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Research Article

Positive Solutions for Second-Order Nonlinear Ordinary Differential Systems with Two Parameters

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By using fixed-point theorem and under suitable conditions, we investigate the existence and multiplicity positive solutions to the following systems: $u''(t)+au(t)+bv(t)+\lambda h_1(t)f(u(t),v(t))=0,\ t\in[0,1],\ v''(t)+cu(t)+dv(t)+\mu h_2(t)g(u(t),v(t))=0,\ t\in[0,1],\ u(0)=u(1)=0,\ v(0)=v(1)=0$, where a,b,c,d are four positive constants and $\lambda>0$, $\mu>0$, $f(u,v),g(u,v)\in C(R^+\times R^+,R^+)$ and $h_1,h_2\in C([0,1],R^+)$. We derive two explicit intervals of λ and μ, such that the existence and multiplicity of positive solutions for the systems is guaranteed.

1. Introduction

In this paper, we are concerned with the existence and multiplicity of positive solutions to the following boundary value problem of second-order nonlinear differential systems with two parameters:

$$\begin{gathered}
u''(t)+au(t)+bv(t)+\lambda h_1(t)f(u(t),v(t))=0,\quad t\in[0,1],\\
v''(t)+cu(t)+dv(t)+\mu h_2(t)g(u(t),v(t))=0,\quad t\in[0,1],\\
u(0)=u(1)=0,\\
v(0)=v(1)=0,
\end{gathered}\tag{1.1}$$

where a,b,c,d are four positive constants.
In addition, we assume the following conditions throughout this paper:

(H_1) $h_1(t),h_2(t)\in C([0,1],R^+)$ does not vanish identically on any subinterval of $[0,1]$;

(H_2) $0 < a < \pi^2,\ 0 < d < \pi^2,\ c > 0,\ b > 0;$

(H_3) $f, g \in C(R^+ \times R^+, R^+)$.

The boundary value problem of ordinary differential systems has attracted much attention, see [1–6] and the references therein. Recently, Fink and Gatica [4] and Ma [7] have studied the existence of positive solutions of the following systems:

$$\begin{gathered} x''(t) + \lambda a(t) f\big(x(t), y(t)\big) = 0, \quad t \in [0,1], \\ y''(t) + \mu b(t) g\big(x(t), y(t)\big) = 0, \quad t \in [0,1], \\ x(0) = x(1) = y(0) = y(1) = 0. \end{gathered} \tag{1.2}$$

In [4], a multiplicity result has been established when $f(0,0) > 0,\ g(0,0) > 0$. In [7], a multiplicity result has been given for the more general case $f(0,0) \geq 0,\ g(0,0) \geq 0$.

Also, by using Krasnosel'skill fixed-point theorem, Ru and An [8] considered the existence of positive solutions of the following systems:

$$\begin{gathered} (-)^p u^{(2p)} = \lambda a(t) f(u(t), v(t)), \quad t \in [0,1], \\ (-)^q v^{(2q)} = \mu b(t) g(u(t), v(t)), \quad t \in [0,1], \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad 0 \leq i \leq p-1, \\ v^{(2j)}(0) = v^{(2j)}(1) = 0, \quad 0 \leq j \leq q-1, \end{gathered} \tag{1.3}$$

where $\lambda >, \mu > 0, p, q \in N$. $f, g : [0,\infty) \times [0,\infty) \to [0,\infty)$.

More recently, Dalbono and Mckenna [5] proved the existence and multiplicity of solutions to a class of asymmetric weakly coupled systems as follows:

$$\begin{gathered} \begin{bmatrix} u_1''(t) \\ u_2''(t) \end{bmatrix} + \begin{bmatrix} b_1 & \varepsilon \\ \varepsilon & b_2 \end{bmatrix} \begin{bmatrix} u_1^+(t) \\ u_2^+(t) \end{bmatrix} = \begin{bmatrix} \sin(t) \\ \sin(t) \end{bmatrix}, \\ u_1(0) = u_2(0) = 0 = u_1(\pi) = u_2(\pi), \end{gathered} \tag{1.4}$$

where ε is suitably small and the positive numbers b_1, b_2 satisfy

$$h^2 < b_1 < (h+1)^2, \quad k^2 < b_2 < (k+1)^2 \quad \text{for some } h,\ k \in N. \tag{1.5}$$

Applying a classical change of variables, the authors transformed the initial problem into an equivalent problem whose solutions can be characterized by their nodal properties. Meanwhile, in [5], there are two open questions.

