Research Article

Positive Solutions for Second-Order Nonlinear Ordinary Differential Systems with Two Parameters

Lan Sun,1 Yukun An,1 and Min Jiang2

1 Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China
2 College of Computer Science and Technology, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China

Correspondence should be addressed to Lan Sun, cliviasun@126.com

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By using fixed-point theorem and under suitable conditions, we investigate the existence and multiplicity positive solutions to the following systems: $u''(t) + au(t) + bv(t) + \lambda h_1(t)f(u(t), v(t)) = 0, \ t \in [0, 1], \ v''(t) + cu(t) + dv(t) + \mu h_2(t)g(u(t), v(t)) = 0, \ t \in [0, 1], \ u(0) = u(1) = 0, \ v(0) = v(1) = 0$, where $a, b, c, d$ are four positive constants and $\lambda > 0, \mu > 0, f(u, v), g(u, v) \in C(R^* \times R^*, R^*)$ and $h_1, h_2 \in C([0, 1], R^*)$. We derive two explicit intervals of $\lambda$ and $\mu$, such that the existence and multiplicity of positive solutions for the systems is guaranteed.

1. Introduction

In this paper, we are concerned with the existence and multiplicity of positive solutions to the following boundary value problem of second-order nonlinear differential systems with two parameters:

\begin{align}
    u''(t) + au(t) + bv(t) + \lambda h_1(t)f(u(t), v(t)) &= 0, \ t \in [0, 1], \\
    v''(t) + cu(t) + dv(t) + \mu h_2(t)g(u(t), v(t)) &= 0, \ t \in [0, 1], \\
    u(0) = u(1) &= 0, \\
    v(0) = v(1) &= 0,
\end{align}

where $a, b, c, d$ are four positive constants.

In addition, we assume the following conditions throughout this paper:

$(H_1)$ $h_1(t), h_2(t) \in C([0, 1], R^*)$ does not vanish identically on any subinterval of $[0, 1]$;
Meanwhile, in an equivalent problem whose solutions can be characterized by their nodal properties. Applying a classical change of variables, the authors transformed the initial problem into the existence of positive solutions of the following systems:

\begin{align*}
x''(t) + \lambda a(t)f(x(t), y(t)) &= 0, \quad t \in [0, 1], \\
y''(t) + \mu b(t)g(x(t), y(t)) &= 0, \quad t \in [0, 1], \\
x(0) &= x(1) = y(0) = y(1) = 0. \tag{1.2}
\end{align*}

In [4], a multiplicity result has been established when \( f(0, 0) > 0, g(0, 0) > 0 \). In [7], a multiplicity result has been given for the more general case \( f(0, 0) \geq 0, g(0, 0) \geq 0 \).

Also, by using Krasnosel’skii fixed-point theorem, Ru and An [8] considered the existence of positive solutions of the following systems:

\begin{align*}
(-)^p u^{(2p)} &= \lambda a(t)f(u(t), v(t)), \quad t \in [0, 1], \\
(-)^q v^{(2q)} &= \mu b(t)g(u(t), v(t)), \quad t \in [0, 1], \\
u^{(2i)}(0) &= u^{(2i)}(1) = 0, \quad 0 \leq i \leq p - 1, \\
v^{(2j)}(0) &= v^{(2j)}(1) = 0, \quad 0 \leq j \leq q - 1, \tag{1.3}
\end{align*}

where \( \lambda > \mu > 0, p, q \in \mathbb{N}, f, g : [0, \infty) \times [0, \infty) \to [0, \infty) \).

More recently, Dalbono and Mckenna [5] proved the existence and multiplicity of solutions to a class of asymmetric weakly coupled systems as follows:

\begin{equation}
\begin{bmatrix}
u_1'(t) \\
u_2'(t)
\end{bmatrix}
+ \begin{bmatrix}
b_1 & \varepsilon \\
\varepsilon & b_2
\end{bmatrix}
\begin{bmatrix}
u_1(t) \\
u_2(t)
\end{bmatrix}
= \begin{bmatrix}
\sin(t) \\
\sin(t)
\end{bmatrix},
\end{equation}

\begin{equation}
u_1(0) = u_2(0) = 0 = u_1(\pi) = u_2(\pi),
\end{equation}

where \( \varepsilon \) is suitably small and the positive numbers \( b_1, b_2 \) satisfy

\begin{equation}
h^2 < b_1 < (h + 1)^2, \quad k^2 < b_2 < (k + 1)^2 \quad \text{for some} \ h, k \in \mathbb{N}.
\end{equation}

Applying a classical change of variables, the authors transformed the initial problem into an equivalent problem whose solutions can be characterized by their nodal properties. Meanwhile, in [5], there are two open questions.

