Research Article

On a Class of PDE Involving p -Biharmonic Operator

Abdelouahed El Khalil

*Department of mathematics, College of Sciences, Al-Imam Muhammad Ibn Saud Islamic University,
 P.O. Box 90950, Riyadh 11623, Saudi Arabia*

Correspondence should be addressed to Abdelouahed El Khalil, alakhil@imamu.edu.sa

Received 18 May 2011; Accepted 15 June 2011

Academic Editor: G. Buttazzo

Copyright © 2011 Abdelouahed El Khalil. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The existence of solution for a fourth-order nonlinear partial differential equation (PDE) class involving p -biharmonic operator $\Delta(|\Delta u|^{p-2}\Delta u) = \lambda\rho(x)|u|^{q-2}u$ in Ω , $u = \Delta u = 0$, on $\partial\Omega$, is proved by applying mountain pass theorem and a local minimization.

1. Introduction and Assumptions

The goal of this paper is to investigate the existence of nontrivial solutions for a class of nonlinear partial differential equations of the fourth order of the form

$$\begin{aligned}\Delta(|\Delta u|^{p-2}\Delta u) &= \lambda\rho(x)|u|^{q-2}u \quad \text{in } \Omega, \\ u = \Delta u &= 0 \quad \text{on } \partial\Omega,\end{aligned}\tag{E}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain in $\mathbb{R}^N$, $\lambda \in \mathbb{R}^+$ is a parameter which plays the role of eigenvalue, and p and q are real numbers such that $p > 1$ and

$$\begin{aligned}1 < q < p^{**} & \quad \text{if } 1 < p < \frac{N}{2}, \\ q < +\infty & \quad \text{if } p \geq \frac{N}{2},\end{aligned}\tag{1.1}$$

where p^{**} is the critical Sobolev exponent defined by $p^{**} = Np/(N - 2p)$ if $1 < p < N/2$ and $+\infty$ if $p \geq N/2$, and ρ is an indefinite weight in $L^r(\Omega)$ with r being so that

$$\begin{aligned} \text{(i) if } 1 < p < \frac{N}{2}, \text{ then } & \begin{cases} r > \frac{N}{2q} & \text{for } 1 < q \leq p, \\ r > p^{**}(p^{**} - q)^{-1} & \text{for } p < q < p^{**}, \end{cases} \\ \text{(ii) } r > q & \text{ if } p = \frac{N}{2}, \\ \text{(iii) } r = 1 & \text{ if } p > \frac{N}{2}, \end{aligned} \quad (1.2)$$

and (the Lebesgue measure)

$$\text{mes}(\{x \in \Omega : \rho(x) > 0\}) \neq 0. \quad (1.3)$$

$\Delta_p^2 u := \Delta(|\Delta u|^{p-2} \Delta u)$ is an operator of the fourth order called the p -biharmonic operator. For $p = 2$, the linear operator $\Delta^2 = \Delta \Delta$ is the iterated Laplace that multiplied with positive constant occurs often in Navier-Stokes equations as being a viscosity coefficient. Its reciprocal operator denoted $(\Delta^2)^{-1}$ is the celebrated Green operator [1]. The boundary condition in (E) is of the compatibility type when $p = q$ which can be considered as the following Hamiltonian system of the two coupled equations:

$$\begin{aligned} -\Delta u &= \phi_p(v) \quad \text{in } \Omega, \\ -\Delta v &= \lambda \rho \phi_p(u) \quad \text{in } \Omega, \\ u &= v = 0 \end{aligned} \quad (1.4)$$

with $\phi_p(t) = |t|^{p-2}t$, and $p' = p/(p - 1)$ is the Hölder conjugate exponent of p , by a transformation of a problem to a known Poisson's problem and using the properties of the operator-solution stated by Agmon-Douglis-Nirenberg; see [2]. Notice that a similar system as was considered recently by [3] in the restrictive case $\rho \equiv 1$. Our approach is quite different. Note that, in the case $p = q$, (E) is $(p - 1)$ -homogeneous, so we are dealing a quasilinear eigenvalue problem which is considered differently. This is few considered by [4] in the particular case $\rho \equiv 1$ and Ω is smooth; and by [5, 6] with $u \in W_0^{2,p}(\Omega)$ as a boundary condition, for any bounded domain.

