Research Article

Unital Four-Dimensional Lattice-Ordered Algebras with a Distributive Basis

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Received 5 May 2011; Accepted 2 June 2011

Academic Editors: V. De Filippis, W. Gao, and T. Nakatsu

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All unital four-dimensional lattice-ordered algebras with a d -basis are constructed.

1. Introduction

Lattice-ordered algebras (ℓ -algebras) with a distributive basis (d -basis) were studied in [5] in an effort to obtain general structure theory for a class of ℓ -algebras that includes some well-known and important examples of ℓ -algebras, such as matrix and triangular matrix ℓ -algebras over a totally ordered field with entrywise lattice order, polynomial ℓ -algebra over a totally ordered field with coordinatewise lattice order, and so forth.

All unital two-dimensional and three-dimensional ℓ -algebras with a d -basis have been constructed [6]. In this paper, we construct all unital four-dimensional ℓ -algebras with a d -basis. We will first review some definitions and results for later use. The reader is referred to [1, 5, 6] for the general theory of lattice-ordered rings (ℓ -rings) and undefined terminologies in this paper. Throughout F always denotes a totally ordered field.

Let R be an ℓ -ring. The *positive cone* of R is $R^+ = \{r \in R : r \geq 0\}$. A *vector lattice* V over F is an ℓ -group and a vector space over F such that $F^+V^+ \subseteq V^+$. An ℓ -algebra A over F is an algebra over F , that is, an ℓ -ring and a vector lattice over F . An ℓ -algebra is called *unital* if it has an identity element 1 and called ℓ -unital if $1 \geq 0$. An element a in an ℓ -ring is called *basic* if $a > 0$ and, for any $b, c \geq 0$, $a \geq b, c$ implies that b and c are comparable, that is, either $b \geq c$ or $c \geq b$. A set of elements $\{x_i : i \in I\}$ is called *disjoint* if each $x_i > 0$ and $x_i \wedge x_j = 0$ for

any two different elements x_i and x_j . An element $x \geq 0$ in an ℓ -ring R is called an f -element if $y \wedge z = 0$ implies that $xy \wedge z = yx \wedge z = 0$, and x is called a d -element if $y \wedge z = 0$ implies that $xy \wedge xz = yx \wedge zx = 0$. An ℓ -ring R is called ℓ -reduced if for any $a \in R^+$, $a^2 = 0$ implies that $a = 0$.

Let A be an ℓ -algebra over F . A nonempty subset S of A is called a d -basis of A if S is a disjoint subset of A that spans A as a vector space over F and each element in S is a d -element [6, Definition 2.1]. Then for each $a \in A$, $a = \alpha_1 s_1 + \cdots + \alpha_p s_p$, where $p \geq 1$, $\alpha_1, \dots, \alpha_p \in F$, and $s_1, \dots, s_p \in S$ are distinct, and $a \geq 0$ if and only if each $\alpha_i \geq 0$, $i = 1, \dots, p$. Thus in a d -basis, each element is basic. Let A be a unital finite-dimensional ℓ -algebra with a d -basis. Then A is ℓ -unital [1], and there exists a d -basis S such that $1 = e_1 + \cdots + e_n$, where $e_1, \dots, e_n \in S$ are distinct.

In the following theorem, we record some results on unital ℓ -algebras with a d -basis from [5, 6].

Theorem 1.1. *Let A be a unital finite-dimensional ℓ -algebra over F with a d -basis S such that $1 = e_1 + \cdots + e_n$, where $n \geq 1$ and $e_1, \dots, e_n \in S$ are distinct.*

- (1) *Each e_i is an idempotent f -element and $e_i e_j = 0$ for $i \neq j$.*
- (2) *Let a be a basic element. Then there exists exactly one e_i such that $e_i a = a$, and for any $j \neq i$, $e_j a = 0$. Similarly, there exists exactly one e_k such that $a e_k = a$, and for any $j \neq k$, $a e_j = 0$.*
- (3) *For each element $a \in S$, if $e_i a = a$ or $a e_i = a$, $1 \leq i \leq n$, then either a is nilpotent or $a^{n_a} = \alpha_a e_i$ for some positive integer n_a and $0 < \alpha_a \in F$.*
- (4) *Let $a, b \in S$. If either a or b is not nilpotent and $ab \neq 0$, then ab is basic.*

2. The Classification of Unital Four-Dimensional ℓ -Algebras with a d -Basis

We first list some ℓ -algebras that we are going to use for the classification of unital four-dimensional ℓ -algebras over F with a d -basis.