(1) Can one replace the near-diagonal matrix with something more general and use information on the eigenvalues of matrix?

(2) Can one replace the near-diagonal matrix with something more general and use information on the eigenvalues of matrix?
Inspired by the above works and the two open questions, we consider the existence and multiplicity of positive solutions to (1.1). The paper is organized as follows. In Section 2, we state some preliminaries. In Sections 3 and 4, we prove the existence and multiplicity results of (1.1).

2. Preliminaries

In order to prove our results, we state the well-known fixed-point theorem [9]:

**Lemma 2.1.** Let $E$ be a Banach space and let $K \subset E$ be a cone in $E$. Assume $\Omega_1, \Omega_2$ are two open subsets of $E$ with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow K$ be a completely continuous operator such that either

(i) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial \Omega_1$ and $\|Tu\| \geq \|u\|$, $u \in K \cap \partial \Omega_2$; or

(ii) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial \Omega_1$ and $\|Tu\| \leq \|u\|$, $u \in K \cap \partial \Omega_2$.

Then, $T$ has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

To be convenient, we introduce the following notations:

$$f_0 = \lim_{u,v \to 0} \frac{f(u, v)}{u + v}, \quad f_\infty = \lim_{u,v \to \infty} \frac{f(u, v)}{u + v},$$

$$g_0 = \lim_{u,v \to 0} \frac{g(u, v)}{u + v}, \quad g_\infty = \lim_{u,v \to \infty} \frac{g(u, v)}{u + v},$$

and suppose that $f_0, g_0, f_\infty, g_\infty \in [0, +\infty]$.

Let $\omega_1 = \sqrt{a}$, $\omega_2 = \sqrt{d}$, $H(t) \in C[0, 1]$, $G_i(t, s)$, $i = 1, 2$ be Green’s function of the corresponding to linear boundary value problem:

$$-u''(t) - \omega_i^2 u(t) = H(t), \quad u(0) = u(1) = 0. \quad (2.2)$$

Then, the solution of (2.2) is given by

$$u(t) = \int_0^1 G_i(t, s)H(s)ds. \quad (2.3)$$

It is well known that $G_i(t, s)$ can be expressed by

$$G_i(t, s) = \begin{cases} \frac{\sin \omega_i t \sin \omega_i (1 - s)}{\omega_i \sin \omega_i} & 0 \leq t \leq s \leq 1, \\ \frac{\sin \omega_i s \sin \omega_i (1 - t)}{\omega_i \sin \omega_i} & 0 \leq s \leq t \leq 1. \end{cases} \quad (2.4)$$

In addition, it can be easily to be checked that

$$M_i = \max_{t \in [0, 1]} G_i(t, t) > 0, \quad m_i = \min_{t \in [1/4, 3/4]} G_i(t, t) > 0, \quad i = 1, 2. \quad (2.5)$$

See [6], we have the following lemma.
Lemma 2.2. $G_i(t,s)$ has the following properties:

(i) $G_i(t,s) > 0$, for all $t, s \in (0,1)$,
(ii) $G_i(t,s) \leq C_i G_i(s,s)$, for all $t, s \in [0,1]$, $C_i = 1/\sin \omega_i$,
(iii) $G_i(t,s) \geq \delta_i G_i(t,t) G_i(s,s)$, for all $t, s \in [0,1]$, $\delta_i = \omega_i \sin \omega_i$, $i = 1,2$.

It is obvious that problem (1.1) is equivalent to the equation:

$$
(u(t), v(t)) = \left( \int_0^1 G_1(t,s)(\lambda h_1(s)f(u(s), v(s)) + b v(s)) \, ds, \right.
\left. \int_0^1 G_2(t,s)(\mu h_2(s)g(u(s), v(s)) + cu(s)) \, ds \right),
$$

and consequently it is equivalent to the fixed-point problem:

$$(u, v) = A(u, v)$$

with $A : C[0,1] \times C[0,1] \rightarrow C[0,1] \times C[0,1]$ given by

$$
A(u, v) = \left( \int_0^1 G_1(t,s)(\lambda h_1(s)f(u(s), v(s)) + b v(s)) \, ds, \int_0^1 G_2(t,s)(\mu h_2(s)g(u(s), v(s)) + cu(s)) \, ds \right).
$$