Note also that the nonhomogeneous case is not considered there. Here we seek nontrivial solutions for (E) by distinguishing between two subcritical cases $1 < q < p$ and $p < q < p^{**}$ which means that the critical points of the associated energy functional

$$\mathcal{A}(u) = \frac{1}{p} \int_{\Omega} |\Delta u|^p dx - \frac{\lambda}{q} \int_{\Omega} \rho(x) |u|^q dx, \quad (1.5)$$

are defined in $X = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$. One solution is obtained by applying classical Mountain Pass Theorem and the other solution by a local minimization technique. The restrictive

case $p \equiv 1$ and Δ_p^2 is substituted by the well-known p -Laplacian $-\Delta_p$ was studied by Azorero and Alonso [7] and Elkhailil and Touzani [8] by using the fundamental properties of the first eigenvalue of the Dirichlet p -Laplacian problem.

It is important to indicate here that condition (1.2) is optimal to ensure the Palais-Smale (PS) condition is satisfied. Notice that our results are investigated without any condition on the parameter λ related to the spectrum of (E) (when $p = q$). On the other hand the condition (1.3) is assumed to have eventually nontrivial solutions.

The rest of this paper is divided in two sections as follows. In Section 2, we introduce some preliminary results and we give some technical lemmas, and in Section 3, we establish our main results.

2. Preliminaries

First, we introduce some preliminary results that we will need and some lemmas that are the key point of our results. We solve the problem (E) in the space $X = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_{\Omega} |\Delta u|^p dx \right)^{1/p}. \quad (2.1)$$

X equipped with this norm is a uniformly convex Banach space for $1 < p < \infty$. Hereafter, $\|\cdot\|_p$ is the L^p -norm, $\langle \cdot, \cdot \rangle$ will denote the duality between X and its dual X' .

Definition 2.1. u is a weak solution of (E) if, and only if, for all $v \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ we have

$$\int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta v dx = \lambda \int_{\Omega} \rho |u|^{q-2} uv dx. \quad (2.2)$$

That is, u is a critical point of the energy functional associated to (E) defined on X by

$$\mathcal{A}(u) = \frac{1}{p} \int_{\Omega} |\Delta u|^p dx - \frac{\lambda}{q} \int_{\Omega} \rho |u|^q dx. \quad (2.3)$$

Lemma 2.2. *For any r verifying (1.2) and q satisfying (1.1) there exists a constant $c = c(p, q, r) > 0$ such that*

$$\left| \int_{\Omega} \rho |u|^q dx \right| \leq c \|\rho\|_r \|u\|^q \quad (2.4)$$

and the map $u \rightarrow \rho |u|^{p-2} u$ is strongly continuous from X into X' .

Proof. To establish (2.4), we will divide the proof to three steps with respect to exponents p, q and N .

Step 1 ($1 < p < N$). Fixing $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ and using Hölder's inequality we have

$$\left| \int \rho |u|^q dx \right| \leq \|\rho\|_r \|u\|_s^{q-1} \|u\|_{p^{**}}, \quad (2.5)$$

where

$$\frac{q-1}{s} = 1 - \frac{1}{r} - \frac{1}{p^{**}}. \quad (2.6)$$

Such exponent s exists. Indeed: if $1 < q \leq p$, we obtain

$$\begin{aligned} \frac{p-1}{s} &\geq \frac{q-1}{s} = 1 - \frac{1}{r} - \frac{1}{p^{**}} \\ &\geq 1 - \frac{2q}{N} - \frac{1}{p^{**}} \\ &\geq 1 - \frac{2p}{N} - \frac{1}{p^{**}} = \frac{p-1}{p^{**}}. \end{aligned} \quad (2.7)$$

Thus, it suffices to take s so that

$$\max(1, q-1) < s < p^{**}. \quad (2.8)$$

If $p < q < p^{**}$, we have $r > p^{**}(p^{**} - q)^{-1}$. So,

$$\frac{1}{r} < 1 - \frac{q}{p^{**}}. \quad (2.9)$$

Hence,

$$\frac{q-1}{s} = 1 - \frac{1}{r} - \frac{1}{p^{**}} > \frac{q-1}{p^{**}}. \quad (2.10)$$

Therefore, it suffices to take s such that

$$\max(1, q-1) < s < p^{**}. \quad (2.11)$$

Step 2 ($p = N/2$). In this case, for any $s > 1$, $W_0^{1,p} \cap W^{2,p}(\Omega) \hookrightarrow L^s(\Omega)$. Thus, for $t = 1/(1 - (r + q')/rq')$ (some t exists because $r > q = q'/(q' - 1)$), we get

$$\frac{1}{t} + \frac{1}{q'} + \frac{1}{r} = 1. \quad (2.12)$$

Hölder's inequality yields

$$\left| \int \rho |u|^q \right| \leq \|\rho\|_r \|u\|_i \|u\|_{q'}. \quad (2.13)$$

