Research Article

Γ -Extension of Binary Matroids

Habib Azanchiler

Department of Mathematics, Faculty of Sciences, University of Urmia, P.O. Box 57135, Urmia, Iran

Correspondence should be addressed to Habib Azanchiler, h.Azanchiler@urmia.ac.ir

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We extend the notion of a point-addition operation from graphs to binary matroids. This operation can be expressed in terms of element-addition operation and splitting operation. We consider a special case of this construction and study its properties. We call the resulting matroid of this special case a Γ -extension of the given matroid. We characterize circuits and bases of the resulting matroids and explore the effect of this operation on the connectivity of the matroids.

1. Introduction

Slater [1] defined few operations for graphs which preserve connectedness of graphs. One such operation is a point-addition (vertex-addition) operation. This operation is defined in the following way. Let G be a graph and $V(G)$ be the set of vertices of G . Let H be the graph obtained from G by adding a new vertex v adjacent to vertices $v_1, v_2, \dots, v_n$ of G . The graph H is said to be obtained from G by point-addition operation. Letting $X = \{v_1, v_2, \dots, v_n\}$, for convenience, we denote the graph H by G^X . Thus, $V(G^X) = V(G) \cup \{v\}$ and $E(G^X) = E(G) \cup \{vv_1, vv_2, \dots, vv_n\}$.

Point-addition operation has several applications in graph theory. For example, Slater classified n -connected graphs using point-addition operation along with some other operations [2].

If $|V(G)| = n$, then the new vertex v can be joined to at most n vertices of the graph. That means, we can add at most n edges in the original graph.

Definition 1.1. Let $M = M[A]$ be a binary matroid of rank r on a set S . Let A'' be the matrix obtained from A by the following way.

- (1) Adjoin k columns to A ($k \leq r$) with labels say $\gamma_1, \gamma_2, \dots, \gamma_k$. Let the resulting matrix be denoted by A' .

- (2) Adjoin a new row to A' with entries zero except in the columns corresponding to $\gamma_1, \gamma_2, \dots, \gamma_k$, where it takes the value 1.

Let M'' be the vector matroid of the matrix A'' . We say that M'' is obtained from M by the point-addition operation. We call the matroid M'' point-addition matroid or Γ -extension of M . Let us denote by Γ , the set of columns $\gamma_1, \gamma_2, \dots, \gamma_k$ which are adjoined to A in the first step. That is, $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$. Then, second step consists of splitting the matroid $M[A']$ with respect to the set Γ (see [3, 4]).

In fact, the matroid M'' is obtained by elements addition and generalized splitting operation [5]. As an immediate consequence of the definition, we have the following result.

Let v_i and v_j be two vertices of G . Then, the addition of an edge $v_i v_j$, results in the smallest supergraph of G containing edge $v_i v_j$.

Proposition 1.2. *Let $M = M(G)$ be a cycle matroid of rank r . Let G' be the graph obtained from G by adding adjacent edges $\gamma_1, \gamma_2, \dots, \gamma_n$ ($n \leq r$) to G . Let $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$. Then, the point-addition matroid M'' is graphic and $M'' = M(G'_\Gamma)$.*

Proof. Let A be representation matrix of M over $GF(2)$. Let the matrix A' be obtained from A by adding column vectors say, $\gamma_1, \gamma_2, \dots, \gamma_n$. Suppose that A'' is obtained from A' by adding a new row where entries are zero, except in the columns corresponding to $\gamma_1, \gamma_2, \dots, \gamma_n$, where it takes the value 1. Thus $M'' = M[A'']$ is a binary matroid with ground set $E(G) \cup \Gamma$. Since $\gamma_1, \gamma_2, \dots, \gamma_n$ are adjacent edges in G' , the splitting of $M(G')$ with respect to Γ is graphic (see [5]), and we have $M(G')_\Gamma = M(G'_\Gamma)$, where G'_Γ is the graph obtained from G by splitting operation with respect to Γ . It follows that $M'' = [M(G)]'' = M(G'_\Gamma)$. $\square$

We assume that the reader is familiar with elementary notions in matroid theory, including minors, binary, and connectivity. For an excellent introduction to the subject, read Oxley [6].

2. Γ -Extension of a Binary Matroid

If a matroid M is obtained from a matroid N by adding a nonempty subset T of $S(N)$, then N is called an extension of M . In particular, if $|T| = 1$, then N is a single-element extension of M (see [6]). Another term, that is sometimes used instead of single-element extension, is addition (see [7]).

Now we consider a special case of the operation that is introduced in the first section.