(1) Can one replace the near-diagonal matrix with something more general and use information on the eigenvalues of matrix?

(2) Can one replace the near-diagonal matrix with something more general and use information on the eigenvalues of matrix?

Inspired by the above works and the two open questions, we consider the existence and multiplicity of positive solutions to (1.1). The paper is organized as follows. In Section 2, we state some preliminaries. In Sections 3 and 4, we prove the existence and multiplicity results of (1.1).

2. Preliminaries

In order to prove our results, we state the well-known fixed-point theorem [9]:

Lemma 2.1. *Let E be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are two open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|,\ u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|,\ u \in K \cap \partial\Omega_2$*; or*

(ii) $\|Tu\| \ge \|u\|,\ u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|,\ u \in K \cap \partial\Omega_2$.

Then, T has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

To be convenient, we introduce the following notations:

$$f_0 = \lim_{u+v\to 0} \frac{f(u,v)}{u+v}, \qquad f_\infty = \lim_{u+v\to \infty} \frac{f(u,v)}{u+v},$$
$$g_0 = \lim_{u+v\to 0} \frac{g(u,v)}{u+v}, \qquad g_\infty = \lim_{u+v\to \infty} \frac{g(u,v)}{u+v}, \tag{2.1}$$

and suppose that $f_0, g_0, f_\infty, g_\infty \in [0,+\infty]$.
Let $\omega_1 = \sqrt{a},\ \omega_2 = \sqrt{d},\ H(t) \in C[0,1]$, $G_i(t,s),\ i = 1,2$ be Green's function of the corresponding to linear boundary value problem:

$$-u''(t) - \omega_i^2 u(t) = H(t), \quad u(0) = u(1) = 0. \tag{2.2}$$

Then, the solution of (2.2) *is given by*

$$u(t) = \int_0^1 G_i(t,s)H(s)ds. \tag{2.3}$$

It is well known that $G_i(t,s)$ can be expressed by

$$G_i(t,s) = \begin{cases} \dfrac{\sin\omega_i t\ \sin\omega_i(1-s)}{\omega_i \sin\omega_i} & 0 \le t \le s \le 1, \\ \dfrac{\sin\omega_i s\ \sin\omega_i(1-t)}{\omega_i \sin\omega_i} & 0 \le s \le t \le 1. \end{cases} \tag{2.4}$$

In addition, it can be easily to be checked that

$$M_i = \max_{t\in[0,1]} G_i(t,t) > 0, \quad m_i = \min_{t\in[1/4,3/4]} G_i(t,t) > 0, \quad i = 1,2. \tag{2.5}$$

See [6], we have the following lemma.

Lemma 2.2. *$G_i(t,s)$ has the following properties:*

(i) $G_i(t,s) > 0$, *for all* $t,s \in (0,1)$,

(ii) $G_i(t,s) \le C_i G_i(s,s)$, *for all* $t,\ s \in [0,1]$, $C_i = 1/\sin\omega_i$,

(iii) $G_i(t,s) \ge \delta_i G_i(t,t) G_i(s,s)$, *for all* $t,\ s \in [0,1]$, $\delta_i = \omega_i \sin\omega_i$, $i = 1,2$.

It is obvious that problem (1.1) *is equivalent to the equation:*

$$(u(t),v(t)) = \left(\int_0^1 G_1(t,s)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds, \int_0^1 G_2(t,s)\big(\mu h_2(s) g(u(s),v(s)) + cu(s)\big)ds\right), \tag{2.6}$$

and consequently it is equivalent to the fixed-point problem:

$$(u,v) = \mathbf{A}(u,v) \tag{2.7}$$

with $\mathbf{A} : C[0,1]\times C[0,1] \to C[0,1]\times C[0,1]$ *given by*

$$\mathbf{A}(u,v) = \left(\int_0^1 G_1(t,s)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds, \int_0^1 G_2(t,s)\big(\mu h_2(s) g(u(s),v(s)) + cu(s)\big)ds\right). \tag{2.8}$$

For convenience, denote

$$\mathbf{A}_\lambda(u,v) = \int_0^1 G_1(t,s)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds, \qquad \mathbf{A}_\mu(u,v) = \int_0^1 G_2(t,s)\big(\mu h_2(s) g(u(s),v(s)) + cu(s)\big)ds. \tag{2.9}$$

It is obvious that $\mathbf{A}$ *is completely continuous. Let* $\sigma_i = \delta_i m_i / C_i = (\sin\omega_i/4)(\sin 3\omega_i/4)(\sin\omega_i)$, $i = 1,2$, $\sigma = \min\{\sigma_1,\sigma_2\}$ *and define a cone in* $C[0,1]\times C[0,1]$ *by*

$$\mathbf{K} = \left\{(u,v) \in C[0,1]\times C[0,1] : u(t) \ge 0,\ v(t) \ge 0, \min_{1/4 \le t \le 3/4}(u(t)+v(t)) \ge \sigma\|(u,v)\|\right\}, \tag{2.10}$$

where $\|(u,v)\| = \|u\| + \|v\| = \sup_{t\in[0,1]} u(t) + \sup_{t\in[0,1]} v(t)$.