For convenience, denote

$$
A_1(u, v) = \int_0^1 G_1(t,s)(\lambda h_1(s)f(u(s), v(s)) + b v(s)) \, ds,
$$

$$
A_2(u, v) = \int_0^1 G_2(t,s)(\mu h_2(s)g(u(s), v(s)) + cu(s)) \, ds.
$$

It is obvious that $A$ is completely continuous. Let $\sigma_i = \delta_1 m_i/C_i = (\sin \omega_i/4)(\sin 3 \omega_i/4)(\sin \omega_i)$, $i = 1,2$, $\sigma = \min\{\sigma_1, \sigma_2\}$ and define a cone in $C[0,1] \times C[0,1]$ by

$$
K = \left\{ (u, v) \in C[0,1] \times C[0,1] : u(t) \geq 0, \ v(t) \geq 0, \ \min_{1/4 \leq t \leq 3/4} (u(t) + v(t)) \geq \sigma \|(u, v)\| \right\},
$$

where $\|(u, v)\| = \|u\| + \|v\| = \sup_{t \in [0,1]} u(t) + \sup_{t \in [0,1]} v(t)$.
Lemma 2.3. \( A(K) \subset K \).

Proof. For any \((t, s) \in [1/4, 3/4] \times [0, 1]\), by Lemma 2.2, we have

\[
A_1(u, v)(t) = \int_0^1 G_1(t, s)(\lambda h_1(s)f(u(s), v(s)) + bv(s))ds
\]
\[
\leq \int_0^1 C_1 G_1(s, s)(\lambda h_1(s)f(u(s), v(s)) + bv(s))ds,
\]
\[
\|A_1(u, v)\| \leq \int_0^1 C_1 G_1(s, s)(\lambda h_1(s)f(u(s), v(s)) + bv(s))ds,
\]
\[
A_1(u, v)(t) = \int_0^1 G_1(t, s)(\lambda h_1(s)f(u(s), v(s)) + bv(s))ds
\]
\[
\geq \delta_1 G_1(t, t) \int_0^1 G_1(s, s)(\lambda h_1(s)f(u(s), v(s)) + bv(s))ds
\]
\[
\geq \frac{\delta_1 G_1(t, t)}{C_1} \|A_1(u, v)\|.
\]

Similarly, for any \((t, s) \in [1/4, 3/4] \times [0, 1]\), we have

\[
A_\mu(u, v)(t) \geq \frac{\delta_2 G_2(t, t)}{C_2} \|A_\mu(u, v)\|.
\]

Then, for any \((t, s) \in [1/4, 3/4] \times [0, 1]\), we have

\[
A_1(u, v)(t) + A_\mu(u, v)(t) = \int_0^1 G_1(t, s)(\lambda h_1 f(u, v) + bv)ds + \int_0^1 G_2(t, s)(\mu h_2 g(u, v) + cu)ds
\]
\[
\geq \frac{\delta_1 G_1(t, t)}{C_1} \|A_1(u, v)\| + \frac{\delta_2 G_2(t, t)}{C_2} \|A_\mu(u, v)\|
\]
\[
\geq \sigma \|A(u, v)\|.
\]

Thus, \( \min_{1/4 \leq t \leq 3/4} A_1(u, v) + A_\mu(u, v) \geq \sigma \|A(u, v)\| \). Therefore, \( A(K) \subset K \). \( \square \)

3. Existence Results

We assume the following:

\((H_4)\) \(0 < b < 2\omega_1 \cos^2(\omega_1/2), 0 < c < 2\omega_2 \cos^2(\omega_2/2);\)

\((H_5)\) \(A_i = \| \int_0^1 h_i(s)ds \|, B_i = \int_{1/4}^{3/4} h_i(s)ds > 0, i = 1, 2;\)

\((H_6)\) \(P_i = C_i M_i = (\sec^2 \omega_i/2)/4\omega_i, Q_i = \delta_i m_i^2 = \sin(\omega_i/4) \sin(3\omega_i/4), i = 1, 2.\)
Theorem 3.1. Assume \((H_1)-(H_6)\) hold. Then one has the following:

1. If \(0 < f_0, f_\infty, g_0, g_\infty < \infty\), \(2P_1A_1f_0 < Q_1\sigma B_1f_\infty(1 - 2P_1b)\), then for each \(\lambda \in (1/Q_1\sigma B_1f_\infty, 1 - 2P_1b/2P_1A_1f_0)\) and \(\mu \in (0, (1 - 2P_2c)/2P_2A_2g_0)\), (1.1) has at least one positive solution.

