta f \left(\frac{1-t_2}{2}a + \frac{1+t_2}{2}b \right) \right] \\
&= \frac{1}{2} \alpha \left[f \left(\frac{1+t_1}{2}a + \frac{1-t_1}{2}b \right) + f \left(\frac{1-t_1}{2}a + \frac{1+t_1}{2}b \right) \right] \\
&\quad + \frac{1}{2} \beta \left[f \left(\frac{1+t_2}{2}a + \frac{1-t_2}{2}b \right) + f \left(\frac{1-t_2}{2}a + \frac{1+t_2}{2}b \right) \right] \\
&= \alpha A_1(t_1) + \beta A_1(t_2),
\end{aligned} \tag{2.4}$$

which proves the convexity of A_1 on $[0, 1]$.

(ii) By the convexity of f on the interval $[a, b]$, we have

$$\begin{aligned}
A_1(t) &\leq \frac{1}{2} \left[\frac{1+t}{2} f(a) + \frac{1-t}{2} f(b) + \frac{1-t}{2} f(a) + \frac{1+t}{2} f(b) \right] = \frac{f(a) + f(b)}{2} = A_1(1), \\
A_1(t) &\geq f \left[\frac{1}{2} \left(\frac{1+t}{2}a + \frac{1-t}{2}b + \frac{1-t}{2}a + \frac{1+t}{2}b \right) \right] = f \left(\frac{a+b}{2} \right) = A_1(0),
\end{aligned} \tag{2.5}$$

the bounds (2.1) hold.

(iii) Let $0 < t_1 < t_2 \leq 1$. By the convexity of the mapping A_1 , we have

$$\frac{A_1(t_2) - A_1(t_1)}{t_2 - t_1} \geq \frac{A_1(t_1) - A_1(0)}{t_1} \geq 0. \tag{2.6}$$

Since we have proved that $A_1(t_1) \geq A_1(0)$ for all $t_1 \in (0, 1]$ in (ii), the monotonicity of A_1 is established.

(iv) We define the function ϕ on $[0, 1]$ by

$$\phi(t) = \int_{((1+t)/2)a + ((1-t)/2)b}^{((1-t)/2)a + ((1+t)/2)b} f(x) dx - t \int_a^b f(x) dx, \tag{2.7}$$

then

$$\begin{aligned}
\phi(0) &= \phi(1) = 0, \\
\phi'(t) &= (b-a)A_1(t) - \int_a^b f(x) dx.
\end{aligned} \tag{2.8}$$

According to the Rolle's mean-value theorem, there is a point $\xi \in (0, 1)$ so that

$$0 = \phi'(\xi) = (b - a)A_1(\xi) - \int_a^b f(x)dx. \quad (2.9)$$

Thus, (2.2) holds.

(v) Considering the properties (ii), (iii), and (iv) above, we have proved the Hadamard's inequality (1.1) and the inequality (2.3). $\square$

3. The Hadamard's Inequality on a Square

We consider a convex function with two variables defined on a square in two-dimensional Euclidean space.

Theorem 3.1. *If f is a convex function defined on an open region $D \subseteq \mathbf{R}^2$, the square*

$$\left\{ (x, y) : |x - a| \leq \frac{h}{2}, |y - b| \leq \frac{h}{2} \right\} \subseteq D, \quad (3.1)$$

and the mapping A_2 is defined by

$$A_2 : [0, 1] \longrightarrow \mathbf{R}, \quad A_2(t) = \frac{1}{4h} \oint_{\substack{|u|=h/2 \\ |v|=h/2}} f(a + tu, b + tv) ds, \quad (3.2)$$

which is a parameter curvilinear integral with respect to arc length, then one has

- (i) the mapping A_2 is convex on $[0, 1]$,
- (ii) one has the bounds

$$\begin{aligned} \inf_{t \in [0, 1]} A_2(t) &= A_2(0) = f(a, b), \\ \sup_{t \in [0, 1]} A_2(t) &= A_2(1) = \frac{1}{4h} \oint_{\substack{|x-a|=h/2 \\ |y-b|=h/2}} f(x, y) ds, \end{aligned} \quad (3.3)$$

- (iii) the mapping A_2 is increasing on $[0, 1]$,
- (iv) there is a point $\xi \in (0, 1)$ so that

$$A_2(\xi) = \frac{1}{h^2} \iint_{\substack{|x-a| \leq h/2 \\ |y-b| \leq h/2}} f(x, y) dx dy, \quad (3.4)$$

- (v) one has the Hadamard's type inequality (1.8) and a refinement

$$A_2(0) \leq A_2(\eta) \leq A_2(\xi) \leq A_2(\mu) \leq A_2(1), \quad \forall \eta \in (0, \xi), \mu \in (\xi, 1). \quad (3.5)$$