- (i) Let G be a finite group, and let $F[G]$ be the group algebra over F . Define an element $\sum_i \alpha_i g_i \geq 0$ if each $\alpha_i \geq 0$. Then $F[G]$ is an ℓ -algebra over F and G is a d -basis. In the following, $F[G]$ always denotes the ℓ -algebra with the lattice order defined above.
- (ii) Let $0 < \alpha \in F$, and let $f(x) = x^n - \alpha \in F[x]$ be irreducible. We use $a = \sqrt[n]{\alpha}$ to denote a fixed root of $f(x)$. Consider the extension field $F[\sqrt[n]{\alpha}]$ of F . Then

$$F[\sqrt[n]{\alpha}] = \left\{ \alpha_{n-1} a^{n-1} + \cdots + \alpha_1 a + \alpha_0 \mid \alpha_i \in F \right\}, \quad (2.1)$$

and $\{a^{n-1}, \dots, a, 1\}$ is a basis of $F[\sqrt[n]{\alpha}]$ over F as a vector space over F . Now define $\alpha_{n-1} a^{n-1} + \cdots + \alpha_1 a + \alpha_0 \geq 0$ if each $\alpha_i \geq 0$, $i = 0, 1, \dots, n-1$. Since $a^n = \alpha > 0$, $F[\sqrt[n]{\alpha}]$ becomes an ℓ -algebra over F with respect to the above lattice order, and a is a d -element. Thus $\{a^{n-1}, \dots, a, 1\}$ is a d -basis of the ℓ -algebra $F[\sqrt[n]{\alpha}]$ over F . In this paper, $n \leq 4$ and $F[\sqrt[n]{\alpha}]$ always denotes the ℓ -algebra defined above.

- (iii) For $n \geq 1$, let $M_n(F)$ and $T_n(F)$ be $n \times n$ matrix algebra and upper triangular matrix algebra, respectively. Define a matrix positive if each of its entries is positive. Then

$M_n(F)$ and $T_n(F)$ become ℓ -algebras. Let e_{ij} denote the matrix with ij th entry equal to 1 and other entries equal to 0. Then $\{e_{ij} \mid 1 \leq i, j \leq n\}$ is a d -basis for $M_n(F)$ over F and $\{e_{ij} \mid 1 \leq i \leq j \leq n\}$ is a d -basis for $T_n(F)$ over F . In the following, $M_n(F)$ and $T_n(F)$ always denote the ℓ -algebras with the above lattice order.

All nonisomorphic unital four-dimensional ℓ -algebras with a d -basis are listed in Theorem 2.1. Suppose that A is a unital four-dimensional ℓ -algebra with a d -basis S such that 1 is a sum of some distinct elements in S . In the proof of Theorem 2.1, we consider four different cases according to the number of elements in this sum, and in each case we consider two cases in which A is ℓ -reduced and A is not ℓ -reduced, respectively. We use $\oplus$ to denote a direct sum of ℓ -rings.

Suppose that A is a four-dimensional ℓ -algebra with a d -basis $S = \{e_1, e_2, e_3, e_4\}$ such that $1 = e_1 + e_2 + e_3 + e_4$. By Theorem 1.1, each e_i is an idempotent f -element, and for $1 \leq i, j \leq 4, i \neq j, e_i e_j = 0$. Thus $A \cong F \oplus F \oplus F \oplus F$ is a four-dimensional f -algebra. This is (1) in Theorem 2.1.

In the following sections, we consider the remaining three cases where 1 is a sum of three, two, or one element from the d -basis S , respectively.

