Research Article

Study of Viscoplastic Flows Governed by the Norton-Hoff Operator

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We deal with viscoplastic flows. The fluid motion is governed by the nonlinear incompressible Norton-Hoff operator with homogeneous boundary conditions. We provide the well-posedness of the model in the static case. The idea is based on the crucial properties of the so-called compliance functional coupled with the min-max theory relayed to a variational analysis of the associated Lagrangian.

1. Introduction and Motivation

In this work, we deal with viscoplastic flows. The fluid's motion is governed by the nonlinear incompressible static Norton-Hoff model with homogeneous Dirichlet boundary conditions. The main purpose of this paper is to establish the existence and uniqueness of the flows solution to the system $\mathcal{P}$. The result is argued in Theorem 3.7.

The main difficulties in this analysis are

- (1) the nonlinearity of the operator is extremely high
- (2) the flow's velocity is not enough regular which does not allow to adopt the tools of the classical analysis.

Many authors were interested in the Norton-Hoff law. It was introduced by Norton [1] in order to describe the unidimensional creep of steel at high temperature and extended by Hoff [2] to the multidimensional solicitations. Dagnieres et al. [3] has generalized the Norton-Hoff law in plasticity and viscoplasticity. Temam [4] has proved that the Prandtl-Reuss law of plasticity is derived from the Norton-Hoff law when the exponent of the material

tends to one, and recently Ferchichi and Zolésio [5] has provided an identification model for a free-boundary problem of a non-Newtonian fluid coupled with the heat equation.

The paper is organized as follows: in Section 2, we introduce the static Norton-Hoff model. We provide the corresponding framework. We supply the equivalence between the norm induced by the set of admissible velocities and the usual one. In Section 3, we analyse the properties of the Norton-Hoff operator and we provide the well-posedness of the static case by using the min-max theory followed by the Lagrangian functional.

2. The Norton-Hoff Static Problem

2.1. Steady Incompressible Model

Consider a bounded open-domain Ω in $\mathbb{R}^N$, $N = 2, 3$, locally on one side of its C^2 -boundary $\partial\Omega$, occupied by a viscoplastic fluid. The fluid motion is governed by the incompressible Norton-Hoff model with homogeneous boundary conditions to the velocity. The Norton-Hoff static problem consists in looking for a velocity field u defined on $\bar{\Omega}$ and fulfills the hereafter equations:

$$p \begin{cases} K_c |\varepsilon(u)|^{p-2} \varepsilon(u) + P \text{Id} = \sigma, & \text{in } \Omega, \\ -\text{div}(\sigma) = f, & \text{in } \Omega, \\ \text{div}(u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

where K_c is the consistency of the material, σ is the Cauchy stress tensor, $\varepsilon(u) = (1/2)(D(u) + D(u)^*)$ is the linearized strain velocity tensor, D is the differential operator, p is the exponent of the material; $1 < p < 2$; it is the sensibility coefficient of the material to the strain velocity tensor, P is the hydrostatic pressure, Id is the identity tensor, and f is the density of the gravitation acting on the fluid. The first equation designates the behavior law, the second one describes the equilibrium state, and the third one prescribes the incompressibility of the fluid during the evolution.

2.2. Functional Setting

Let us introduce the functional framework

$$\mathcal{W} = W_0^{1,p}(\Omega, \mathbb{R}^N)^3, \quad \mathcal{W}_{\text{div}} = \{v \in \mathcal{W}, \text{ s.t. } \text{div}(v) = 0 \text{ on } \partial\Omega\}, \quad L_0^2(\Omega) = \frac{L^2(\Omega)}{\mathbb{R}}. \quad (2.2)$$

Further, we introduce the set of the admissible right-hand side

$$W^{-1,p'}(\Omega, \mathbb{R}^N)^3, \quad (2.3)$$

which is the topological dual space of $\mathcal{W}$ endowed with its natural norm, where p' is the algebraic dual of p . We define $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{V^*, V}$ with V^* being the topological dual space of V .

The following result provides a Banach space structure for the spaces $\mathcal{W}$ and $\mathcal{W}_{\text{div}}$.

