Research Article

Note on Neighborhoods of Some Classes of Analytic Functions with Negative Coefficients

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In this paper, we prove several inclusion relations associated with the (n, δ) neighborhoods of some subclasses of starlike and convex functions with negative coefficients.

1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U ,

$$A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\} \quad (1.1)$$

and $S = \{f \in A : f \text{ is univalent in } U\}$.

In [1], the subfamily T of S consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \quad a_j \geq 0, \quad j = 2, 3, \dots, \quad z \in U, \quad (1.2)$$

was introduced, where

$$T = \left\{ f \in S : f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \quad a_j \geq 0, \quad j \geq 2, \quad z \in U \right\}. \quad (1.3)$$

For $f(z)$ belonging to A , Sălăgean [2] has introduced the following operator called the Sălăgean operator:

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z), \quad D^n f(z) = D(D^{n-1} f(z)), \quad n \in \mathbb{N}^*. \quad (1.4)$$

The object of the present paper is to derive some properties of neighborhoods for some subclasses of analytic functions with negative coefficients, which we have already studied.

2. Preliminary Results

Remark 2.1. In [3], we have introduced the following operator concerning the functions of form (1.2):

$$D_\lambda^\beta : A \longrightarrow A, \quad D_\lambda^\beta f(z) = z - \sum_{j=n+1}^{\infty} [1 + (j-1)\lambda]^\beta a_j z^j. \quad (2.1)$$

Definition 2.2 (see [3]). Let $f \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, and $a_j \geq 0$, $j = 2, 3, \dots$, $z \in U$. We say that f is in the class $TL_\beta(\alpha)$ if

$$\operatorname{Re} \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} > \alpha, \quad \alpha \in [0, 1), \lambda \geq 0, \beta \geq 0, z \in U. \quad (2.2)$$

Theorem 2.3 (see [3]). Let $\alpha \in [0, 1)$, $\lambda \geq 0$, $n \in \mathbb{N}^*$, and $\beta \geq 0$. The function $f \in A(n)$ of the form (1.2) is in the class $T^*L_{\beta,n}(\alpha)$ if and only if

$$\sum_{j=n+1}^{\infty} \left[(1 + (j-1)\lambda)^\beta (1 + (j-1)\lambda - \alpha) \right] \cdot a_j < 1 - \alpha. \quad (2.3)$$

Remark 2.4. Using the condition (1.3), we can to prove that

$$T^*L_{\beta+1,n}(\alpha) \subset T^*L_{\beta,n}(\alpha), \quad (2.4)$$

$\beta \geq 0$, $\alpha \in [0, 1)$, $\lambda \geq 0$, and $n \in \mathbb{N}^*$.

Definition 2.5 (see [4]). Let $f \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, and $a_j \geq 0$, $j = 2, 3, \dots$, $z \in U$. We say that f is in the class $T^cL_\beta(\alpha)$ if

$$\operatorname{Re} \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} > \alpha, \quad \alpha \in [0, 1), \lambda \geq 0, \beta \geq 0, z \in U. \quad (2.5)$$

Theorem 2.6 (see [4]). Let $\alpha \in [0, 1)$, $\lambda \geq 0$, and $\beta \geq 0$. The function $f \in T$ of the form (1.2) is in the class $T^c L_\beta(\alpha)$ if and only if

$$\sum_{j=2}^{\infty} \left[(1 + (j-1)\lambda)^{\beta+1} (1 + (j-1)\lambda - \alpha) \right] a_j < 1 - \alpha. \quad (2.6)$$

Remark 2.7. Using the condition (2.2), we can prove that

$$T^c L_{\beta+1,n}(\alpha) \subset T^c L_{\beta,n}(\alpha), \quad (2.7)$$

$\beta \geq 0$, $\alpha \in [0, 1)$, $\lambda \geq 0$, and $n \in \mathbb{N}^*$.

Let $A(n)$ be the class of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0; n \in \mathbb{N} - \{0\}), \quad (2.8)$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. For any $f(z) \in A(n)$ and $\delta \geq 0$, we define

$$N_{n,\delta} = \left\{ g \in A(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k, \sum_{k=n+1}^{\infty} k \cdot |a_k - b_k| \leq \delta \right\}, \quad (2.9)$$

which was called (n, δ) -neighborhood of $f(z)$. So, for $e(z) = z$, we observe that

$$N_{n,\delta} = \left\{ g \in A(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k, \sum_{k=n+1}^{\infty} k \cdot |b_k| \leq \delta \right\}. \quad (2.10)$$

The concept of neighborhoods was first introduced by Goodman in [5] and then generalized by Ruscheweyh in [6].