Theorem 2.1. *Let A be a unital four-dimensional ℓ -algebra over F with a d -basis. Then A is isomorphic to one of following ℓ -algebras over F :*

- (1) $F \oplus F \oplus F \oplus F$,
- (2) $F[G] \oplus F \oplus F$, where G is a group of order two,
- (3) $F[\sqrt{\alpha}] \oplus F \oplus F$, where $0 < \alpha \in F$,
- (4) $(F1 + Fa) \oplus F \oplus F$, where $a^2 = 0$,
- (5) $T_2(F) \oplus F$,
- (6) $F[G] \oplus F[G]$, where G is a group of order two,
- (7) $F[G] \oplus F[\sqrt{\alpha}]$, where $0 < \alpha \in F$ and G is a group of order two,
- (8) $F[\sqrt{\alpha}] \oplus F[\sqrt{\beta}]$, where $0 < \alpha, \beta \in F$,
- (9) $F[G] \oplus F$, where G is a group of order three,
- (10) $F[\sqrt[3]{\alpha}] \oplus F$, where $0 < \alpha \in F$,
- (11) $M_2(F)$,
- (12) $(F1 + Fa + Fb) \oplus F$, where $a^2 = b^2 = ab = ba = 0$,
- (13) $(F1 + Fa) \oplus (F1 + Fb)$, where $a^2 = b^2 = 0$,
- (14) $T_2(F) + Fa$, where $e_{11}a = ae_{11} = a, ae_{12} = 0$, and $a^2 = 0$,
- (15) $T_2(F) + Fa$, where $e_{11}a = ae_{22} = a$, and $a^2 = 0$,
- (16) $T_2(F) + Fa$, where $e_{22}a = ae_{11} = a, e_{12}a = ae_{12} = 0$, and $a^2 = 0$,
- (17) $T_2(F) + Fa$, where $e_{22}a = ae_{22} = a, e_{12}a = 0$, and $a^2 = 0$,
- (18) $(F1 + Fa + Fa^2) \oplus F$, where $a^3 = 0$,
- (19) $F[G] \oplus (F1 + Fa)$, where $a^2 = 0$,
- (20) $F[\sqrt{\alpha}] \oplus (F1 + Fa)$, where $a^2 = 0$ and $0 < \alpha \in F$,
- (21) $F[G]$, where G is a cyclic group of order four,
- (22) $F[\sqrt[4]{\alpha}]$, where $0 < \alpha \in F$,

- (23) $F[G] + F[G]b$, where $G = \{e, a\}$ is a group of order two and $b^2 = \alpha a$ with $0 < \alpha \in F$ and $\sqrt{\alpha} \notin F$,
- (24) $F[G]$, where G is the Klein four group,
- (25) $F[\sqrt{\alpha}][G]$, where $0 < \alpha \in F$ and G is a group of order two,
- (26) $F[\sqrt{\alpha}][\sqrt{\beta}]$, where $0 < \alpha, \beta \in F$, $\sqrt{\beta} \notin F[\sqrt{\alpha}]$,
- (27) $F1 + Fa + Fb + Fc$, where $a^2 = b + c$ and the product of any other two elements of a, b, c is zero,
- (28) $F1 + Fa + Fa^2 + Fa^3$, where $a^4 = 0$,
- (29) $F1 + Fa + Fa^2 + Fb$, where $b^2 = \alpha a^2$, $0 < \alpha \in F$, and the product of any two elements not both a or b is zero,
- (30) $F1 + Fa + Fa^2 + Fb$, where the product of any two elements not both a is zero,
- (31) $F1 + Fa + Fb + Fc$, where $Fa + Fb + Fc$ is a zero ring,
- (32) $F1 + Fa + Fb + F(ab)$, where $a^2 = b^2 = ba = 0$,
- (33) $F1 + Fa + Fb + F(ab)$, where $a^2 = b^2 = 0$ and $ba = \alpha(ab)$ with $0 < \alpha \in F$,
- (34) $F[G] + F[G]a$, where G is a group of order two, $a^2 = 0$, and a commutes with each element in $F[G]$,
- (35) $F[\sqrt{\alpha}] + F[\sqrt{\alpha}]a$, where $a^2 = 0$ and a commutes with each element in $F[\sqrt{\alpha}]$.

3. 1 Is a Sum of Three Disjoint Basic Elements

Let A be a unital four-dimensional ℓ -algebra with a d -basis $S = \{e_1, e_2, e_3, a\}$ such that $1 = e_1 + e_2 + e_3$. Then by Theorem 1.1, each e_i is an idempotent f -element, $i = 1, 2, 3$, and $e_i e_j = 0$ for any $1 \leq i, j \leq 3, i \neq j$. By Theorem 1.1, we may assume that $e_1 a = a$ and $e_2 a = e_3 a = 0$. We consider two cases according to whether A is ℓ -reduced or not.