Lemma 2.3. $A(K) \subset K$.

Proof. For any $(t,s) \in [1/4,3/4] \times [0,1]$, by Lemma 2.2, we have

$$\begin{aligned}
A_\lambda(u,v)(t) &= \int_0^1 G_1(t,s)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds \\
&\le \int_0^1 C_1 G_1(s,s)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds, \\
\|A_\lambda(u,v)\| &\le \int_0^1 C_1 G_1(s,s)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds, \\
A_\lambda(u,v)(t) &= \int_0^1 G_1(t,s)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds \\
&\ge \delta_1 G_1(t,t) \int_0^1 G_1(s,s)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds \\
&\ge \frac{\delta_1 G_1(t,t)}{C_1}\|A_\lambda(u,v)\|.
\end{aligned} \tag{2.11}$$

Similarly, for any $(t,s) \in [1/4,3/4] \times [0,1]$, we have

$$A_\mu(u,v)(t) \ge \frac{\delta_2 G_2(t,t)}{C_2}\big\|A_\mu(u,v)\big\|. \tag{2.12}$$

Then, for any $(t,s) \in [1/4,3/4] \times [0,1]$, we have

$$\begin{aligned}
A_\lambda(u,v)(t) + A_\mu(u,v)(t) &= \int_0^1 G_1(t,s)\big(\lambda h_1 f(u,v) + bv\big)ds + \int_0^1 G_2(t,s)\big(\mu h_2 g(u,v) + cu\big)ds \\
&\ge \frac{\delta_1 G_1(t,t)}{C_1}\|A_\lambda(u,v)\| + \frac{\delta_2 G_2(t,t)}{C_2}\big\|A_\mu(u,v)\big\| \\
&\ge \sigma\|A(u,v)\|.
\end{aligned} \tag{2.13}$$

Thus, $\min_{1/4\le t\le 3/4} A_\lambda(u,v) + A_\mu(u,v) \ge \sigma\|A(u,v)\|$. Therefore, $A(K) \subset K$. □

3. Existence Results

We assume the following:

(H_4) $0 < b < 2\omega_1\cos^2(\omega_1/2)$, $0 < c < 2\omega_2\cos^2(\omega_2/2)$;

(H_5) $A_i = \|\int_0^1 h_i(s)ds\|$, $B_i = \int_{1/4}^{3/4} h_i(s)ds > 0$, $i = 1,2$;

(H_6) $P_i = C_i M_i = (\sec^2\omega_i/2)/4\omega_i$, $Q_i = \delta_i m_i^2 = \sin(\omega_i/4)\sin(3\omega_i/4)$, $i = 1,2$.

Theorem 3.1. *Assume* (H_1)–(H_6) *hold. Then one has the following:*

(1) *If* $0 < f_0, f_\infty, g_0, g_\infty < \infty$, $2P_1A_1f_0 < Q_1\sigma B_1 f_\infty(1-2P_1b)$, *then for each* $\lambda \in (1/Q_1\sigma B_1 f_\infty, 1-2P_1b/2P_1A_1f_0)$ *and* $\mu \in (0,(1-2P_2c)/2P_2A_2g_0)$, (1.1) *has at least one positive solution.*

(2) *If* $0 < f_0, f_\infty, g_0, g_\infty < \infty$, $2P_2A_2g_0 < Q_2\sigma B_2 g_\infty(1-2P_2c)$, *then for each* $\lambda \in (0,(1-2P_1b)/2P_1A_1f_0)$ *and* $\mu \in (1/Q_2\sigma B_2 g_\infty, (1-2P_2c)/2P_2A_2g_0)$, (1.1) *has at least one positive solution.*

(3) *If* $f_0 = 0$, $f_\infty = \infty$, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0,(1-2P_2c)/2P_2A_2g_0)$, (1.1) *has at least one positive solution.*