2. If \(0 < f_0, f_\infty, g_0, g_\infty < \infty\), \(2P_2A_2g_0 < Q_2\sigma B_2g_\infty(1 - 2P_2c)\), then for each \(\lambda \in (0, (1 - 2P_1b)/2P_1A_1f_0)\) and \(\mu \in (1/Q_2\sigma B_2g_\infty, (1 - 2P_2c)/2P_2A_2g_0)\), (1.1) has at least one positive solution.

3. If \(f_0 = 0\), \(f_\infty = \infty\), then for each \(\lambda \in (0, \infty)\) and \(\mu \in (0, (1 - 2P_2c)/2P_2A_2g_0)\), (1.1) has at least one positive solution.

4. If \(g_0 = 0\), \(g_\infty = \infty\), then for each \(\lambda \in (0, (1 - 2P_1b)/2P_1A_1f_0)\) and \(\mu \in (0, \infty)\), (1.1) has at least one positive solution.

5. If \(f_0 = 0\), \(f_\infty = \infty\) and \(g_0 = 0\), \(g_\infty = \infty\), then for each \(\lambda \in (0, \infty)\) and \(\mu \in (0, \infty)\), (1.1) has at least one positive solution.

6. If \(f_\infty = \infty\), \(0 < f_0 < \infty\) or \(g_\infty = \infty\), \(0 < g_0 < \infty\), then for each \(\lambda \in (0, (1 - 2P_1b)/2P_1A_1f_0)\) and \(\mu \in (0, (1 - 2P_2c)/2P_2A_2g_0)\), (1.1) has at least one positive solution.

7. If \(f_0 = 0\), \(0 < f_\infty < \infty\) and \(g_0 = 0\), \(0 < g_\infty < \infty\), then for each \(\lambda \in (1/Q_1\sigma B_1f_\infty, \infty)\), \(\mu \in (0, \infty)\) or \(\lambda \in (0, \infty)\), \(\mu \in ((1/Q_2\sigma B_2g_\infty, \infty))\), (1.1) has at least one positive solution.

Proof. We only prove case (1.1). The other cases can be proved similarly. In order to apply the Lemma 2.1, we construct the sets \(\Omega_1, \Omega_2\).

Let \(\lambda \in (1/Q_1\sigma B_1f_\infty, (1 - 2P_1b)/2P_1A_1f_0)\), \(\mu \in (0, (1 - 2P_2c)/2P_2A_2g_0)\), and we choose \(\epsilon > 0\) such that

\[
\frac{1}{\sigma Q_1B_1(f_\infty - \epsilon)} \leq \lambda \leq \frac{1 - 2P_1b}{2P_1A_1(f_0 + \epsilon)}, \quad 0 < \mu \leq \frac{1 - 2P_2c}{2P_2A_2(g_0 + \epsilon)}. \quad (3.1)
\]

By the definition of \(f_0\) and \(g_0\), there exists \(R_1 > 0\), such that

\[
f(u, v) \leq (f_0 + \epsilon)(u + v), \quad g(u, v) \leq (g_0 + \epsilon)(u + v), \quad \text{for } u + v \in [0, R_1]. \quad (3.2)
\]

Choosing \((u, v) \in K\) with \(\|(u, v)\| = R_1\), we have

\[
A_1(u, v)(t) = \int_0^1 G_1(t, s)(\lambda h_1(s)f(u(s), v(s)) + bv(s))ds
\]

\[
\leq C_1M_1 \int_0^1 (\lambda h_1(s)(f_0 + \epsilon)(u(s) + v(s)) + bv(s))ds
\]

\[
\leq C_1M_1\|(u, v)\| \int_0^1 (\lambda h_1(s)(f_0 + \epsilon) + b)ds
\]

\[
= P_1\|(u, v)\|(\lambda A_1(f_0 + \epsilon) + b)
\]

\[
\leq \frac{1}{2}\|(u, v)\|,
\]