Definition 2.1. Let $M = M[A]$ be a binary matroid of rank r on a set S , and let $A = [I_r \mid J]$ be the standard representation of M over $GF(2)$. Let B be a base of M , and let $X = \{e_{i_1}, e_{i_2}, \dots, e_{i_m}\}$ be a subset of B . We obtain the matrix A^X by the following way.

- (1) Obtain a matrix A_1 from A by adjoining m ($m \leq r$) columns say $\gamma_{i_1}, \gamma_{i_2}, \dots, \gamma_{i_m}$ to A , parallel to $e_{i_1}, e_{i_2}, \dots, e_{i_m}$, respectively.
- (2) Split the matrix A_1 with respect to the set Γ , where $\Gamma = \{\gamma_{i_1}, \gamma_{i_2}, \dots, \gamma_{i_m}\}$. Denote the resulting matrix by A^X .

Let M^X be the vector matroid of the matrix A^X . We say that M^X is Γ -extension of M . Note that M^X is a binary matroid with ground set $S \cup \Gamma$, where $S \cap \Gamma = \emptyset$, and $|X| = |\Gamma|$. The

transition from M to M^X is called Γ -extension operation on M . In particular, if $|X| = \ell$, it is called ℓ - Γ -extension operation, and, for $|X| = 1$, we call it single- Γ -extension operation.

The next example illustrates this construction for the dual of Fano matroid.

Example 2.2. Let $M = F_7^*$ be the dual of the Fano matroid F_7 , and let $S = \{1, 2, 3, 4, 5, 6, 7\}$ be the ground set of M . The matrix A that represents M over $GF(2)$ is given by.

$$A = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} & \cdot \end{matrix} \quad (2.1)$$

Consider the set $X = \{1, 3, 4\}$ contained in the base of M . Then, the corresponding matrix A^X is given by

$$A^X = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \gamma_1 & \gamma_3 & \gamma_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} & \cdot \end{matrix} \quad (2.2)$$

The vector matroid of A^X is the matroid $(F_7^*)^X$.

Corollary 2.3. Let $M = M[A]$ be a binary matroid on S . Let X be a subset of a base of M , and M^X be the Γ -extension of M on the set $S \cap \Gamma$. Then, $M^X \setminus \Gamma = M$, that is, M^X is an extension of M .

Corollary 2.4. Let r and r' be the rank functions of the matroids M and M^X , respectively. Then $r'(M^X) = r(M) + 1$.

With the help of Lemma 2.5, we characterize the circuits of the matroid M^X .

Lemma 2.5. (1) Every circuit of M is a circuit of M^X .

(2) Every circuit of M^X contains at least one element of S .

(3) Every circuit of M^X contains even number of elements of Γ .

The proof follows from the construction of the matrix A^X .

Remark 2.6. Let M^X be a single- Γ -extension of M (i.e., $|X| = 1$). Then every circuit of M^X is a circuit of M and vice versa. In fact, the added element γ is a coloop in the resulting matroid.

Theorem 2.7. Let M be a binary matroid on S with representation matrix $A = [I_r \mid E]$ over $GF(2)$ and X be a subset of a base of M . Then, a subset Z of $S \cup \Gamma$ is a circuit of M^X if and only if one of the following conditions hold:

- (1) $Z = \{e_i, e_j, \gamma_i, \gamma_j\}$, where $i \neq j$, $\gamma_i, \gamma_j \in \Gamma$ and $e_i, e_j \in X$ for $1 \leq i, j \leq r$,
- (2) $Z = J \cup D$, where $J \subseteq \Gamma$, $|J|$ is an even integer and $\phi \neq D \subseteq S$ is such that $D \cup X_J$ is a circuit in M , where $X_J = \{e_i \in X \mid \gamma_i \in J\}$.

Proof. If $Z = \{e_i, e_j, \gamma_i, \gamma_j\}$, then, by Definition 2.1 of A^X , Z is a circuit of M^X . Now, let $Z = J \cup D$ be as stated in (2). If $|J| = 0$, then $J = \phi$, $X_J = \phi$, and $Z = D$ is a circuit of M . Suppose that $J \subseteq \Gamma$, $J \neq \phi$, and $X_J \cup D$ is a circuit of M . Then clearly, $Z = D \cup J$ is a circuit of M^X .