(4) *If* $g_0 = 0$, $g_\infty = \infty$, *then for each* $\lambda \in (0,(1-2P_1b)/2P_1A_1f_0)$ *and* $\mu \in (0,\infty)$, (1.1) *has at least one positive solution.*

(5) *If* $f_0 = 0$, $f_\infty = \infty$ *and* $g_0 = 0$, $g_\infty = \infty$, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0,\infty)$, (1.1) *has at least one positive solution.*

(6) *If* $f_\infty = \infty$, $0 < f_0 < \infty$ *or* $g_\infty = \infty$, $0 < g_0 < \infty$, *then for each* $\lambda \in (0,(1-2P_1b)/2P_1A_1f_0)$ *and* $\mu \in (0,(1-2P_2c)/2P_2A_2g_0)$, (1.1) *has at least one positive solution.*

(7) *If* $f_0 = 0$, $0 < f_\infty < \infty$ *and* $g_0 = 0$, $0 < g_\infty < \infty$, *then for each* $\lambda \in (1/Q_1\sigma B_1 f_\infty, \infty)$, $\mu \in (0,\infty)$ *or* $\lambda \in (0,\infty)$, $\mu \in ((1/Q_2\sigma B_2 g_\infty)\infty)$, (1.1) *has at least one positive solution.*

Proof. We only prove case (1.1). The other cases can be proved similarly. In order to apply the Lemma 2.1, we construct the sets Ω_1, Ω_2.
Let $\lambda \in (1/Q_1\sigma B_1 f_\infty, (1-2P_1b)/2P_1A_1f_0)$, $\mu \in (0,(1-2P_2c)/2P_2A_2g_0)$, and we choose $\varepsilon > 0$ such that

$$\frac{1}{\sigma Q_1 B_1(f_\infty - \varepsilon)} \le \lambda \le \frac{1-2P_1b}{2P_1A_1(f_0+\varepsilon)}, \qquad 0 < \mu \le \frac{1-2P_2c}{2P_2A_2(g_0+\varepsilon)}. \tag{3.1}$$

By the definition of f_0 and g_0, there exists $R_1 > 0$, such that

$$f(u,v) \le (f_0+\epsilon)(u+v), \ g(u,v) \le (g_0+\epsilon)(u+v), \quad \text{for } u+v \in [0,R_1]. \tag{3.2}$$

Choosing $(u,v) \in K$ with $\|(u,v)\| = R_1$, we have

$$\begin{aligned} A_\lambda(u,v)(t) &= \int_0^1 G_1(t,s)(\lambda h_1(s) f(u(s),v(s)) + bv(s))ds \\ &\le C_1M_1 \int_0^1 (\lambda h_1(s)(f_0+\varepsilon)(u(s)+v(s)) + bv(s))ds \\ &\le C_1M_1\|(u,v)\| \int_0^1 (\lambda h_1(s)(f_0+\varepsilon)+b)ds \\ &= P_1\|(u,v)\|(\lambda A_1(f_0+\varepsilon)+b) \\ &\le \frac{1}{2}\|(u,v)\|, \end{aligned} \tag{3.3}$$

namely, $\|A_\lambda(u,v)(t)\| \le (1/2)\|(u,v)\|$. In the same way, we also have

$$\left\|A_\mu(u,v)(t)\right\| \le \frac{1}{2}\|(u,v)\|. \tag{3.4}$$

then $\|A_\lambda(u,v)(t)\| \le \|(u,v)(t)\|$. Thus, if we set $\Omega_1 = \{(u,v) \in K : \|(u,v)\| < R_1\}$, then

$$\|A(u,v)(t)\| \le \|(u,v)\|, \quad \forall (u,v) \in K \cap \partial\Omega_1. \tag{3.5}$$

On the other hand, by the definition of f_∞, there exists $R_2' > 0$, such that $f(u,v) \ge (f_\infty - \epsilon)(u+v)$, for $u+v \in [R_2', +\infty)$. Let $R_2 = \max\{2R_1, \sigma^{-1}R_2'\}$ and $\Omega_2 = \{(u,v) \in K : \|(u,v)\| < R_2\}$. If we choose $(u,v) \in K$ with $\|(u,v)\| = R_2$, such that $\min_{1/4\le t\le 3/4}(u(t)+v(t)) \ge \sigma\|(u,v)\| \ge R_2$, then we have

$$\begin{aligned}
A_\lambda(u,v)\left(\frac{1}{4}\right) &= \int_0^1 G_1\left(\frac{1}{4}, s\right)\left(\lambda h_1(s) f(u(s),v(s)) + bv(s)\right)ds \\
&\ge \delta_1 G_1\left(\frac{1}{4},\frac{1}{4}\right)\int_{1/4}^{3/4} G_1(s,s)\left(\lambda h_1(s) f(u(s),v(s)) + bv(s)\right)ds \\
&\ge \delta_1 m_1^2 \int_{1/4}^{3/4}\left(\lambda h_1(s)\left(f_\infty - \varepsilon\right)\right)(u(s)+v(s))ds \\
&\ge Q_1\sigma B_1\lambda\left(f_\infty - \varepsilon\right)\|(u,v)\| \\
&\ge \|(u,v)\|.
\end{aligned} \tag{3.6}$$