\(\Box\)
namely, \( \| A_1(u, v)(t) \| \leq (1/2) \| (u, v) \| \). In the same way, we also have
\[
\| A_2(u, v)(t) \| \leq \frac{1}{2} \| (u, v) \|. \tag{3.4}
\]
then \( \| A_1(u, v)(t) \| \leq \| (u, v)(t) \|. \) Thus, if we set \( \Omega_1 = \{(u, v) \in K : \| (u, v) \| < R_1 \} \), then
\[
\| A(u, v)(t) \| \leq \| (u, v) \|, \quad \forall (u, v) \in K \cap \partial \Omega_1. \tag{3.5}
\]
On the other hand, by the definition of \( f_\infty \), there exists \( R'_0 > 0 \), such that \( f(u, v) \geq (f_\infty - \varepsilon)(u + v) \), for \( u + v \in [R'_0, +\infty) \). Let \( R_2 = \max \{2R_1, \sigma^{-1} R'_2 \} \) and \( \Omega_2 = \{(u, v) \in K : \| (u, v) \| < R_2 \} \). If we choose \( (u, v) \in K \) with \( \| (u, v) \| = R_2 \), such that \( \min_{1/4 \leq t \leq 3/4} (u(t) + v(t)) \geq \sigma \| (u, v) \| \geq R_2 \), then we have
\[
A_1(u, v) \left( \frac{1}{4} \right) = \int_0^{1/4} G_1 \left( \frac{1}{4}, s \right) (\lambda h_1(s) f(u(s), v(s)) + b v(s)) ds \\
\geq \delta_1 G_1 \left( \frac{1}{4}, \frac{1}{4} \right) \int_{1/4}^{3/4} G_1(s, s) (\lambda h_1(s) f(u(s), v(s)) + b v(s)) ds \\
\geq \delta_1 m^2 \int_{1/4}^{3/4} (\lambda h_1(s) (f_\infty - \varepsilon))(u(s) + v(s)) ds \\
\geq Q_1 \sigma B_1 \lambda (f_\infty - \varepsilon) \| (u, v) \| \\
\geq \| (u, v) \|. \tag{3.6}
\]
Hence
\[
\| A(u, v)(t) \| \geq \| A_1(u, v)(t) \| \geq \| (u, v) \|, \quad \forall (u, v) \in K \cap \partial \Omega_2. \tag{3.7}
\]
Therefore, it follows from (3.5) to (3.7) and Lemma 2.2, \( A \) has a fixed point \( (u, v) \in K \cap (\overline{\Omega_2} \setminus \Omega_1) \), which is a positive solution of (1.1).

Similarly, we also have the following results.

**Theorem 3.2.** Assume \((H_1)-(H_6)\) hold. Then one has the following:

(1) If \( 0 < f_0, f_\infty, g_0, g_\infty < \infty, 2P_1 A_1 f_\infty < Q_1 \sigma B_1 f_0 (1 - 2P_1 b) \), then for each \( \lambda \in (1/Q_1 \sigma B_1 f_0, (1 - 2P_1 b) / 2P_1 A_1 f_\infty) \) and \( \mu \in (0, (1 - 2P_2 c) / 2P_2 A_2 g_\infty) \), \( (1.1) \) has at least one positive solution.

(2) If \( 0 < f_0, f_\infty, g_0, g_\infty < \infty, 2P_2 A_2 g_\infty < Q_2 \sigma B_2 g_0 (1 - 2P_2 c) \), then for each \( \lambda \in (0, (1 - 2P_1 b) / 2P_1 A_1 f_\infty) \) and \( \mu \in (1/Q_2 \sigma B_2 g_0, (1 - 2P_2 c) / 2P_2 A_2 g_\infty) \), \( (1.1) \) has at least one positive solution.

(3) If \( f_0 = \infty, f_\infty = 0 \), then for each \( \lambda \in (0, \infty) \) and \( \mu \in (0, (1 - 2P_2 c) / 2P_2 A_2 g_0) \), \( (1.1) \) has at least one positive solution.