We propose to investigate the (n, δ) -neighborhoods of the subclasses $T^* L_{\beta,n}(\alpha)$ and $T^c L_{\beta,n}(\alpha)$ of the class $A(n)$ of normalized analytic functions in U with negative coefficients, where $T^* L_{\beta,n}(\alpha)$ is the subclass of n -starlike functions with negative coefficients of order α and type β introduced in [3] and $T^c L_{\beta,n}(\alpha)$ is the subclass of n -convex functions with negative coefficients of order α and type β studied in [4].

3. Main Results

We start by considering the linear operator (2.1) and conclude the study with several general inclusion relations associated with the (n, δ) neighborhoods for some subclasses of starlike and convex functions with negative coefficients.

Using the inequality (1.3) from Definition 2.5 and the inequality (2.1) from Definition 2.2, we obtain the subclasses $T^* L_{\beta,n}(\alpha)$, and $T^c L_{\beta,n}(\alpha)$ and from Theorem 2.3, we derive the corresponding results.

Theorem 3.1. *Let*

$$\delta = \frac{(1 - \alpha)(n + 1)}{(1 + n\lambda)^\beta (1 + n\lambda - \alpha)}, \quad (3.1)$$

where $\alpha \in [0, 1)$, $\lambda \geq 0$, $\beta \geq 0$, and $n \in \mathbb{N}^*$; then

$$T^*L_{\beta, n}(\alpha) \subset N_{n, \delta}(e). \quad (3.2)$$

Proof. For $f(z) \in T^*L_{\beta, n}(\alpha)$ and making use of the condition (2.2), we obtain $[(1 + n\lambda)^\beta (1 + n\lambda - \alpha)] \sum_{j=n+1}^{\infty} a_j < 1 - \alpha$, so that

$$\sum_{j=n+1}^{\infty} a_j < \frac{1 - \alpha}{(1 + n\lambda)^\beta (1 + n\lambda - \alpha)}. \quad (3.3)$$

On the other hand, we also find from (2.2) and (3.3) that

$$\begin{aligned} \sum_{j=n+1}^{\infty} \left[(1 + (j-1)\lambda)^\beta (1 + (j-1)\lambda - \alpha) \right] a_j &< 1 - \alpha, \\ (1 + n\lambda)^\beta \sum_{j=n+1}^{\infty} (1 + (j-1)\lambda - \alpha) a_j &< 1 - \alpha, \\ \lambda(1 + n\lambda)^\beta \sum_{j=n+1}^{\infty} j \cdot a_j &< 1 - \alpha - \left[(1 - \lambda - \alpha)(1 + n\lambda)^\beta \right] \cdot \sum_{j=n+1}^{\infty} a_j. \end{aligned} \quad (3.4)$$

Thus,

$$\sum_{j=n+1}^{\infty} j \cdot a_j < \frac{(1 - \alpha)(n + 1)}{(1 + n\lambda)^\beta (1 + n\lambda - \alpha)} = \delta, \quad (3.5)$$

which in view of definition (2.10), proves Theorem 3.1. $\square$

Remark 3.2. If $\lambda = 1$ and $\beta = 1$, we obtain that $\sum_{j=n+1}^{\infty} j \cdot a_j < (1 - \alpha)/(1 + n - \alpha) = \delta$ for $D_1^1 f(z) = zf'(z)$, $f(z) \in T_n^*(\alpha)$.

In a similar way, applying (2.5) instead of (2.2), we can prove the following.

Theorem 3.3. *Let*

$$\delta = \frac{(1 - \alpha)(n + 1)}{(1 + n\lambda)^{\beta+1} (1 + n\lambda - \alpha)}, \quad (3.6)$$

where $\alpha \in [0, 1)$, $\lambda \geq 0$, $\beta \geq 0$, and $n \in \mathbb{N}^*$; then

$$T^c L_{\beta,n}(\alpha) \subset N_{n,\delta}(e). \quad (3.7)$$

Consequently, we determine the neighborhood for each of the classes $T^{*(v)} L_{\beta,n}(\alpha)$ and $T^{c(v)} L_{\beta,n}(\alpha)$, which we define as follows. A function $f(z) \in A(n)$ defined by (2.8) is said to be in the class $T^{*(v)} L_{\beta,n}(\alpha)$ if there exists a function $g(z) \in T^* L_{\beta,n}(\alpha)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \alpha, \quad z \in U, \quad \alpha \in [0, 1). \quad (3.8)$$

Analogously, a function $f(z) \in A(n)$ defined by (2.8) is said to be in the class $T^{c(v)} L_{\beta,n}(\alpha)$ if there exists a function $g(z) \in T^c L_{\beta,n}(\alpha)$ such the inequality (3.8) holds.

Remark 3.4. If $\lambda = 1$ and $\beta = 1$, we obtain that $\sum_{j=n+1}^{\infty} j \cdot a_j < (1 - \alpha) / ((1 + n)(1 + n - \alpha)) = \delta$ for $D_1^1 f(z) = z f'(z)$, $f(z) \in T_n^c(\alpha)$.

Further, we consider the inclusion relations just studied and generalize them by taking into account the relation (2.9).