Hence

$$\|A(u,v)(t)\| \ge \|A_\lambda(u,v)(t)\| \ge \|(u,v)\|, \quad \forall (u,v) \in K \cap \partial\Omega_2. \tag{3.7}$$

Therefore, it follows from (3.5) to (3.7) and Lemma 2.2, A has a fixed point $(u,v) \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which is a positive solution of (1.1). □

Similarly, we also have the following results.

Theorem 3.2. *Assume (H_1)–(H_6) hold. Then one has the following:*

(1) *If $0 < f_0, f_\infty, g_0, g_\infty < \infty$, $2P_1A_1f_\infty < Q_1\sigma B_1 f_0(1-2P_1b)$, then for each $\lambda \in (1/Q_1\sigma B_1 f_0, (1-2P_1b)/2P_1A_1f_\infty)$ and $\mu \in (0,(1-2P_2c)/2P_2A_2g_\infty)$, (1.1) has at least one positive solution.*

(2) *If $0 < f_0, f_\infty, g_0, g_\infty < \infty$, $2P_2A_2g_\infty < Q_2\sigma B_2 g_0(1-2P_2c)$, then for each $\lambda \in (0,(1-2P_1b)/2P_1A_1f_\infty)$ and $\mu \in (1/Q_2\sigma B_2 g_0, (1-2P_2c)/2P_2A_2g_\infty)$, (1.1) has at least one positive solution.*

(3) *If $f_0 = \infty$, $f_\infty = 0$, then for each $\lambda \in (0,\infty)$ and $\mu \in (0,(1-2P_2c)/2P_2A_2g_0)$, (1.1) has at least one positive solution.*

(4) *If $g_0 = \infty$, $g_\infty = 0$, then for each $\lambda \in (0,(1-2P_1b)/2P_1A_1f_0)$ and $\mu \in (0,\infty)$, (1.1) has at least one positive solution.*

(5) *If* $f_0 = \infty$, $f_\infty = 0$ *and* $g_0 = \infty$, $g_\infty = 0$, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0,\infty)$, (1.1) *has at least one positive solution.*

(6) *If* $f_0 = \infty$, $0 < f_\infty < \infty$ *or* $g_0 = \infty$, $0 < g_\infty < \infty$ *then for each* $\lambda \in (0,(1-2P_1b)/2P_1A_1f_\infty)$ *and* $\mu \in (0,(1-2P_2c)/2P_2A_2g_\infty)$, (1.1) *has at least one positive solution.*

(7) *If* $f_\infty = 0$, $0 < f_0 < \infty$ *and* $g_\infty = 0$, $0 < g_0 < \infty$ *then for each* $\lambda \in (1/Q_1\sigma B_1 f_0,\infty)$, $\mu \in (0,\infty)$ *or* $\lambda \in (0,\infty)$, $\mu \in (1/Q_2\sigma B_2 g_0)$, (1.1) *has at least one positive solution.*

Proof. We only prove case (1). The other cases can be proved similarly.
Let $\lambda \in (1/Q_1\sigma B_1 f_0,(1-2P_1b)/2P_1A_1f_\infty)$, $\mu \in (0,(1-2P_2c)/2P_2A_2g_\infty)$, and we choose $\varepsilon > 0$ such that

$$\frac{1}{\sigma Q_1 B_1(f_0-\varepsilon)} \le \lambda \le \frac{1-2P_1b}{2P_1A_1(f_\infty+\varepsilon)}, \qquad 0 < \mu \le \frac{1-2P_2c}{2P_2A_2(g_\infty+\varepsilon)}, \tag{3.8}$$

by the definition of f_0, there exists $R_1 > 0$, such that

$$f(\mathrm{u},v) \ge (f_0-\epsilon)(u+v), \quad \text{for } u+v \in [0,R_1]. \tag{3.9}$$