3.1. A Is ℓ -Reduced

If $ae_1 = 0$, then $ae_i = a$ for $i = 2$ or 3 by Theorem 1.1(2), and hence $a^2 = (ae_i)(e_1 a) = 0$, which contradicts the fact that A is ℓ -reduced. Thus $ae_1 = a$ and $ae_2 = ae_3 = 0$. Since a is not nilpotent, by Theorem 1.1(4), a^2 is a basic element, so $a^2 = \alpha s$ for some $0 < \alpha \in F$ and $s \in S$. Since $e_1 \wedge a = 0$ and $e_1 a = a$, $s \neq a$. If $s = e_2$ or e_3 , then $a^3 = (\alpha s)a = \alpha(sa) = 0$, which is a contradiction. Thus $a^2 = ae_1$ for some $0 < \alpha \in F$. There are two cases now.

- (1) There exists $0 < \beta \in F$ such that $\beta^2 = \alpha$.

Let $b = a/\beta$. Then $\{e_1, e_2, e_3, b\}$ is also a d -basis with the following multiplication table:

	e_1	b	e_2	e_3
e_1	e_1	b	0	0
b	b	e_1	0	0
e_2	0	0	e_2	0
e_3	0	0	0	e_3

(3.1)

Thus $A \cong F[G] \oplus F \oplus F$, where G is a group of order two. This is (2) in Theorem 2.1.

(2) The polynomial $f(x) = x^2 - \alpha \in F[x]$ has no root in F .

Then we have the following multiplication table for S :

$$\begin{array}{c|cccc}
 & e_1 & a & e_2 & e_3 \\
 \hline
 e_1 & e_1 & a & 0 & 0 \\
 a & a & \alpha e_1 & 0 & 0 \\
 e_2 & 0 & 0 & e_2 & 0 \\
 e_3 & 0 & 0 & 0 & e_3
 \end{array} \tag{3.2}$$

Thus $A \cong F[\sqrt{\alpha}] \oplus F \oplus F$. This is (3) in Theorem 2.1.

3.2. A Is Not ℓ -Reduced

In this case, a must be nilpotent. If $a^2 \neq 0$, then a^2 must be a positive linear combination of elements in S . Since each of e_1, e_2, e_3 is not nilpotent, we must have $a^2 = \beta a$ for some $0 < \beta \in F$. Then a is nilpotent implying that $a^2 = \beta a = 0$, which is a contradiction. Thus we must have $a^2 = 0$. There are two cases that we need to consider.

(1) $ae_1 = a$.

Then $ae_2 = ae_3 = 0$, and the multiplication table for $S = \{e_1, e_2, e_3, a\}$ is given below:

$$\begin{array}{c|cccc}
 & e_1 & a & e_2 & e_3 \\
 \hline
 e_1 & e_1 & a & 0 & 0 \\
 a & a & 0 & 0 & 0 \\
 e_2 & 0 & 0 & e_2 & 0 \\
 e_3 & 0 & 0 & 0 & e_3
 \end{array} \tag{3.3}$$

Thus $A \cong (F1 + Fa) \oplus F \oplus F$, where $a^2 = 0$ and $1a = a1 = a$. This is (4) in Theorem 2.1.

(2) $ae_1 = 0$.

Without loss of generality, we may assume that $ae_2 = a$ and $ae_3 = 0$. Then we have the following multiplication table for elements in S :

$$\begin{array}{c|cccc}
 & e_1 & e_2 & a & e_3 \\
 \hline
 e_1 & e_1 & 0 & a & 0 \\
 e_2 & 0 & e_2 & 0 & 0 \\
 a & 0 & a & 0 & 0 \\
 e_3 & 0 & 0 & 0 & e_3
 \end{array} \tag{3.4}$$

Let $T_2(F)$ be the 2×2 upper triangular matrix algebra over F with the entrywise lattice order. Then $T_2(F)$ becomes an ℓ -algebra over F with a d -basis $\{e_{11}, e_{22}, e_{12}\}$. The multiplication table for $\{e_{11}, e_{22}, e_{12}\}$ is given below:

$$\begin{array}{c|ccc}
 & e_{11} & e_{22} & e_{12} \\
 \hline
 e_{11} & e_{11} & 0 & e_{12} \\
 e_{22} & 0 & e_{22} & 0 \\
 e_{12} & 0 & e_{12} & 0
 \end{array} \tag{3.5}$$

Since $\{e_1, e_2, a\}$ and $\{e_{11}, e_{22}, e_{12}\}$ have the same multiplication tables, we have $A \cong T_2(F) \oplus F$. This is (5) in Theorem 2.1.