(4) If \( g_0 = \infty, g_\infty = 0 \), then for each \( \lambda \in (0, (1 - 2P_1 b) / 2P_1 A_1 f_0) \) and \( \mu \in (0, \infty) \), \( (1.1) \) has at least one positive solution.
We only prove case (1). The other cases can be proved similarly. Let \( \lambda \in (1/Q_1\sigma B_1 f_0, 1-2P_1b)/2P_1A_1 f_\infty \), \( \mu \in (0, (1-2P_2c)/2P_2A_2g_\infty) \), and we choose \( \varepsilon > 0 \) such that
\[
\frac{1}{\sigma Q_1 B_1 (f_0 - \varepsilon)} \leq \lambda \leq \frac{1-2P_1b}{2P_1A_1 (f_\infty + \varepsilon)},
\]
(3.8)
by the definition of \( f_0 \), there exists \( R_1 > 0 \), such that
\[
f(u, v) \geq (f_0 - \varepsilon)(u + v), \quad \text{for } u + v \in [0, R_1].
\]
Choosing \( (u, v) \in K \) with \( \|u, v\| = R_1 \), such that \( \min_{1/4 \leq t \leq 3/4} (u(t) + v(t)) \geq \sigma \|u, v\| \), then we have
\[
A_\lambda(u, v) \left( \frac{1}{4} \right) = \int_0^{1/4} G_1 \left( \frac{1}{4}, s \right) \left( \lambda h_1(s) f(u(s), v(s)) + b v(s) \right) ds
\]
\[
\geq \delta_1 G_1 \left( \frac{1}{4}, \frac{1}{4} \right) \int_{1/4}^{3/4} G_1(s, s) (\lambda h_1(s) f(u(s), v(s)) + b v(s)) ds
\]
\[
\geq \delta_1 m_2^2 \int_{1/4}^{3/4} (\lambda h_1(s)(f_0 - \varepsilon))(u(s) + v(s)) ds
\]
\[
\geq Q_1 \sigma B_1 \lambda (f_0 - \varepsilon) \|u, v\| \geq \|u, v\|.
\]
(3.10)
So, if we set \( \Omega_1 = \{ (u, v) \in K : \|u, v\| < R_1 \} \), then
\[
\|A(u, v)(t)\| \geq \|u, v\|, \quad \forall (u, v) \in K \cap \partial \Omega_1.
\]
(3.11)
On the other hand, by the definition of \( f_\infty \) and \( g_\infty \), there exists \( R_2 > 2R_1 \), such that
\[
f(u, v) \leq (f_\infty + \varepsilon)(u + v), \quad g(u, v) \leq (g_\infty + \varepsilon)(u + v) \quad \text{for } u + v \in [R_2, + \infty).
\]
(3.12)
Choosing \((u, v) \in K\) with \(\|(u, v)\| = R_2\), we have

\[
A_1(u, v)(t) = \int_0^1 G_1(t, s)(\lambda h_1(s)f(u(s), v(s)) + bv(s))\,ds \\
\leq \int_0^1 C_1 G_1(s, s)(\lambda h_1(s)f(u(s), v(s)) + bv(s))\,ds \\
\leq C_1 M_1 \int_0^1 \lambda h_1(s) (f_\infty + \varepsilon)(u(s) + v(s)) + bv(s))\,ds \leq P_1(\lambda A_1(f_\infty + \varepsilon) + b)\|(u, v)\| \\
\leq \frac{1}{2}\|(u, v)\|.
\]

In the same way, we also have

\[
\|A_\mu(u, v)(t)\| \leq \frac{1}{2}\|(u, v)\|. \tag{3.14}
\]

Hence, if we set \(\Omega_2 = \{(u, v) \in K : \|(u, v)\| < R_2\}\), then

\[
\|A(u, v)(t)\| \leq \|(u, v)(t)\|, \quad \forall (u, v) \in K \cap \partial\Omega_2. \tag{3.15}
\]

Therefore, it follows from (3.11) to (3.15) and Lemma 2.2 that \(A\) has a fixed point \((u, v) \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)\), which is a positive solution of (1.1). \(\square\)

4. Multiplicity Results

**Theorem 4.1.** Assume \((H_1)-(H_6)\) hold. In addition, assume that there exist three constants \(r, M, N\), where \(N\) is sufficient small with \(2P_1NA_1 \leq Q_1B_1M(1 - 2P_1b)\), \(2P_2NA_2 \leq Q_2B_2M(1 - 2P_2c)\) such that

(i) \(f_0 = f_\infty = 0, \, g_0 = g_\infty = 0\),

(ii) \(f(u, v) \geq Mr\) or \(g(u, v) \geq Mr\), for \(\sigma r \leq \|(u, v)\| \leq r\).