Choosing $(u,v) \in K$ with $\|(u,v)\| = R_1$, such that $\min_{1/4\le t\le 3/4}(u(t)+v(t)) \ge \sigma\|(u,v)\|$, then we have

$$\begin{aligned}
A_\lambda(u,v)\left(\frac{1}{4}\right) &= \int_0^1 G_1\left(\frac{1}{4},s\right)\left(\lambda h_1(s)f(u(s),v(s)) + bv(s)\right)ds \\
&\ge \delta_1 G_1\left(\frac{1}{4},\frac{1}{4}\right)\int_{1/4}^{3/4} G_1(s,s)\left(\lambda h_1(s)f(u(s),v(s)) + bv(s)\right)ds \\
&\ge \delta_1 m_1^2 \int_{1/4}^{3/4}\left(\lambda h_1(s)(f_0-\varepsilon)\right)(u(s)+v(s))ds \\
&\ge Q_1\sigma B_1\lambda(f_0-\varepsilon)\|(u,v)\| \\
&\ge \|(u,v)\|.
\end{aligned} \tag{3.10}$$

So, if we set $\Omega_1 = \{(u,v) \in K : \|(u,v)\| < R_1\}$, then

$$\|A(u,v)(t)\| \ge \|(u,v)\|, \quad \forall (u,v) \in K \cap \partial\Omega_1. \tag{3.11}$$

On the other hand, by the definition of f_∞ and g_∞, there exists $R_2 > 2R_1$, such that

$$f(u,v) \le (f_\infty+\epsilon)(u+v), \quad g(u,v) \le (g_\infty+\epsilon)(u+v) \quad \text{for } u+v \in [R'_{2,}+\infty). \tag{3.12}$$

Choosing $(u,v) \in K$ with $\|(u,v)\| = R_2$, we have

$$\begin{aligned} A_\lambda(u,v)(t) &= \int_0^1 G_1(t,s)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds \\ &\le \int_0^1 C_1 G_1(s,s)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds \\ &\le C_1 M_1 \int_0^1 \big(\lambda h_1(s)\big(f_\infty + \varepsilon\big)(u(s)+v(s)) + bv(s)\big)ds \\ &\le P_1\big(\lambda A_1\big(f_\infty + \varepsilon\big) + b\big)\|(u,v)\| \\ &\le \frac{1}{2}\|(u,v)\|. \end{aligned} \tag{3.13}$$

In the same way, we also have

$$\big\|A_\mu(u,v)(t)\big\| \le \frac{1}{2}\|(u,v)\|. \tag{3.14}$$

Hence, if we set $\Omega_2 = \{(u,v) \in K : \|(u,v)\| < R_2\}$, then

$$\|A(u,v)(t)\| \le \|(u,v)(t)\|, \quad \forall (u,v) \in K \cap \partial\Omega_2. \tag{3.15}$$

Therefore, it follows from (3.11) to (3.15) and Lemma 2.2 that A has a fixed point $(u,v) \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which is a positive solution of (1.1). □

4. Multiplicity Results

Theorem 4.1. *Assume (H_1)–(H_6) hold. In addition, assume that there exist three constants r, M, N, where N is sufficient small with $2P_1NA_1 \le Q_1B_1M(1-2P_1b)$, $2P_2NA_2 \le Q_2B_2M(1-2P_2c)$ such that*

(i) $f_0 = f_\infty = 0$, $g_0 = g_\infty = 0$,

(ii) $f(u,v) \ge Mr$ *or* $g(u,v) \ge Mr$, *for* $\sigma r \le \|(u,v)\| \le r$.

Then, for any $\lambda \in (1/Q_1B_1M, (1-2P_1b)/2P_1A_1N)$, $\mu \in (0, (1-2P_2c)/2P_2A_2N)$, or $\lambda \in (0, (1-2P_1b)/2P_1A_1N)$, $\mu \in (1/Q_2B_2M, (1-2P_2c)/2P_2A_2N)$, the problem (1.1) *has at least two positive solutions.*

Proof. We only prove the case of $\lambda \in (1/Q_1\sigma B_1M, (1-2P_1b)/2P_1A_1N)$, $\mu \in (0, (1-2P_2c/2P_2A_2N))$, The other case is similar.