Proposition 2.1. *The mapping $\|\cdot\|$ defined from $\mathcal{W}$ to $\mathbb{R}^+$ by*

$$\|v\| = \left(\int_{\Omega} |\varepsilon(v)|^p \right)^{1/p} \quad (2.4)$$

is a norm on $\mathcal{W}$, where $|\cdot|$ means the Euclidian norm of a matrix in $\mathbb{R}^{N^2}$.

Poincare's and Korn's inequalities prove the equivalence between this norm and the usual one (see [6]). All the other functional spaces are used with their natural structure, unless mentioned.

Proposition 2.2 (Poincare's Inequality). *For any adequate domain Ω , there exists a constant $c_p(\Omega)$ such that, for any v in $\mathcal{W}$,*

$$\|v\|_{L^p(\Omega, \mathbb{R}^N)} \leq c_p(\Omega) \|Dv\|_{L^p(\Omega, \mathbb{R}^{2N})}. \quad (2.5)$$

Proof. Since this proposition is classical, we briefly give the main steps of the proof. We consider, for any nonzero v in $\mathcal{W}$, the quotient

$$Q(v) = \frac{\|Dv\|_{L^p}^p}{\|v\|_{L^p}^p}. \quad (2.6)$$

This functional has a minimum over $\mathcal{W}/\{0\}$ which coincides with the minimum over the subset B of elements of $\mathcal{W}$ with L^p -norm equals to 1. Considering a minimizing sequence $(v_n)_{n \in \mathbb{N}}$ in B , it is bounded in $\mathcal{W}$ so it converges weakly towards a v_* . Due to the lower semicontinuity in L^p -norm, if $Q(v_*) \neq 0$, then $c_p(\Omega) = Q(v_*)^{-1}$ is convenient. This holds since $Q(v_*) = 0$ if and only if v_* is a non-zero constant, which is not possible because $v = 0$ on the boundary. $\square$

Proposition 2.3 (Korn's Inequality). *For any adequate domain Ω , there exists a constant $c_K(\Omega)$, such that, for any v in $\mathcal{W}$,*

$$\|Dv\|_{L^p(\Omega, \mathbb{R}^{N^2})} \leq c_K(\Omega) \left(\|\varepsilon(v)\|_{L^p(\Omega, \mathbb{R}^{N^2})} + \|v\|_{L^p(\Omega, \mathbb{R}^N)} \right). \quad (2.7)$$

We refer to the study of functional spaces for Norton-Hoff materials made by Geymonat and Suquet in [6] and for the proof of Korn's inequality. It is important to notice that this inequality does not hold for $p = 1$ (see a counterexample of Orstein in [7]). The proof of Poincare's inequality is classical and may be found in [4].

We can now provide the proof of Proposition 2.1.

Proof of Proposition 2.1. For shortness, we denote $\|\cdot\|_{1,p} = \|\cdot\|_{W^{1,p}(\Omega, \mathbb{R}^N)}$ the usual norm of $W^{1,p}(\Omega, \mathbb{R}^N)$.

For any v in $\mathcal{W}$, we have

$$\begin{aligned}\|v\| &= \frac{1}{2} \|Dv + {}^*Dv\|_{L^p(\Omega, \mathbb{R}^{N^2})} \leq \|Dv\|_{L^p(\Omega, \mathbb{R}^{N^2})}, \\ \|v\| &\leq \|Dv\|_{L^p(\Omega, \mathbb{R}^{N^2})} + \|v\|_{L^p(\Omega, \mathbb{R}^N)} = \|v\|_{1,p},\end{aligned}\tag{2.8}$$

which proves that $W^{1,p}(\Omega, \mathbb{R}^N) \subset \mathcal{W}(\Omega)$. The converse inclusion is a mere consequence of Korn's inequality.