Theorem 3.5. *If $g(z) \in T^* L_{\beta,n}(\alpha)$ and*

$$v = 1 - \frac{\delta \cdot \left[(1 + n\lambda)^\beta (1 + n\lambda - \alpha) \right]}{(n + 1) \cdot \left[(1 + n\lambda)^\beta (1 + n\lambda - \alpha) - (1 - \alpha) \right]}, \quad (3.9)$$

then

$$N_{n,\delta}(g) \subset T^{*(v)} L_{\beta,n}(\alpha), \quad (3.10)$$

$\alpha \in [0, 1)$, $\lambda \geq 0$, $\beta \geq 0$, and $n \in \mathbb{N}^*$.

Proof. Let $f(z) \in N_{n,\delta}(g)$. Making use of (2.9), we find that

$$\sum_{j \geq n+1} j \cdot |a_j - b_j| < \delta, \quad n \in \mathbb{N}^*, \quad (3.11)$$

which readily implies the coefficients of inequality

$$\sum_{j \geq n+1} |a_j - b_j| < \frac{\delta}{n + 1}, \quad n \in \mathbb{N}. \quad (3.12)$$

Furthermore, since $g(z) \in T^* L_{\beta,n}(\alpha)$, we have

$$\sum_{j \geq n+1} b_j < \frac{1 - \alpha}{(1 + n\lambda)^\beta (1 + n\lambda - \alpha)}, \quad (3.13)$$

so that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum_{j \geq n+1} |a_j - b_j|}{1 - \sum_{j \geq n+1} b_j} < \frac{\delta \cdot \left[(1 + n\lambda)^\beta (1 + n\lambda - \alpha) \right]}{(n+1) \cdot \left[(1 + n\lambda)^\beta (1 + n\lambda - \alpha) - (1 - \alpha) \right]} = 1 - \nu, \quad (3.14)$$

provided that ν is given precisely by (3.9), which evidently completes our proof of Theorem 3.5. $\square$

Example 3.6. For a given $g(z) = z - \sum_{j \geq n+1} b_j z^j \in T^* L_{\beta, n}(\alpha)$, $n \in \mathbb{N}$, $n > 1$, we consider $f(z) = z - \sum_{j \geq n+1} a_j z^j \in T$ with $a_j = (1 - \nu)/(j^2(j-1)) + b_j$, $j \geq n+1$, $n \in \mathbb{N} \setminus \{0\}$, (for $\beta = 0$, $\lambda = 1$), where $1 - \nu$ is given by (3.9).

Then we have that

$$\sum_{j \geq n+1} j |a_j - b_j| = \sum_{j \geq n+1} j \left| \frac{1 - \nu}{j^2(j-1)} \right| = (1 - \nu) \left\{ \sum_{j \geq n+1} \left(\frac{1}{j-1} - \frac{1}{j} \right) \right\} = 1 - \nu, \quad (3.15)$$

where $1 - \nu = \delta(1 + n - \alpha)/(n(n+1))$.

Therefore, $f(z) \in N_{n, \delta}(g)$.

In a similar way, we can prove Theorem 3.7.

Theorem 3.7. If $g(z) \in T^c L_{\beta, n}(\alpha)$ and

$$\nu = 1 - \frac{\delta \cdot \left[(1 + n\lambda)^{\beta+1} (1 + n\lambda - \alpha) \right]}{(n+1) \cdot \left[(1 + n\lambda)^{\beta+1} (1 + n\lambda - \alpha) - (1 - \alpha) \right]}, \quad (3.16)$$

then

$$N_{n, \delta}(g) \subset T^{c(\nu)} L_{\beta, n}(\alpha), \quad (3.17)$$

$\alpha \in [0, 1)$, $\lambda \geq 0$, $\beta \geq 0$, and $n \in \mathbb{N}^*$.

Example 3.8. For a given $g(z) = z - \sum_{j \geq n+1} b_j z^j \in T^c L_{\beta, n}(\alpha)$, $n \in \mathbb{N}$, $n > 1$, we consider $f(z) = z - \sum_{j \geq n+1} a_j z^j \in T$ with $a_j = (1 - \nu)/(j^2(j-1)) + b_j$, $j \geq n+1$, $n \in \mathbb{N} \setminus \{0\}$, (for $\beta = 0$, $\lambda = 1$), where $1 - \nu$ is given by (3.16).

Then we have that

$$\sum_{j \geq n+1} j |a_j - b_j| = \sum_{j \geq n+1} j \left| \frac{1 - \nu}{j^2(j-1)} \right| = (1 - \nu) \left\{ \sum_{j \geq n+1} \left(\frac{1}{j-1} - \frac{1}{j} \right) \right\} = 1 - \nu, \quad (3.18)$$

where $1 - \nu = \delta(1 + n - \alpha)/(n(n+1))$.

Therefore, $f(z) \in N_{n, \delta}(g)$.

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