Step 1. Since $f_0 = g_0 = 0$, there exists $r_1 \in (0,r)$ such that

$$f(u,v) \le N(u+v), \quad g(u,v) \le N(u+v), \quad \text{for } 0 < u+v < r_1. \tag{4.1}$$

Set $\Omega_1 = \{(u,v) \in K : \|(u,v)\| < r_1\}$, for all $(u,v) \in K \cap \partial\Omega_1$, then we have

$$\begin{aligned} A_\lambda(u,v)(t) &= \int_0^1 G_1(t,s)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds \\ &\le \int_0^1 C_1 M_1\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds \\ &\le C_1 M_1 \int_0^1 (\lambda h_1(s) N(u(s)+v(s)) + bv(s))ds \\ &\le P_1(\lambda A_1 N + b)\|(u,v)\| \\ &\le \frac{1}{2}\|(u,v)\|. \end{aligned} \tag{4.2}$$

In the same way, we also have

$$\big\|A_\mu(u,v)(t)\big\| \le \frac{1}{2}\|(u,v)\|. \tag{4.3}$$

Hence,

$$\|A(u,v)\| \le \|(u,v)\|, \quad \forall (u,v) \in K \cap \partial\Omega_1. \tag{4.4}$$

Step 2. Since $f_\infty = g_\infty = 0$, there exists $\overline{R} > r$ such that

$$f(u,v) \le N(u+v), \quad g(u,v) \le N(u+v), \quad \text{for } u+v \ge \overline{R}. \tag{4.5}$$

Similarly, set $\Omega_2 = \{(u,v) \in K : \|(u,v)\| < r_2\}$, then

$$\|A(u,v)\| \le \|(u,v)\|, \quad \forall (u,v) \in K \cap \partial\Omega_2. \tag{4.6}$$

Step 3. Let $\Omega_3 = \{(u,v) \in K : \|(u,v)\| < r\}$, we can see that

$$\begin{aligned} A_\lambda(u,v)\left(\frac{1}{4}\right) &= \int_0^1 G_1\left(\frac{1}{4},s\right)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds \\ &\ge \int_0^1 \delta_1 G_1\left(\frac{1}{4},\frac{1}{4}\right) G_1(s,s)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds \\ &\ge \delta_1 G_1\left(\frac{1}{4},\frac{1}{4}\right)\int_{1/4}^{3/4} G_1(s,s)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds \\ &\ge\ Q_1 \int_{1/4}^{3/4} \big(\lambda h_1(s)\big(f(u(s),v(s)) + bv(s)\big)\big)ds \\ &\ge\ Q_1 \lambda B_1 M r \\ &\ge r. \end{aligned} \tag{4.7}$$

Then,

$$\|A(u,v)\| \geq \|(u,v)\|, \quad \forall (u,v) \in K \cap \partial\Omega_3. \tag{4.8}$$

Consequently, by Lemma 2.1 and from (4.4)–(4.8), A has two fixed points: $(u_1, v_1) \in K \cap (\overline{\Omega}_3 \setminus \Omega_1)$ and $(u_2, v_2) \in K \cap (\overline{\Omega}_2 \setminus \Omega_3)$, so (1.1) has at least two positive solutions satisfying $0 < (u_1, v_1) < r < (u_2, v_2)$. □

Theorem 4.2. *Assume (H_1)–(H_6) hold. In addition, assume that there exist constants r, M, N, where N is sufficient large with $2P_1MA_1 \leq Q_1\sigma B_1 N(1-2P_1b)$, $2P_2MA_2 \leq Q_2\sigma B_2 N(1-2P_2c)$ such that*

(i) *$f_0 = f_\infty = \infty$ or $g_0 = g_\infty = \infty$,*

(ii) *$f(u,v) \leq Mr$ or $g(u,v) \leq Mr$, for $\sigma r \leq \|(u,v)\| \leq r$,*

then, for any $\lambda \in (1/Q_1\sigma B_1 N, (1-2P_1b)/2P_1A_1M)$, $\mu \in (0, (1-2P_2c)/2P_2A_2M)$, or $\lambda \in (0, (1-2P_1b)/2P_1A_1M)$, $\mu \in (1/Q_2\sigma B_2 N, (1-2P_2c)/2P_2A_2M)$, the problem (1.1) *has at least two positive solutions.*

Proof. We only prove the case of $\lambda \in (1/Q_1\sigma B_1 N, (1-2P_1b)/2P_1A_1M)$, $\mu \in (0, (1-2P_2c)/2P_2A_2M)$, The other case is similar.