Proof. For a fixed $t \in [0, 1]$, let $u = z$ if $|v| = h/2$, and $v = z$ if $|u| = h/2$, we have

$$\begin{aligned} A_2(t) &= \frac{1}{4h} \oint_{\substack{|u|=h/2 \\ |v|=h/2}} f(a+tu, b+tv) ds \\ &= \frac{1}{4h} \int_{-h/2}^{h/2} \left[f\left(a+tz, b-\frac{th}{2}\right) + f\left(a+tz, b+\frac{th}{2}\right) \right. \\ &\quad \left. + f\left(a-\frac{th}{2}, b+tz\right) + f\left(a+\frac{th}{2}, b+tz\right) \right] dz, \end{aligned} \quad (3.6)$$

which indicates the mapping $A_2(t)$ is continuous on $[0, 1]$.

(i) Let $t_1, t_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. By the convexity of f , we have

$$\begin{aligned} &f\left[a + (\alpha t_1 + \beta t_2)z, b - \frac{(\alpha t_1 + \beta t_2)h}{2}\right] \\ &= f\left[\alpha(a + t_1 z) + \beta(a + t_2 z), \alpha\left(b - \frac{t_1 h}{2}\right) + \beta\left(b - \frac{t_2 h}{2}\right)\right] \\ &\leq \alpha f\left(a + t_1 z, b - \frac{t_1 h}{2}\right) + \beta f\left(a + t_2 z, b - \frac{t_2 h}{2}\right). \end{aligned} \quad (3.7)$$

Similarly, we have

$$\begin{aligned} &f\left[a + (\alpha t_1 + \beta t_2)z, b + \frac{(\alpha t_1 + \beta t_2)h}{2}\right] \\ &\leq \alpha f\left(a + t_1 z, b + \frac{t_1 h}{2}\right) + \beta f\left(a + t_2 z, b + \frac{t_2 h}{2}\right), \\ &f\left[a - \frac{(\alpha t_1 + \beta t_2)h}{2}, b + (\alpha t_1 + \beta t_2)z\right] \\ &\leq \alpha f\left(a - \frac{t_1 h}{2}, b + t_1 z\right) + \beta f\left(a - \frac{t_2 h}{2}, b + t_2 z\right), \\ &f\left[a + \frac{(\alpha t_1 + \beta t_2)h}{2}, b + (\alpha t_1 + \beta t_2)z\right] \\ &\leq \alpha f\left(a + \frac{t_1 h}{2}, b + t_1 z\right) + \beta f\left(a + \frac{t_2 h}{2}, b + t_2 z\right). \end{aligned} \quad (3.8)$$

Thus,

$$A_2(\alpha t_1 + \beta t_2) \leq \alpha A_2(t_1) + \beta A_2(t_2), \quad (3.9)$$

which proves the convexity of A_2 on $[0, 1]$.

(ii) By the convexity of f and (3.6), we have

$$A_2(t) \geq \frac{1}{2th} \left(\int_{a-th/2}^{a+th/2} f(x, b) dx + \int_{b-th/2}^{b+th/2} f(a, y) dy \right) \geq f(a, b) = A_2(0), \quad (3.10)$$

the last inequality holds owing to (1.1); as the convexity of A_2 we have

$$A_2(t) \leq tA_2(1) + (1-t)A_2(0) \leq A_2(1), \quad (3.11)$$

and note that

$$A_2(1) = \frac{1}{4h} \oint_{\substack{|u|=h/2 \\ |v|=h/2}} f(a+u, b+v) ds = \frac{1}{4h} \oint_{\substack{|x-a|=h/2 \\ |y-b|=h/2}} f(x, y) ds, \quad (3.12)$$

the bounds (3.3) hold.

(iii) The monotonicity of A_2 follows as in the proof of Theorem 2.1, and we omit the detail.

(iv) We define the function ϕ on $[0, 1]$ by

$$\phi(t) = \iint_{\substack{|x-a| \leq th/2 \\ |y-b| \leq th/2}} f(x, y) dx dy - t^2 \iint_{\substack{|x-a| \leq h/2 \\ |y-b| \leq h/2}} f(x, y) dx dy, \quad (3.13)$$

then

$$\begin{aligned} \phi(0) &= \phi(1) = 0, \\ \phi'(t) &= \frac{h}{2} \left(\int_{a-th/2}^{a+th/2} \left[f\left(x, b + \frac{th}{2}\right) + f\left(x, b - \frac{th}{2}\right) \right] dx \right. \\ &\quad \left. + \int_{b-th/2}^{b+th/2} \left[f\left(a + \frac{th}{2}, y\right) + f\left(a - \frac{th}{2}, y\right) \right] dy \right) \\ &\quad - 2t \iint_{\substack{|x-a| \leq h/2 \\ |y-b| \leq h/2}} f(x, y) dx dy, \\ &= \frac{h}{2} \oint_{\substack{|x-a|=th \\ |y-b|=th}} f(x, y) ds - 2t \iint_{\substack{|x-a| \leq h/2 \\ |y-b| \leq h/2}} f(x, y) dx dy. \end{aligned} \quad (3.14)$$