We assume that there exists a sequence $(v_n)_{n \in \mathbb{N}}$ in $\mathcal{W}$ and a sequence $(c_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^+$ which tends to infinity such that $\|v_n\|_{1,p} \geq c_n \|v_n\|$. Without loss of generality, we suppose $\|v_n\|_{L^p(\Omega, \mathbb{R}^N)} = 1$. By Korn's inequality, we have $c_K(\Omega) \|v_n\| + c_K(\Omega) + 1 \geq c_n \|v_n\|$ for any n . Hence,

$$\begin{aligned}\frac{c_K(\Omega) + 1}{c_n - c_K(\Omega)} &\geq \|v_n\|, \\ \frac{c_K(\Omega)(c_K(\Omega) + 1)}{c_n - c_K(\Omega)} + c_K(\Omega) &\geq \|Dv_n\|_{L^p(\Omega, \mathbb{R}^{N^2})}.\end{aligned}\tag{2.9}$$

Thus, the sequence $(v_n)_{n \in \mathbb{N}}$ is bounded in $\mathcal{W}$ endowed with its classical topology. So it, up a subsequence, converges weakly towards v_* in $\mathcal{W}$. Since $\|\cdot\|$ is lower semicontinuous,

$$\|v_*\| \leq \liminf_{n \rightarrow \infty} \|v_n\| = 0.\tag{2.10}$$

So, $v_* = 0$ which is contradictory with $\|v_n\|_{L^p(\Omega, \mathbb{R}^N)} = 1$. Hence, there exists a constant c such that

$$\forall v \in \mathcal{W}, \quad \|v\|_{1,p} \leq c \|v\|.\tag{2.11}$$

Thus, the equivalence is provided. $\square$

3. Properties of the Norton-Hoff Operator

Definition 3.1. The Norton-Hoff functional or the so-called compliance functional is given by

$$\begin{aligned}\Phi_\Omega : \mathcal{W}(\Omega) &\longrightarrow \mathbb{R}, \\ v &\longrightarrow \int_\Omega \frac{K_c}{p} |\varepsilon(v)|^p - f v.\end{aligned}\tag{3.1}$$

The existence and uniqueness of a solution to the Norton-Hoff problem $\mathcal{P}$ is derived from the properties of Φ_Ω . As a preliminary result, we have the following differentiability property.

Lemma 3.2. *The functional $J : v \rightarrow \|v\|^p$ is Gateaux differentiable in $\mathcal{W}$. Its Gateaux derivative at a point u in direction v is given by*

$$J'(u; v) = \int_{\Omega} p|\varepsilon(u)|^{p-2}\varepsilon(u) \cdots \varepsilon(v), \quad (3.2)$$

where the expression $|\varepsilon(u)|^{p-2}\varepsilon(u) \cdots \varepsilon(v)$ is to be understood as continuously extended with 0 at any point x with $|\varepsilon(u)|(x) = 0$.

Proof. Let u and v be in $\mathcal{W}$. Let λ be a positive real number. The differential quotient is

$$\frac{J(u + \lambda v) - J(u)}{\lambda} = \frac{1}{\lambda} \int_{\Omega} (|\varepsilon(u) + \lambda \varepsilon(v)|^p - |\varepsilon(u)|^p). \quad (3.3)$$

The function $\rho : X \rightarrow \mathbb{R}^+$ is differentiable on $\mathbb{R}^+$, with derivative $\rho'(X) = pX^{p-1}$ for $X > 0$ and $\rho'(0) = 0$. Hence, for almost every point x in Ω , we can apply Taylor's formula in order to get

$$\left| \frac{\rho(|\varepsilon(u)(x) + \lambda \varepsilon(v)(x)|) - \rho(|\varepsilon(u)(x)|)}{\lambda} \right| \leq p(|\varepsilon(u)(x)| + \lambda |\varepsilon(v)(x)|)^{p-1} |\varepsilon(v)(x)|. \quad (3.4)$$

If we assume that $\lambda \in]0, 1]$, then we can overestimate this quotient,

$$\left| \frac{\rho(|\varepsilon(u)(x) + \lambda \varepsilon(v)(x)|) - \rho(|\varepsilon(u)(x)|)}{\lambda} \right| \leq p(|\varepsilon(u)(x)| + |\varepsilon(v)(x)|)^{p-1} |\varepsilon(v)(x)|. \quad (3.5)$$