Step 3 ($p > N/2$). In this case we have $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ and $r = 1$. Thus,

$$\int_{\Omega} \rho |u|^q dx \leq \|u\|_{\infty}^q \|\rho\|_1 \leq c \|u\|^q \|\rho\|_1. \quad (2.14)$$

To prove the continuity of $u \rightarrow \rho |u|^{q-2} u$, we argue as follows.

Let $(u_n)_{n \geq 0} \subset X$ such that $u_n \rightarrow u$ in X . Thus, $\rho |u_n|^{q-2} u_n \rightarrow \rho |u|^{q-2} u$ in X' , that is,

$$\sup_{v \in X, \|v\| \leq 1} \left| \int_{\Omega} \rho \left[|u_n|^{q-2} u_n - |u|^{q-2} u \right] v dx \right| = o(1). \quad (2.15)$$

We prove it in the case $1 < p < N/2$, $1 < q < p^{**}$ and $\rho \in L^r(\Omega)$ with r satisfying (i).

Using (2.6) or (2.10), we obtain by calculation

$$\begin{aligned} & \sup_{v \in X, \|v\|_{1,p} \leq 1} \left| \int_{\Omega} \rho \left[|u_n|^{q-2} u_n - |u|^{q-2} u \right] v dx \right| \\ & \leq \sup_{v \in X, \|v\|_{1,p} \leq 1} \left[\|\rho\|_r \left\| \left(|u_n|^{q-2} u_n - |u|^{q-2} u \right) \right\|_{s/(q-1)} \|v\|_{p^{**}} \right] \\ & \leq c \|\rho\|_r \left\| \left(|u_n|^{q-2} u_n - |u|^{q-2} u \right) \right\|_{s/(q-1)}. \end{aligned} \quad (2.16)$$

Therefore, the desired result can be obtained since the limit

$$\left\| \left(|u_n|^{q-2} u_n - |u|^{q-2} u \right) \right\|_{s/(q-1)} = o(1) \quad (2.17)$$

holds by using the continuity of the Nemytskii operator $u \rightarrow |u|^{q-2} u$ from $L^s(\Omega) \rightarrow L^{s/(q-1)}(\Omega)$ and the right embedding of Sobolev space. $\square$

Remark 2.3. If u is a solution of (E) associated to $\lambda = 1$, then, for any $\alpha > 0$, $\alpha^{1/(p-q)} u$ is also a weak solution of (E) associated to α . Hence, we can reduce the problem to $\lambda = 1$.

Lemma 2.4. For $p \in (1, +\infty)$ we have the following assertions

- (i) if $1 < q < p$ then $\mathcal{A}$ is bounded from below,
- (ii) if $p < q < p^{**}$ thus, there exist two real $\sigma > 0$ and $\delta > 0$ such that:

$$A(u) > \delta \text{ if } 0 < \|u\| < \sigma,$$

$$A(u) \geq 0 \text{ if } \|u\| = \sigma.$$

Proof.

Step 1 ($1 < q < p$). We have

$$\begin{aligned}\mathcal{A}(u) &= \frac{1}{p}\|u\|^p - \frac{1}{q} \int_{\Omega} \rho |u|^q \\ &\geq \frac{1}{p}\|u\|^p - \frac{c}{q} \|\rho\|_r \|u\|^q \\ &\geq \|u\|^q \left[\frac{1}{p}\|u\|^{p-q} - \frac{c}{q} \|\rho\|_r \right],\end{aligned}\tag{2.18}$$

for a positive constant c .