Conversely, let $Z \subseteq S \cup \Gamma$ be a circuit of M^X , we have two cases:

- (I) $Z \cap \Gamma = \phi$. Then $J = \phi$, $X_J = \phi$. Thus, $Z = D$ is a circuit in M and the condition (2) in the result holds.
- (II) Let $Z \cap \Gamma \neq \phi$, and suppose that $Z \cap \Gamma = J$. We have two subcases:
 - (i) $Z \cap X \neq \phi$. Then, $Z \cap X = \{e_i, e_j\} = D = X_J$ and $J = \{\gamma_i, \gamma_j\}$. Thus, $Z = \{e_i, e_j, \gamma_i, \gamma_j\}$ and condition (1) in the result holds.
 - (ii) $Z \cap X = \phi$. Take $Z \cap S = D$. Then $D \cup X_J$ is a circuit of M and $D \cup J$ is a circuit of M^X . Thus, $Z = D \cup J$, and the condition (2) in the result holds. $\square$

We characterize the independent sets of M^X in terms of independent sets of M . Firstly, we have the following lemma.

Lemma 2.8. (1) Every independent set of M is independent in M^X .
(2) Every subset of Γ is independent in M^X .

The proof is straightforward.

Remark 2.9. Let M^X be a single- Γ -extension of M . Then, every independent set in M^X is also independent in M and vice versa.

Theorem 2.10. Let M be a binary matroid on S and M^X be the Γ -extension matroid of M with respect to X . Let $\mathcal{O}$ be a collection of independent sets of M . Then, a subset I of $S \cup \Gamma$ is an independent set of M^X if and only if one of the following conditions hold:

- (1) $I = I_1 \cup \{\gamma\}$, where $\gamma \in \Gamma$ and $I_1 \in \mathcal{O}$.
- (2) $I = I_1 \cup J$, where $J \subseteq \Gamma$, $I_1 \in \mathcal{O}$ and $I_1 \cup X_J$ contains no circuit of M .

Proof. If $I_1 \in \mathcal{O}$, then clearly $I_1 \cup \{\gamma\}$ for $\gamma \in \Gamma$ is an independent set in M^X . Now, suppose that $I_1 \cup X_J$ contains no circuit of M . On the contrary, suppose that $I_1 \cup J$ contains a circuit say C' of M^X . Then $C' \setminus J \subseteq I_1$ and $(C' \setminus J) \cup X_J \subseteq I_1 \cup X_J$. But $(C' \setminus J) \cup X_J$ is a circuit of M and is contained in $I_1 \cup X_J$, a contradiction.

Conversely, let I be an independent set in M^X and $I \subseteq S \cup \Gamma$. We have two cases.

- (I) Let $I \cap \Gamma = \phi$. Then $I \subseteq S$ and I is independent in M .
- (II) Let $I \cap \Gamma \neq \phi$, and let $I \cap \Gamma = J$. Then $J \subseteq \Gamma$ and $J \subseteq I$.

We prove that $I - J \subseteq S$ is an independent set in M . On the contrary, suppose that $I - J$ contains a circuit of M , say C , then $C \subseteq I - J$ and $C \subseteq I$ gives a contradiction.

Letting $I - J = I_1$, we have $I = I_1 \cup J$. We claim that $I_1 \cup X_J$ does not contain any circuit of M . If $I_1 \cup X_J$ contains a circuit of M , say C' , then $C' \setminus X_J \subseteq I_1$. Further, $(C' \setminus X_J) \cup J \subseteq I_1 \cup J$, and thus $(C' \setminus X_J) \cup J \subseteq I$ leads to a contradiction as $(C' \setminus X_J) \cup J$ is a circuit of M^X . This completes the proof of the theorem. $\square$

Corollary 2.11. *Let $\mathcal{O}$ and $\mathcal{O}^X$ denote the collection of independent sets of M and M^X , respectively. If $|X| = 1$, then $\mathcal{O}^X = \{I \cup \{\gamma\} \mid I \in \mathcal{O}, \gamma \in \Gamma\}$.*

Corollary 2.12. *A subset I of S is independent in M if and only if $I \cup \{\gamma\}$ for $\gamma \in \Gamma$ is independent in M^X .*

Corollary 2.13. *Let r and r' be the rank functions of M and M^X , respectively. Then $r'(Z \cup \{\gamma\}) = r(Z) + 1$ for $Z \subseteq S$.*

In the next theorem, we characterize the bases of the matroid M^X in terms of the bases of M .

Lemma 2.14. *Let $Z \subseteq S$, then Z is an independent set in M^X if and only if Z is an independent set in M .*

The proof is straightforward.