So, Lebesgue's theorem supplies [8]:

$$\lim_{\lambda \rightarrow 0} \frac{J(u + \lambda v) - J(u)}{\lambda} = \int_{\Omega} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (|\varepsilon(u) + \lambda \varepsilon(v)|^p - |\varepsilon(u)|^p). \quad (3.6)$$

Because of

$$\frac{|\varepsilon(u)(x) + \lambda \varepsilon(v)(x)|^2 - |\varepsilon(u)(x)|^2}{\lambda} = \lambda \varepsilon(v) \cdots \varepsilon(v) + \varepsilon(u) \cdots \varepsilon(v), \quad (3.7)$$

the function $u \rightarrow |\varepsilon(u)|^2$ is Gateaux differentiable and its Gateaux derivative at u in a direction v is $2\varepsilon(u) \cdots \varepsilon(v)$. By composition with the real function $Y \rightarrow Y^{p/2}$, we come to

$$\lim_{\lambda \rightarrow 0} \frac{|\varepsilon(u) + \lambda \varepsilon(v)|^p - |\varepsilon(u)(x)|^p}{\lambda} = p|\varepsilon(u)|^{p-2}\varepsilon(u) \cdots \varepsilon(v), \quad (3.8)$$

which achieves the proof. □

Remark 3.3. Notice that we also can prove the Fréchet differentiability of the norm, with derivative that coincides (up to the coefficient K_c/p) with the Norton-Hoff operator.

Proposition 3.4. *The functional Φ_Ω is strictly convex, weakly l.s.c., Gateaux differentiable, and coercive. Moreover, for any u and v in $\mathcal{W}(\Omega)$, its directional derivative at u in direction v is given by*

$$\Phi'_\Omega(u, v) = \int_\Omega K_c |\varepsilon(u)|^{p-2} \varepsilon(u) \cdots \varepsilon(v) - f v. \quad (3.9)$$

Proof. Since $\Phi_\Omega(v) = (K_c/p) \|v\|^p - \langle f, v \rangle_{\mathcal{W}' \times \mathcal{W}}$, the continuity for the strong topology is obvious and so is the lower semi-continuity (l.s.c.) for the weak topology of $\mathcal{W}$. By this equality, the strict convexity of Φ_Ω derives from the strict convexity of $\|\cdot\|^p$, which is a mere consequence of both $\|\cdot\|$ and $x \rightarrow x^p$ from $\mathbb{R}^+$ to itself are convex functions.

The coercivity of Φ_Ω comes from

$$\forall v \in \mathcal{W}, \quad \Phi_\Omega(v) \geq \frac{K_c}{p} \|v\|^p - \|f\|_{\mathcal{W}'} \|v\|_{\mathcal{W}}. \quad (3.10)$$

Let u and v be in $\mathcal{W}$ and let λ be a positive number. For any $x \in \Omega$, we denote

$$Q_\lambda(x) = \frac{|\varepsilon(u)(x) + \lambda \varepsilon(v)(x)|^p - |\varepsilon(u)(x)|^p}{\lambda}. \quad (3.11)$$

Using the convexity of $X \rightarrow |X|$ from the space of $n \times n$ matrices to $\mathbb{R}$ and the convexity of $x \rightarrow x^p$, we have

$$\begin{aligned} Q_\lambda(x) &= \frac{|\lambda(\varepsilon(u)(x) + \varepsilon(v)(x)) + (1-\lambda)\varepsilon(u)(x)|^p - |\varepsilon(u)(x)|^p}{\lambda} \\ &\leq \frac{\lambda|\varepsilon(u+v)(x)|^p + (1-\lambda)|\varepsilon(u)(x)|^p - |\varepsilon(u)(x)|^p}{\lambda} \\ &\leq |\varepsilon(u+v)(x)|^p - |\varepsilon(u)(x)|^p = Q_1(x). \end{aligned} \quad (3.12)$$

Moreover, classical calculus yields

$$\lim_{\lambda \rightarrow 0} Q_\lambda(x) = p |\varepsilon(u)(x)|^{p-2} \varepsilon(u)(x) \cdots \varepsilon(v)(x), \quad (3.13)$$

which is continuously extended by 0 if $\varepsilon(u)(x) = 0$.