Let $x = a + tu$ and $y = b + tv$, we have

$$\begin{aligned} \phi'(t) &= \frac{th}{2} \oint_{\substack{|u|=h/2 \\ |v|=h/2}} f(a+tu, b+tv) ds - 2t \iint_{\substack{|x-a| \leq h/2 \\ |y-b| \leq h/2}} f(x, y) dx dy \\ &= 2th^2 A_2(t) - 2t \iint_{\substack{|x-a| \leq h/2 \\ |y-b| \leq h/2}} f(x, y) dx dy. \end{aligned} \quad (3.15)$$

According to the Rolle's mean-value theorem, there is a point $\xi \in (0, 1)$ so that

$$0 = \phi'(\xi) = 2h^2 \xi A_2(\xi) - 2\xi \iint_{\substack{|x-a| \leq h/2 \\ |y-b| \leq h/2}} f(x, y) dx dy, \quad (3.16)$$

thus, (3.4) holds.

(v) Considering the properties (ii), (iii), and (iv) above, we have proved the Hadamard's type inequality (1.8) and the inequality (3.5). $\square$

4. Remark

In recent years, several authors have discussed the problem of extending the classical inequality (1.1) to a convex function f on a general convex body Ω of n -dimensional space [6–9]. For the left side of (1.1) (i.e., Jensen's inequality), the following inequality was showed in [6, 9] (also see [7]):

$$f(c) \leq \frac{1}{\mu(\Omega)} \int_{\Omega} f(x) dx, \quad (4.1)$$

where c is the barycenter of Ω , and $\mu(\Omega)$ denotes n -dimensional Lebesgue's measure of Ω .

It is a much more delicate problem to extend the right side of (1.1) on a general convex body of n -dimensional space [7]. On a closed hyperball of the n -dimensional space, de la Cal and Cárcamo [7] obtained the inequality

$$\frac{1}{\text{Vol}(K)} \int_K f(x) dx \leq \frac{1}{\text{Area}(K^*)} \oint_{K^*} f(x) ds, \quad (4.2)$$

where $\text{Vol}(K)$ denotes the volumes of K , and $\text{Area}(K^*)$ denotes the areas of the hypersphere K^* , for $n = 2, 3$ which was early obtained by Dragomir [4, 5] based on Calculus.

When the convex function f was supposed to be differentiable on a closed convex body Ω , the following inequality was proved in [9] based on Stokes' Formula

$$\frac{1}{\text{Vol}(\Omega)} \int_{\Omega} f(x) dx \leq \frac{1}{n+1} \frac{1}{\text{Area}(\partial\Omega)} \oint_{\partial\Omega} \left(1 + \frac{\text{Area}(\partial\Omega)}{\text{Vol}(\Omega)} (x - \theta) \cdot \pi(x) \right) f(x) ds, \quad (4.3)$$

where $\partial\Omega$ denotes the piecewisely smooth boundary of Ω , θ denotes the barycenter of $\partial\Omega$, and $\pi(x)$ denotes the exterior unit normal vector in the point x .

For a closed hyperball Ω , the inequality (4.2) is able to be deduced from (4.3); and for a hypercube Ω , the inequality (1.9) is also able to be deduced from (4.3); however, these are on the condition that f is differentiable.

Now, let f be a convex function defined on an open region $D \subseteq \mathbf{R}^n$, the hypercube $\{x : \|x - c\|_{\infty} \leq h/2\} \subseteq D$, the mapping A_n is defined by

$$A_n : [0, 1] \longrightarrow \mathbf{R}, \quad A_n(t) = \frac{1}{2nh^{n-1}} \oint_{\|u\|_{\infty}=h/2} f(c + tu) ds, \quad (4.4)$$

which is a parameter surface integral with respect to area of $n - 1$ dimension, and let

$$\phi(t) = \int_{\|x-c\|_{\infty} \leq th/2} f(x)dx - t^n \int_{\|x-c\|_{\infty} \leq h/2} f(x)dx, \quad (4.5)$$

in the same way as the proof of Theorem 3.1, by the induction we will be able to obtain the inequality (1.9).

Some mappings associated to Hadamard's inequality have been established in [1–5], which refine the classical Hadamard's inequality. These mappings are based on Hadamard's inequality. However, our method in this paper to establish mappings is different from theirs: we first proved (1.1) by the properties of A_1 and subsequently established A_2 and proved (1.8) based on (1.1). Thus, using the induction, we will be able to establish the mapping A_n and prove Hadamard's inequality on a hypercube.

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