Corollary 2.15. *Let Z be a subset of S . Then $r'(Z) = r(Z)$ where r' and r are rank functions of M^X and M , respectively.*

Theorem 2.16. *A subset B' of $S \cup \Gamma$ is a base for M^X if and only if $B' = (B \setminus D) \cup J$, where $\phi \neq J \subseteq \Gamma$, $D \subset B$, $|D| = |J| - 1$ and $(B \setminus D) \cup X_J$ contains no circuit of M .*

Proof. Let B be a base for M . Then B is an independent set in M , and so $B \setminus D$ is independent in M . Let $B \setminus D = I_1$. Then, by Theorem 2.10, $I_1 \cup J$, where $J \subseteq \Gamma$, $|D| = |J| - 1$ and $(B \setminus D) \cup X_J$ contains no circuit of M and hence is independent in M^X . Moreover, $r'((B \setminus D) \cup J) = |(B \setminus D) \cup J| = |B| - |D| + |J| = |B| + 1 = r'(M^X)$.

We conclude that $B' = (B \setminus D) \cup J$ is a base for M^X .

Conversely, let $B' \subseteq S \cup \Gamma$ be a base for M^X . Firstly, we show that $B' \cap \Gamma \neq \phi$. On the contrary, suppose that $B' \cap \Gamma = \phi$. Then $B' \subseteq S$ and is independent in M^X . So by Lemma 2.14, B' is independent in M . Also by Corollary 2.15,

$$r'(B') = r(B) = |B'| = r(M) + 1. \quad (2.3)$$

This shows that $B' \subseteq S$ and $r(B') > r(M)$; a contradiction.

Now, let $B' \cap \Gamma = J$. Then $B' - J \subseteq S$ is independent in M^X as well as in M . It can be extended to form the base of M . Let $D \subseteq S - (B' - J)$ be such that $(B' - J) \cup D = B$ is a base for M . Then $B' = (B \setminus D) \cup J$. We claim that $|D| = |J| - 1$. Now,

$$|B| = |(B' - J) \cup D| = |B'| - |J| + |D| = r(M) + 1 - |J| + |D|. \quad (2.4)$$

Since $|B| = r(M)$, we conclude that $1 - |J| + |D| = 0$, that is, $|D| = |J| - 1$. Finally, we show that $(B \setminus D) \cup X_J$ contains no circuit of M . On the contrary, suppose that $(B \setminus D) \cup X_J$ contains a

circuit, say C of M . Then $C \setminus X_J \subseteq B \setminus D$ and $(C \setminus X_J) \cup J \subseteq (B \setminus D) \cup J$. Thus, $(C \setminus X_J) \cup J \subseteq B'$ leads to a contradiction. This completes the proof of theorem. $\square$

Corollary 2.17. *Every base of M^X contains at least one element of Γ .*

3. Connectivity of M^X

Let M be a binary matroid on a set S and A be the representation matrix of M over $GF(2)$. If M is bridgeless, then Γ -extension of M with respect to a singleton subset X of B yields a disconnected matroid.

Lemma 3.1. *Let $e_i \in S$ be a coloop in a matroid M and $e_i \notin X$. Then e_i is a coloop in M^X .*

The proof is straightforward.

Corollary 3.2. *Suppose that no element of $S - X$ is a coloop of M . Then M^X has no coloops.*

Theorem 3.3. *Let $|X| \geq 2$. If M is connected matroid, then, so is M^X .*

Proof. Assume that M is connected. We show that for every pair of elements $u, v \in S \cup \Gamma$ there is a circuit of M^X containing u and v . We have three cases.

- (1) Let $u, v \in S$. By hypothesis, M is connected. So there is a circuit of M say C , containing u and v . Since C is a circuit in M^X , we are through.
- (2) Let $u, v \in \Gamma$ and let $u = \gamma_i$ and $v = \gamma_j$. Then the 4-circuit $\{e_i, e_j, \gamma_i, \gamma_j\}$ in M^X contains u and v .
- (3) Let $u \in S$ and $v \in \Gamma$. By assumption $|X| \geq 2$. So there is an element say $w \in \Gamma$. Let C_1 be a circuit of M^X containing v and w . By Lemma 2.5, C_1 contains at least one element of S , say e . Now $u, e \in S$, and M is connected, so there is a circuit of M , say C_2 which contains u and e . Thus $C_1 \cap C_2 \neq \emptyset$ and $u \in C_1, v \in C_2$. Then there is a circuit in M^X , say C_3 , such that $C_3 \subseteq C_1 \cup C_2$ and $u, v \in C_3$. This completes the proof of the theorem. $\square$

Remark 3.4. Converse of the above theorem is not true.