Since all the Q_λ belong to $L^1(\Omega)$, we have

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{\Phi_\Omega(u + \lambda v) - \Phi_\Omega(u)}{\lambda} &= \lim_{\lambda \rightarrow 0} \int_\Omega \frac{K_c}{p} Q_\lambda(x) dx - \langle f, v \rangle_{\mathcal{W}' \times \mathcal{W}} \\ &= \int_\Omega \lim_{\lambda \rightarrow 0} \frac{K_c}{p} Q_\lambda(x) dx - \langle f, v \rangle_{\mathcal{W}' \times \mathcal{W}} \\ &= \int_\Omega K_c |\varepsilon(u)(x)|^{p-2} \varepsilon(u)(x) \cdots \varepsilon(v)(x) - f v. \end{aligned} \quad (3.14)$$

Hence, we get the proof. $\square$

3.1. Well-Posedness via Min-Max Theory

In order to look for a solution to the Norton Hoff problem $\mathcal{P}$, we adopt the min-max theory (see [9]); mainly we use the following theorem.

Theorem 3.5. *Let E and F be two reflexive spaces and L a function from $E \times F$ to $\mathbb{R}$. One assumes the following:*

- (i) *For any a in E , the function $L(a, \cdot)$ is concave and upper semi-continuous (u.s.c.).*
- (ii) *For any b in F , the function $L(\cdot, b)$ is convex and lower semi-continuous (l.s.c.).*
- (iii) *There exists b_0 in F such that*

$$\lim_{\|a\| \rightarrow \infty} L(a, b_0) = +\infty. \quad (3.15)$$

- (iv)

$$\lim_{\|b\| \rightarrow \infty} \inf_{a \in E} L(a, b) = -\infty. \quad (3.16)$$

Then, L admits a saddle point $(\bar{a}, \bar{b})$, that is,

$$\forall (a, b) \text{ in } E \times F; \quad L(\bar{a}, b) \leq L(\bar{a}, \bar{b}) \leq L(a, \bar{b}). \quad (3.17)$$

Moreover, the set of saddle points is convex and is a Cartesian product (i.e., if $(\bar{a}_1, \bar{b}_1)$ and $(\bar{a}_2, \bar{b}_2)$ are saddle points of L , then so do $(\bar{a}_1, \bar{b}_2)$ and $(\bar{a}_2, \bar{b}_1)$).

We refer to [4] for the proof.

Let L be the following Lagrangian associated with the Norton-Hoff operator:

$$\begin{aligned} L : \mathcal{W} \times Q &\longrightarrow \mathbb{R}, \\ (v, q) &\longrightarrow \int_{\Omega} \left(\frac{K_c}{p} |\varepsilon(v)|^p - f x - q \operatorname{div} v \right). \end{aligned} \quad (3.18)$$

Proposition 3.6. *The function L has at least a saddle point.*

Proof. It is enough to apply Theorem 3.5 to the previous Lagrangian associated with our problem. Since the operator L is a linear with respect to q , condition (i) obviously holds.

Condition (ii) is satisfied. In fact, for any q in Q , $L(\cdot, q)$ is the sum of Φ_{Ω} and a linear functional; hence, with the fact that the mapping Φ_{Ω} is convex and lower semi-continuous, we get the assertion.

The choice $q_0 = 0$ is convenient for condition (iii) since $L(\cdot, 0)$ is coercive. We end the proof by noticing that

$$\begin{aligned} \lim_{\|q\|_{\uparrow\infty}} \inf_{v \in \mathcal{W}} L(v, q) &\leq \lim_{\|q\|_{\uparrow\infty}} \left(\inf_{v \in \mathcal{W}} \Phi_{\Omega}(v) + \inf_{v \in \mathcal{W}} - \int_{\Omega} q \operatorname{div} v \right) \\ &\leq \inf_{v \in \mathcal{W}} \Phi_{\Omega}(v) + \lim_{\|q\|_{\uparrow\infty}} \left(\inf_{v \in \mathcal{W}} - \langle q, \operatorname{div} v \rangle \right). \end{aligned} \quad (3.19)$$