If $\|u\| \geq ((cp/q)\|\rho\|_r)^{1/(p-q)}$, then

$$\mathcal{A}(u) \geq 0.\tag{2.19}$$

If $\|u\| < ((cp/q)\|\rho\|_r)^{1/(p-q)}$, then

$$\mathcal{A}(u) \geq c' \quad \text{with } c' = -\frac{c}{q} \|\rho\|_r \left(\frac{cp}{q} \|\rho\|_r \right)^{1/(p-q)}.\tag{2.20}$$

Step 2 ($p < q < p^{**}$). From (2.18) we deduce that

$$A(u) \geq \frac{1}{p}\|u\|^q \left[1 - \frac{pc}{q} \|\rho\|_r \|u\|^{q-p} \right].\tag{2.21}$$

If $\|u\| \leq ((q/pc)(1/\|\rho\|_r))^{1/(q-p)}$, then there exists $\delta > 0$ such that

$$\mathcal{A}(u) > \delta.\tag{2.22}$$

If $\|u\| = ((q/pc)(1/\|\rho\|_r))^{1/(q-p)}$, we get

$$\mathcal{A}(u) > 0.\tag{2.23}$$

The estimation above completes the proof. $\square$

Proposition 2.5. *If the pair (p, q) satisfies (1.1) with $p \neq q$, then $\mathcal{A}$ satisfies the (PS) condition.*

Proof. Let $(u_j)_j$ be a sequence of Palais-Smale of $\mathcal{A}$ in X . Thus there exists $M > 0$ such that

$$|\mathcal{A}(u_j)| \leq M \tag{PS}_1$$

and for any $\varepsilon > 0$ there is $j_0 > 0$ such that

$$|\langle \mathcal{A}'(u_j), u_j \rangle| \leq \varepsilon \|u_j\|, \quad \forall j \geq j_0. \quad (\text{PS})_2$$

Claim that $(u_j)_j$ is a bounded sequence in X .

Step 1 ($1 < p < q$). By (2.18) we deduce that $\mathcal{A}$ is coercive, then the claim follows.

Step 2 ($p < q < p^{**}$). Equation $((\text{PS})_2)$ implies that, for all $j > j_0$,

$$-\varepsilon \|u_j\| \leq \langle \mathcal{A}'(u_j), u_j \rangle \leq \varepsilon \|u_j\|. \quad (2.24)$$

Thanks to $((\text{PS})_1)$, we obtain,

$$-M \leq \frac{1}{p} \int_{\Omega} |\Delta u_j|^p dx - \frac{1}{q} \int_{\Omega} \rho |u_j|^q dx \leq M. \quad (2.25)$$

Multiplying (2.24) by $(-1/q)$ and adding with (2.18), we obtain,

$$\left(\frac{1}{p} - \frac{1}{q}\right) \|u_j\|^p \leq M + \frac{\varepsilon}{q} \|u_j\|. \quad (2.26)$$

Hence, there exists a positive constant $c > 0$ such that

$$\|u_j\| \leq c \quad \forall j \geq j_0. \quad (2.27)$$

So (u_j) is bounded in the two cases in X and the claim is concluded. Consequently, there exists a subsequence still denoted $(u_j)_j$ converges weakly for some $u \in X$ and strongly in $L^p(\Omega)$ and in $L^q(\Omega)$ for all (p, q) satisfies (1.2).

Let

$$\langle Ju, v \rangle = \int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta v dx. \quad (2.28)$$

Thus

$$\langle A'(u_j) - A'(u), u_j - u \rangle + \int_{\Omega} \rho (|u_j|^{q-2} u_j - |u|^{q-2} u) (u_j - u) = \langle Ju_j - Ju, u_j - u \rangle. \quad (2.29)$$

Since $u \rightarrow \rho |u|^{q-2} u$ is strongly continuous from X into X' , we deduce from $((\text{PS})_2)$, as $j \rightarrow +\infty$ in (2.29), that

$$\lim_{j \rightarrow +\infty} \langle Ju_j - Ju, u_j - u \rangle = 0. \quad (2.30)$$

On the other hand, we have

$$\begin{aligned} \langle Ju_j - Ju, u_j - u \rangle &\geq \left(\|\Delta u_j\|_p^{p-1} - \|\Delta u\|_p^{p-1} \right) \left(\|\Delta u_j\|_p - \|\Delta u\|_p \right). \\ &\geq 0. \end{aligned} \quad (2.31)$$

This and (2.30) imply that

$$\|\Delta u_j\|_p \longrightarrow \|\Delta u\|_p \quad \text{as } j \longrightarrow +\infty. \quad (2.32)$$

Finally, since $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ is uniformly convex, the proof is achieved. $\square$

3. Main Results

3.1. The Case $1 < q < p$: Local Minimization Technique

In this subsection, we show that the problem (E) has a sequence of weak solution by using the results of Lusternik-Shnirelman [9]. In other words, we use a local minimization for the corresponding energy functional.