4. 1 Is a Sum of Two Disjoint Basic Elements

Let A be a unital ℓ -algebra with a d -basis $S = \{e_1, e_2, a, b\}$ such that $1 = e_1 + e_2$. By Theorem 1.1, $ae_i = a$ for some $i = 1$ or 2 . Then $e_i \wedge b = 0$ implies that $a \wedge ab = 0$. Similarly $b \wedge ab = 0$. Thus $ab = \alpha e_1 + \beta e_2$ for some $\alpha, \beta \in F^+$. By a similar argument, $ba = \gamma e_1 + \delta e_2$, for some $\gamma, \delta \in F^+$.

4.1. A Is ℓ -Reduced

We may assume that $e_1 a = a$ and $e_2 a = 0$. Since A contains no nonzero positive nilpotent elements, we also have $ae_1 = a$ and $ae_2 = 0$. Then $ab = \alpha e_1 + \beta e_2$ implies that $ab = e_1 ab = \alpha e_1$. Similarly, $ba = \gamma e_1$.

Since A is ℓ -reduced, a is not a nilpotent element, so a^2 is basic by Theorem 1.1. Thus $a^2 = \sigma s$ for some $s \in S$ and $0 < \sigma \in F$. Since $e_1 \wedge a = 0$ and $e_1 a = a$, $s \neq a$. If $s = e_2$, then $a^3 = \sigma(ae_2) = 0$, which is a contradiction. Thus $s = e_1$ or $s = b$. We consider these two cases separately.

(1) $a^2 = \sigma e_1$.

Suppose that $ab \neq 0$. Since $ab = \alpha e_1$, $a^2 b = \alpha a$, so $\sigma(e_1 b) = \alpha a$, which is a contradiction. Thus $ab = 0$, and hence $ba = 0$ since $(ba)^2 = 0$. Since $0 = ba^2 = \sigma b e_1$, $b e_1 = 0$. Thus $b e_2 = b$, and hence $e_1 b = 0$ and $e_2 b = b$. Now b is not nilpotent implying that b^2 is a basic element, so $b^2 = \lambda s$ for some $s \in S$ and $0 < \lambda \in F$. Since $e_2 \wedge b = 0$ and $e_2 b = b$, $s \neq b$. If $b^2 = \lambda e_1$ or $b^2 = \lambda a$, then $\lambda a = 0$ or $\lambda a^2 = 0$, which are contradictions. Thus we must have $b^2 = \lambda e_2$. Then we have the following multiplication table for the elements of S :

	e_1	a	e_2	b	
e_1	e_1	a	0	0	
a	a	σe_1	0	0	
e_2	0	0	e_2	b	
b	0	0	0	λe_2	(4.1)

Depending on $\sqrt{\sigma} \in F$ or $\sqrt{\sigma} \notin F$ and $\sqrt{\lambda} \in F$ or $\sqrt{\lambda} \notin F$, we have the following cases:

$$A \cong F[G] \oplus F[G], \quad A \cong F[G] \oplus F[\sqrt{\sigma}], \quad \text{or} \quad A \cong F[\sqrt{\sigma}] \oplus F[\sqrt{\lambda}], \quad (4.2)$$

where G is a group of order two. These are (6), (7), and (8) in Theorem 2.1.

(2) $a^2 = \sigma b$.

Since $a^2 = \sigma b$ with $0 < \sigma \in F$, $T = \{e_1, e_2, a, a^2\}$ is also a d -basis for the ℓ -algebra A

over F and $a^3 = \sigma(ab) = (\alpha\sigma)e_1$. Let $\mu = \alpha\sigma$. Then $\mu > 0$ since $a^3 \neq 0$, so we have the following multiplication table for elements in T :

$$\begin{array}{c|cccc}
 & e_1 & a & a^2 & e_2 \\
 \hline
 e_1 & e_1 & a & a^2 & 0 \\
 a & a & a^2 & \mu e_1 & 0 \\
 a^2 & a^2 & \mu e_1 & \mu a & b \\
 e_2 & 0 & 0 & 0 & e_2
 \end{array} \tag{4.3}$$

Now it is clear that if $\sqrt[3]{\mu} \in F$, then $A \cong F[G] \oplus F$, where G is a group of order three, and if $\sqrt[3]{\mu} \notin F$, then $A \cong F[\sqrt[3]{\mu}] \oplus F$, where $F[\sqrt[3]{\mu}]$ is the ℓ -field extension of F of order three. Those are (9) and (10) in Theorem 2.1.