Then, for any \(\lambda \in (1/Q_1B_1M, (1 - 2P_1b)/2P_1A_1N), \mu \in (0, (1-2P_2c)/2P_2A_2N)\), or \(\lambda \in (0, (1 - 2P_1b)/2P_1A_1N), \mu \in (1/Q_2B_2M, (1-2P_2c)/2P_2A_2N)\), the problem (1.1) has at least two positive solutions.

**Proof.** We only prove the case of \(\lambda \in (1/Q_1\sigma B_1M, (1 - 2P_1b)/2P_1A_1N), \mu \in (0, (1 - 2P_2c)/2P_2A_2N)\). The other case is similar.

**Step 1.** Since \(f_0 = g_0 = 0\), there exists \(r_1 \in (0, r)\) such that

\[
f(u, v) \leq N(u + v), \quad g(u, v) \leq N(u + v), \quad \text{for } 0 < u + v < r_1. \tag{4.1}
\]
Set $\Omega_1 = \{(u, v) \in K : \|(u, v)\| < r_1\}$, for all $(u, v) \in K \cap \partial \Omega_1$, then we have

$$A_1(u, v)(t) = \int_0^1 G_1(t, s) (\lambda h_1(s) f(u(s), v(s)) + bv(s)) \, ds$$

$$\leq \int_0^1 C_1 M_1 (\lambda h_1(s) f(u(s), v(s)) + bv(s)) \, ds$$

$$\leq C_1 M_1 \int_0^1 (\lambda h_1(s) N(u(s) + v(s)) + bv(s)) \, ds$$

$$\leq P_1 (\lambda A_1 N + b) \|(u, v)\|$$

$$\leq \frac{1}{2} \|(u, v)\|.$$

In the same way, we also have

$$\left\| A_\mu(u, v)(t) \right\| \leq \frac{1}{2} \|(u, v)\|. \quad (4.3)$$

Hence,

$$\|A(u, v)\| \leq \|(u, v)\|, \; \forall (u, v) \in K \cap \partial \Omega_1. \quad (4.4)$$

**Step 2.** Since $f_\infty = g_\infty = 0$, there exists $\overline{R} > r$ such that

$$f(u, v) \leq N(u + v), \quad g(u, v) \leq N(u + v), \quad \text{for } u + v \geq \overline{R}. \quad (4.5)$$

Similarly, set $\Omega_2 = \{(u, v) \in K : \|(u, v)\| < r_2\}$, then

$$\|A(u, v)\| \leq \|(u, v)\|, \; \forall (u, v) \in K \cap \partial \Omega_2. \quad (4.6)$$

**Step 3.** Let $\Omega_3 = \{(u, v) \in K : \|(u, v)\| < r\}$, we can see that

$$A_1(u, v) \left(\frac{1}{4}\right) = \int_0^1 G_1 \left(\frac{1}{4}, s\right) (\lambda h_1(s) f(u(s), v(s)) + bv(s)) \, ds$$

$$\geq \int_0^1 \delta_1 G_1 \left(\frac{1}{4}, \frac{1}{4}\right) G_1(s, s) (\lambda h_1(s) f(u(s), v(s)) + bv(s)) \, ds$$

$$\geq \delta_1 G_1 \left(\frac{1}{4}, \frac{1}{4}\right) \int_{1/4}^{3/4} G_1(s, s) (\lambda h_1(s) f(u(s), v(s)) + bv(s)) \, ds$$

$$\geq Q_1 \int_{1/4}^{3/4} (\lambda h_1(s) f(u(s), v(s)) + bv(s)) \, ds$$

$$\geq Q_1 \lambda B_1 M \overline{r}$$

$$\geq r. \quad (4.7)$$
Then,

$$\|A(u, v)\| \geq \|(u, v)\|, \quad \forall (u, v) \in K \cap \partial \Omega_3. \quad (4.8)$$

Consequently, by Lemma 2.1 and from (4.4)–(4.8), $A$ has two fixed points: $(u_1, v_1) \in K \cap (\overline{\Omega_3} \setminus \Omega_1)$ and $(u_2, v_2) \in K \cap (\overline{\Omega_2} \setminus \Omega_3)$, so (1.1) has at least two positive solutions satisfying $0 < (u_1, v_1) < r < (u_2, v_2)$.