Let $\Gamma_n = \{B \subset W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega), \text{ compact symmetric and } \gamma(B) \geq n\}$ and set

$$c_n = \inf_{B \in \Gamma_n} \left(\sup_{u \in B} \mathcal{A}(u) \right), \quad \forall n \in \mathbb{N}^*. \quad (3.1)$$

We will show that the sequence c_n , defined by (3.1) is critical values of $\mathcal{A}$. Here and in the following $\gamma(B) = k$ is the genus of B , that is, the smallest integer k such that there is an odd continuous map from B into $\mathbb{R}^k \setminus \{0\}$.

Lemma 3.1. *For any $k \in \mathbb{N}^*$,*

$$\Gamma_k \neq \emptyset. \quad (3.2)$$

Proof. $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ is separable. Therefore, for any $k \in \mathbb{N}^*$, there exists a finite sequence of functions $u_1, \dots, u_k$ in $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ linearly independent such that $\text{supp } u_i \cap \text{supp } u_j = \emptyset$ for $i \neq j$ and thanks to (1.3) we can choose u_i such that $\int_{\Omega} g|u_i|^q dx = 1$. Let $F_k = \text{span}(u_1, \dots, u_k)$ be a subspace in $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ spanned by u_i of dimension k .

If $v \in F_k$, then, there exist $\beta_1, \dots, \beta_k$ real numbers such that $v = \sum_{i=1}^k \beta_i u_i$.

Thus, $\int_{\Omega} \rho|v|^q dx = \sum_{i=1}^k |\beta_i|^q$. Hence, the map

$$v \longrightarrow \left(\int_{\Omega} \rho|u|^q dx \right)^{1/q} \quad (3.3)$$

is a norm in F_k . Consequently, there exists a constant $c > 0$ such that

$$c\|u\| \leq \left(\int \rho|u|^q dx \right)^{1/q} \leq \frac{1}{c}\|u\|, \quad \forall u \in F_k. \quad (3.4)$$

Set

$$U_k = F_k \cap \left\{ v \in X : \frac{\theta}{2} \leq \int_{\Omega} \rho|v|^q dx \leq \theta \right\} \quad (3.5)$$

with $\theta = (p/qc^p)^{q/(p-1)}$. It is clear that U_k is a closed neighborhood, symmetric, compact not containing 0. Finally, by the property of genus we get $\gamma(U_k) = k$ and $\Gamma_k \neq \emptyset$. $\square$

Theorem 3.2. *Let c_n be a sequence defined by (3.1). Then, we have the following statements:*

- (i) *For any $n \in \mathbb{N}^*$, there exist n distinct pairs critical points of the functional $\mathcal{A}$.*
- (ii) *If*

$$-\infty < c = c_n = c_{n+1} = \dots = c_{n+k} \quad (3.6)$$

then

$$\gamma(K_c) \geq k + 1, \quad (3.7)$$

where

$$K_c = \left\{ u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \text{ such that } \mathcal{A}'(u) = 0, \mathcal{A}(u) = c \right\}. \quad (3.8)$$

Proof. From the result of [10], it suffices to prove (i).

Recall that the functional A is even, bounded from below and $\sup_X A(u) < 0$.

In view of Lemma 2.4, A is of class C^1 on X ; satisfying the (PS) condition.

Hence, these properties confirm the hypotheses of Clark's Lemma cf. [11] which proves (i). $\square$

Remark 3.3. The set $K = \{u \in W_0^{1,p}(\Omega) \cap W^{1,p}(\Omega) : \mathcal{A}'(u) = 0\}$ is compact. Indeed, let $(u_n)_n$ be a sequence in K , that is,

$$\mathcal{A}'(u_n) = 0, \quad \forall n \in \mathbb{N}. \quad (3.9)$$

It is clear that, for $p > q$,

$$\mathcal{A}(u_n) \leq \langle \mathcal{A}'(u_n), u_n \rangle = 0, \quad \forall n. \quad (3.10)$$

Since A is bounded from below, by Lemma 2.4, $(u_n)_n$ is a sequence of Palais-Smale (PS). Thus, $(u_n)_n$ possesses a convergent subsequence.

Consequently, K is compact in $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ and the functional $\mathcal{A}$ has an infinity of critical points.

3.2. The Case $p < q < p^{**}$: Mountain Pass Theorem

Here, we prove the existence of solutions of problem (E), by using Mountain Pass theorem [11].