While $\inf_{v \in \mathcal{W}} \Phi_{\Omega}(v) \leq \Phi_{\Omega}(0) = 0$, then

$$\inf_{v \in \mathcal{W}} \Phi_{\Omega}(v) + \lim_{\|q\|_{\uparrow\infty}} \inf_{v \in \mathcal{W}} - \langle q, \operatorname{div} v \rangle = -\infty, \quad (3.20)$$

which achieves the proof of the existence for a saddle point. $\square$

Theorem 3.7. *The Norton-Hoff mixte problem $\mathcal{P}$ has a unique solution (u, P) that belongs to $\mathcal{W} \times L_0^2$.*

Proof. For the existence, it is sufficient to prove that a saddle point of the operator L provides a solution to the problem $\mathcal{P}$.

Let (u, P) be a saddle point of L , then one has

$$\forall (v, q) \in \mathcal{W} \times L_0^2; \quad L(u, q) \leq L(u, P) \leq L(v, P). \quad (3.21)$$

The first inequality yields

$$\int_{\Omega} \frac{K_c}{p} |\varepsilon(u)|^p - fu - q \operatorname{div} u \leq \int_{\Omega} \frac{K_c}{p} |\varepsilon(u)|^p - fu - P \operatorname{div} v, \quad (3.22)$$

which holds for any q in L_0^2 . Hence, $\operatorname{div} u = 0$ so u belongs to $\mathcal{W}_{\operatorname{div}}$.

We notice that, as a simple consequence of the announced properties of Φ_{Ω} the Lagrangian L is Gateaux differentiable both with respect to its first and second variable. According to [9], first-order optimality conditions hold. As a consequence, the second inequality implies that, for any v in $\mathcal{W}$, the Gateaux derivative of L at point (u, P) in direction v with respect to the first variable vanishes.

Hence,

$$\int_{\Omega} \frac{K_c}{p} |\varepsilon(u)|^{p-2} \varepsilon(u) \cdots \varepsilon(v) - fv - P \operatorname{div} v = 0. \quad (3.23)$$

Using Green's formula, see [8], we come to

$$\left\langle -\operatorname{div} \left(\frac{K_c}{p} |\varepsilon(u)|^{p-2} \varepsilon(u) \right) - fu + \nabla P, v \right\rangle_{\mathcal{W}' \times \mathcal{W}} = 0, \quad (3.24)$$

which is the weak formulation of the problem $\mathcal{P}$ for any v in $\mathcal{W}$. Eventually, we have a solution of $\mathcal{P}$ in $\mathcal{W} \times L_0^2$.

The converse is easily proven as a mere application of Green's formula: any solution of $\mathcal{P}$ in $\mathcal{W} \times \mathcal{Q}$ is a saddle point of L .

Consequently, in order to provide the uniqueness of solution, we proceed as follows. If (u', P') is another solution of $\mathcal{P}$, then it is a saddle point of L . It is well known that the set of saddle points is a cartesian product (see, e.g., [9]). Hence, (u', P) is a third saddle point of L . But both u and u' are solution of

$$\min_{v \in \mathcal{W}_{\text{div}}} \Phi_{\Omega}(v). \quad (3.25)$$

Or Φ_{Ω} is strictly convex, continuous, and coercive functional, so it has a unique minimum. One can deduce that $u' = u$. If we assume that (u, P) and (u', P') are both solutions of $\mathcal{P}$, then we come to $\nabla(P - P') = 0$ in Ω . Whence P and P' are equal; up to a constant that is, they are equal in L_0^2 .

This achieves the proof. □

4. Conclusion

We have studied the problem of well-posedness of the Norton-Hoff operator that models visco-plastic flows. We have supplied an existence and uniqueness result to the considered system. We were faced by the fact that the flow is governed by a less-regular velocity field. Nevertheless, we have come over to this difficulty by using the theory of min-max applied to the so-called compliance functional. The results that were obtained encourage further analysis of the proposed techniques including identification of free interfaces.

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