Step 1. Since $f_0 = \infty$, there exists $r_1 \in (0, r)$ such that

$$f(u,v) \geq N(u+v), \quad \text{for } 0 < u+v \leq r_1. \tag{4.9}$$

Set $\Omega_1 = \{(u,v) \in K : \|(u,v)\| < r_1\}$, then we have

$$\begin{aligned}
A_\lambda(u,v)\left(\frac{1}{4}\right) &= \int_0^1 G_1\left(\frac{1}{4}, s\right)\big(\lambda h_1(s) f(u(s), v(s)) + bv(s)\big)ds \\
&\geq \int_0^1 \delta_1 G_1\left(\frac{1}{4}, \frac{1}{4}\right) G_1(s,s)\big(\lambda h_1(s) f(u(s), v(s)) + bv(s)\big)ds \\
&\geq \delta_1 G_1\left(\frac{1}{4}, \frac{1}{4}\right) \int_{1/4}^{3/4} G_1(s,s)\big(\lambda h_1(s) f(u(s), v(s)) + bv(s)\big)ds \\
&\geq Q_1 \int_{1/4}^{3/4} \big(\lambda h_1(s)\big(f(u(s), v(s)) + bv(s)\big)\big)ds \\
&\geq Q_1 \lambda B_1 N \sigma \|(u,v)\| \\
&\geq r.
\end{aligned} \tag{4.10}$$

Hence,

$$\|A(u,v)\| \geq \|(u,v)\|, \quad \forall (u,v) \in K \cap \partial\Omega_1. \tag{4.11}$$

Step 2. Since $f_\infty = \infty$, there exists $\overline{R} > r$ such that

$$f(u,v) \geq N(u+v), \quad \text{for } u+v \geq \overline{R}. \tag{4.12}$$

Similarly, set $\Omega_2 = \{(u,v) \in K : \|(u,v)\| < r_2\}$, then

$$\|A(u,v)\| \geq \|(u,v)\|, \quad \forall (u,v) \in K \cap \partial\Omega_2. \tag{4.13}$$

Step 3. Let $\Omega_3 = \{(u,v) \in K : \|(u,v)\| < r\}$, for all $(u,v) \in K \cap \partial\Omega_3$, we can see that

$$\begin{aligned} A_\lambda(u,v)(t) &= \int_0^1 G_1(t,s)\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds \\ &\leq \int_0^1 C_1 M_1\big(\lambda h_1(s) f(u(s),v(s)) + bv(s)\big)ds \\ &\leq P_1(\lambda A_1 M + b)\|(u,v)\| \\ &\leq \frac{1}{2}\|(u,v)\|. \end{aligned} \tag{4.14}$$

In the same way, we also have

$$\left\|A_\mu(u,v)(t)\right\| \leq \frac{1}{2}\|(u,v)\|. \tag{4.15}$$

Hence,

$$\|A(u,v)\| \leq \|(u,v)\|, \quad \forall (u,v) \in K \cap \partial\Omega_3. \tag{4.16}$$

□

Consequently, by Lemma 2.1 and from (4.11)–(4.16), A has two fixed points: $(u_1,v_1) \in K \cap (\overline{\Omega}_3 \setminus \Omega_1)$ and $(u_2,v_2) \in K \cap (\overline{\Omega}_2 \setminus \Omega_3)$, so (1.1) has at least two positive solutions satisfying $0 < (u_1,v_1) < r < (u_2,v_2)$.

References

[1] Y. K. An and X. L. Fan, "One the coupled systems of second and fourth order elliptic equations," *Applied Mathematics and Computation*, vol. 140, no. 2-3, pp. 341–351, 2003.

[2] H. Y. Wang, "Positive periodic solutions of singular systems with a parameter," *Journal of Differential Equations*, vol. 249, no. 12, pp. 2986–3002, 2010.

[3] K. C. Chang, "Ambrosetti-Prodi type results in elliptic systems," *Nonlinear Analysis*, vol. 51, no. 4, pp. 553–566, 2002.

[4] A. M. Fink and J. A. Gatica, "Positive solutions of second order systems of boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 180, no. 1, pp. 93–108, 1993.
[5] F. Dalbono and P. J. McKenna, "Multiplicity results for a class of asymmetric weakly coupled systems of second-order ordinary differential equations," *Boundary Value Problems*, no. 2, pp. 129–151, 2005.
[6] Y. X. Li, "Positive solutions for fourth-order boundary value problems with two parameters," *Journal of Mathematical Analysis and Applications*, vol. 281, pp. 477–484, 2003.
[7] R. Ma, "Multiplicity nonnegtive solutions of second-order systems of boundary value problems," *Nonlinear Analysis*, vol. 42, pp. 1003–1010, 2000.
[8] Y. F. Ru and Y. K. An, "Positive solutions for 2p-order and 2q-order nonlinear ordinary differential systems," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 1093–1104, 2006.
[9] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, NY, USA, 1988.