Theorem 3.5. *Let M^X be a 3- Γ -extension matroid of M and $\Gamma = \{\gamma_i, \gamma_j, \gamma_k\}$. If M is a 3-connected matroid on S , then M^X is 3-connected.*

Proof. On the contrary, suppose that M^X is not 3-connected, then M^X has a 1-separated or 2-separated partition. Let (A, B) be a 2-separated partition of $S \cup \Gamma$. That means, $\min\{|A|, |B|\} \geq 2$ and

$$r'(A) + r'(B) - r'(M^X) \leq 1. \quad (*)$$

We consider three cases.

- (i) Let $A = \Gamma$ and $B = S$. By Lemma 2.8, A is independent in M^X , so

$$r'(A) = |A| = |\Gamma| = 3. \quad (3.1)$$

Also, by Lemma 2.14,

$$r'(B) = r'(S) = r(S) = r(M). \quad (3.2)$$

Thus,

$$r'(A) + r'(B) - r'(M^X) = 3 + r(M) - r(M) - 1 = 2. \quad (3.3)$$

This is a contradiction to (*).

(ii) Let $A = \{\gamma_i, \gamma_j\}$ and $B = S \cup \{\gamma_k\}$.

By Lemma 2.14, $r'(B) = r(M) + 1$. So

$$r'(A) + r'(B) - r'(M^X) = 2 + r(M) + 1 - r(M) - 1 = 2 \quad (3.4)$$

gives a contradiction to (*).

(iii) Let $A = S_1 \cup \{\gamma_i\}$ and $B = S_2 \cup \{\gamma_j, \gamma_k\}$, where $S_1, S_2 \subseteq S$, and $S_1 \neq \emptyset, S_2 \neq \emptyset$. Then $\min\{|S_1|, |S_2|\} \geq 1$ and $r(S_1) + r(S_2) - r(M) \leq r'(A) - 1 + r'(B) - 1 - r'(M^X) + 1 \leq 0$. Moreover,

$$r(S_1) + r(S_2) \geq r(M). \quad (3.5)$$

Thus, $r(S_1) + r(S_2) - r(M) = 0$, and we conclude that (S_1, S_2) is a 1-separated partition for M .

This is a contradiction to the fact that M is 3-connected. By the same argument, we can show that M^X does not have 1-separated partition. $\square$

In the last theorem, the condition that $|X| = 3$ is necessary. Consider the following example.

Example 3.6. $M(K_4)$ is a 3-connected matroid. Let a representation matrix of $M(K_4)$ be

$$A = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} & \end{matrix}. \quad (3.6)$$

Let $X \subseteq B$ and $X = \{1, 2\}$, $\Gamma = \{\gamma_1, \gamma_2\}$. Then

$$A^X = \begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 & \gamma_1 & \gamma_2 \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix} \quad (3.7)$$

By row operations on A^X , we can show that

$$A^X = \left[I_4 \left| \begin{array}{cccc} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right. \right] \quad (3.8)$$

By $[\]$, $M^X = M[A^X]$ is not 3-connected.

If $|X| = 1$, then M^X has a coloop, and it is not 3-connected.

In general, we state the following result whose proof is immediate.

Corollary 3.7. *Let M be a n -connected binary matroid and $|X| < n$. Then M^X is not n -connected.*

References

- [1] P. J. Slater, "A classification of 4-connected graphs," *Journal of Combinatorial Theory. Series B*, vol. 17, pp. 281–298, 1974.
- [2] P. J. Slater, "Soldering and point splitting," *Journal of Combinatorial Theory. Series B*, vol. 24, no. 3, pp. 338–343, 1978.
- [3] T. T. Raghunathan, M. M. Shikare, and B. N. Waphare, "Splitting in a binary matroid," *Discrete Mathematics*, vol. 184, no. 1–3, pp. 267–271, 1998.
- [4] M. M. Shikare, "Splitting operation and connectedness in binary matroids," *Indian Journal of Pure and Applied Mathematics*, vol. 31, no. 12, pp. 1691–1697, 2000.
- [5] M. M. Shikare, G. Azadi, and B. N. Waphare, "Generalized splitting operation and its applications to binary matroids," *Preprint*.
- [6] J. G. Oxley, *Matroid Theory*, Oxford Science Publications, The Clarendon Press/Oxford University Press, New York, NY, USA, 1992.
- [7] K. Truemper, *Matroid Decomposition*, Academic Press, Boston, Mass, USA, 1992.