Lemma 3.4. *Suppose that $p < q < p^{**}$ and $\rho \in L^r(\Omega)$ with r satisfying (1.2). Hence there exists $e \in X \setminus \{0\}$ such that*

$$\mathcal{A}(e) \leq 0. \quad (3.11)$$

Proof. We have for any $u \in X$ and $t \in \mathbb{R}$,

$$\mathcal{A}(tu) = \frac{t^p}{p} \int_{\Omega} |\Delta u|^p dx - \frac{t^q}{q} \int_{\Omega} \rho |u|^q dx. \quad (3.12)$$

From (1.3), there exists $u_0 \in X$ such that $\int_{\Omega} \rho |u_0|^q dx = 1$. Thus,

$$\lim_{t \rightarrow +\infty} \mathcal{A}(tu_0) = -\infty, \quad \text{because } p < q. \quad (3.13)$$

Hence, there is $t_0 > 0$ large enough so that $\mathcal{A}(t_0 u_0) < 0$. To achieve the proof, it suffices to take $e = t_0 u_0$. $\square$

Consequently, we conclude the following statements.

- (a) $\mathcal{A}$ is C^1 of class C^1 , even and $\mathcal{A}(0) = 0$.
- (b) $\mathcal{A}$ satisfies the (PS) condition.
- (c) There exist two positive real $l > 0$, $\delta > 0$ such that

$$\begin{aligned} \mathcal{A}(u) &> \delta \quad \text{if } 0 < \|u\| < l, \\ \mathcal{A}(u) &\geq 0 \quad \text{if } \|u\| = l. \end{aligned} \quad (3.14)$$

- (d) There exists $e \in X \setminus \{0\}$ such that $\mathcal{A}(e) \leq 0$.

Now, we can establish the following theorem.

Theorem 3.5. *If $p < q < p^{**}$, then the value c defined by*

$$c = \inf_{\gamma \in G} \max_{t \in [0,1]} \mathcal{A}(\gamma(t)) \quad (3.15)$$

is a critical value of $\mathcal{A}$.

Here,

$$G = \left\{ \gamma \in C([0,1]; W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)), \gamma(0) = 0, \gamma(1) = e \right\}. \quad (3.16)$$

Proof. In view of Lemma 2.4., Proposition 2.5, and properties (a)–(d), c is a critical value of $\mathcal{A}$ by applying the Mountain Pass Theorem due to [11]. $\square$

Acknowledgment

The author gratefully acknowledges the financial support provided by Al-Imam Muhammed Ibn Saud Islamic University during this research.

References

- [1] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Lineaires*, Dunod, Paris, France, 1969.
- [2] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, Germany, 2nd edition, 1983.
- [3] J. Marcos do Ó and P. Ubilla, "A multiplicity result for a class of superquadratic Hamiltonian systems," *Electronic Journal of Differential Equations*, vol. 2003, no. 15, 14 pages, 2003.
- [4] P. Drábek and M. Ôtani, "Global bifurcation result for the p -biharmonic operator," *Electronic Journal of Differential Equations*, vol. 2001, no. 48, 19 pages, 2001.
- [5] A. El khalil, S. Kelati, and A. Touzani, "On the spectrum of the p -biharmonic operator," *Electronic Journal of Differential Equations*, pp. 191–170, 2002, 2002-Fez, Conference on Partial Differential Equation.
- [6] A. El khalil, S. Kelati, and A. Touzani, "A nonlinear boundary problem involving the p -biharmonic operator," *International Journal of Mathematics and Mathematical Sciences*, vol. 10, pp. 1525–1537, 2005.
- [7] J. P. García Azorero and I. Peral Alonso, "Existence and nonuniqueness for the p -Laplacian: nonlinear eigenvalues," *Communications in Partial Differential Equations*, vol. 12, no. 12, pp. 1389–1430, 1987.
- [8] A. El khalil and A. Touzani, "On the first eigencurve of the p -Laplacian," in *Partial Differential Equations*, vol. 229 of *Lecture Notes in Pure and Applied Mathematics*, pp. 195–205, Dekker, New York, NY, USA, 2002.
- [9] A. Szulkin, "Ljusternik-Schnirelmann theory on C^1 -manifolds," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 5, no. 2, pp. 119–139, 1988.
- [10] D. G. Costa, "Tópicos em análise não linear e equações diferenciais," in *Proceedings of ELAM VIII*, Springer Lecture Notes, Rio de Janeiro, Brazil, 1986.
- [11] A. Ambrosetti and P. H. Rabinowitz, "Dual variational methods in critical point theory and applications," *Journal of Functional Analysis*, vol. 14, pp. 349–381, 1973.