**Theorem 4.2.** Assume $(H_1)$–$(H_6)$ hold. In addition, assume that there exist constants $r, M, N$, where $N$ is sufficient large with $2P_1MA_1 \leq Q_1\sigma B_1N(1 - 2P_1b)$, $2P_2MA_2 \leq Q_2\sigma B_2N(1 - 2P_2c)$ such that

(i) $f_0 = f_\infty = \infty$ or $g_0 = g_\infty = \infty$,

(ii) $f(u, v) \leq Mr$ or $g(u, v) \leq Mr$, for $\sigma r \leq \|(u, v)\| \leq r$,

then, for any $\lambda \in (1/Q_1\sigma B_1N, (1 - 2P_1b)/2P_1A_1M)$, $\mu \in (0, (1 - 2P_2c)/2P_2A_2M)$, or $\lambda \in (0, (1 - 2P_1b)/2P_1A_1M)$, $\mu \in (1/Q_2\sigma B_2N, (1 - 2P_2c)/2P_2A_2M)$, the problem (1.1) has at least two positive solutions.

**Proof.** We only prove the case of $\lambda \in (1/Q_1\sigma B_1N, (1 - 2P_1b)/2P_1A_1M)$, $\mu \in (0, (1 - 2P_2c)/2P_2A_2M)$, The other case is similar.

**Step 1.** Since $f_0 = \infty$, there exists $r_1 \in (0, r)$ such that

$$f(u, v) \geq N(u + v), \quad \text{for } 0 < u + v \leq r_1. \quad (4.9)$$

Set $\Omega_1 = \{(u, v) \in K : \|(u, v)\| < r_1\}$, then we have

$$A_\lambda(u, v) \left(\frac{1}{4}\right) = \int_0^1 G_1 \left(\frac{1}{4}, s\right) (\lambda h_1(s)f(u(s), v(s)) + bv(s))ds$$

$$\geq \int_0^1 \delta_1 G_1 \left(\frac{1}{4}, \frac{1}{4}\right) G_1(s, s) (\lambda h_1(s)f(u(s), v(s)) + bv(s))ds$$

$$\geq \delta_1 G_1 \left(\frac{1}{4}, \frac{1}{4}\right) \int_{1/4}^{3/4} G_1(s, s) (\lambda h_1(s)f(u(s), v(s)) + bv(s))ds \quad (4.10)$$

$$\geq Q_1 \int_{1/4}^{3/4} (\lambda h_1(s)(f(u(s), v(s)) + bv(s)))ds$$

$$\geq Q_1 \lambda B_1 N \sigma \|(u, v)\|$$

$$\geq r.$$
Hence,

$$\|A(u, v)\| \geq \|(u, v)\|, \quad \forall (u, v) \in K \cap \partial \Omega_1. \tag{4.11}$$

**Step 2.** Since $f_{\infty} = \infty$, there exists $\bar{R} > r$ such that

$$f(u, v) \geq N(u + v), \quad \text{for } u + v \geq \bar{R}. \tag{4.12}$$

Similarly, set $\Omega_2 = \{(u, v) \in K : \|(u, v)\| < r_2\}$, then

$$\|A(u, v)\| \geq \|(u, v)\|, \quad \forall (u, v) \in K \cap \partial \Omega_2. \tag{4.13}$$

**Step 3.** Let $\Omega_3 = \{(u, v) \in K : \|(u, v)\| < r\}$, for all $(u, v) \in K \cap \partial \Omega_3$, we can see that

$$A_1(u, v)(t) = \int_0^1 G_1(t, s)(\lambda h_1(s)f(u(s), v(s)) + bv(s))ds$$

$$\leq \int_0^1 C_1M_1(\lambda h_1(s)f(u(s), v(s)) + bv(s))ds$$

$$\leq P_1(\lambda A_1 M + b)\|(u, v)\|$$

$$\leq \frac{1}{2}\|(u, v)\|.$$ \hspace{1cm} (4.14)

In the same way, we also have

$$\|A_p(u, v)(t)\| \leq \frac{1}{2}\|(u, v)\|. \tag{4.15}$$

Hence,

$$\|A(u, v)\| \leq \|(u, v)\|, \quad \forall (u, v) \in K \cap \partial \Omega_3. \tag{4.16}$$

Consequently, by Lemma 2.1 and from (4.11)–(4.16), $A$ has two fixed points: $(u_1, v_1) \in K \cap (\Omega_3 \setminus \Omega_1)$ and $(u_2, v_2) \in K \cap (\Omega_2 \setminus \Omega_3)$, so (1.1) has at least two positive solutions satisfying $0 < (u_1, v_1) < r < (u_2, v_2)$.

**References**


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