4.2. A Is Not ℓ -Reduced

Since A is not ℓ -reduced, a or b is nilpotent. Let's assume that $e_1a = a$, and hence $e_2a = 0$. Suppose that a is nilpotent and $a^2 \neq 0$. Then

$$a^2 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 a + \alpha_4 b, \quad \alpha_i \in F^+. \tag{4.4}$$

Let k be the smallest positive integer such that $a^k = 0$. Then $k \geq 3$ and

$$0 = a^k = a^2 a^{k-2} = \alpha_1 e_1 a^{k-2} + \alpha_2 e_2 a^{k-2} + \alpha_3 a^{k-1} + \alpha_4 b a^{k-2}, \tag{4.5}$$

so $\alpha_1 a^{k-2} = \alpha_3 a^{k-1} = 0$. Thus $\alpha_1 = \alpha_3 = 0$, and hence $a^2 = \alpha_2 e_2 + \alpha_4 b$. Then $a^2 = e_1 a^2 = \alpha_4 (e_1 b)$, which implies that $\alpha_2 = 0$ and $a^2 = \alpha_4 b$. Then $a^2 \neq 0$ implies that $\alpha_4 > 0$. So $\{e_1, e_2, a, a^2\}$ is a d -basis of A over F . Similarly if b is nilpotent and $b^2 \neq 0$, then $\{e_1, e_2, b, b^2\}$ is a d -basis of A over F . Therefore we only need to consider the following cases.

- (i) $a^2 = b^2 = 0$.
- (ii) a is nilpotent, but $a^2 \neq 0$. By the above argument, $a^2 = \alpha_4 b$ for some $0 < \alpha_4 \in F$. Thus $T = \{e_1, e_2, a, a^2\}$ is also a d -basis for A over F .
- (iii) $a^2 = 0$ and b is not nilpotent.

Below, we consider each case in detail.

4.2.1. $a^2 = b^2 = 0$

From $ba = \gamma e_1 + \delta e_2$, we have $0 = ba^2 = \gamma e_1 a + \delta e_2 a = \gamma e_1 a$, so $\gamma = 0$, and hence $ba = \delta e_2$.

- (1) $\delta > 0$. Then $ba = \delta e_2 \neq 0$ implies that $e_2 b = b$, $e_1 b = 0$, $ae_2 = a$, and $ae_1 = 0$. Then $ab = ae_1 + \beta e_2$ implies that $0 = a^2 b = \alpha ae_1 + \beta ae_2 = \beta a$, so $\beta = 0$ and $ab = ae_1$. Thus $aba = ae_1 a = \delta ae_2$ implies that $\alpha = \delta$. Since $ab = ae_1 \neq 0$, $be_1 = b$ and $be_2 = 0$.

Let $a' = a/\alpha$. Then $\{e_1, e_2, a', b\}$ is also a d -basis of A with the following multiplication table:

$$\begin{array}{c|cccc}
 & e_1 & e_2 & a' & e_2 \\
 \hline
 e_1 & e_1 & a & a' & 0 \\
 e_2 & 0 & e_2 & 0 & b \\
 a' & 0 & a' & 0 & e_1 \\
 b & b & 0 & e_2 & 0
 \end{array} \tag{4.6}$$

It is clear now that $A \cong M_2(F)$, the 2×2 matrix ℓ -algebra over F . This is (11) in Theorem 2.1.

- (2) $\delta = 0$. Then $ba = 0$. From $ab = \alpha e_1 + \beta e_2$, we have $0 = a(ba) = (ab)a = \alpha e_1 a + \beta e_2 a = \alpha a$, so $\alpha = 0$. Thus $ab = \beta e_2$, and hence $0 = e_2 ab = \beta e_2$, which implies that $\beta = 0$, so $ab = 0$. Therefore, $Fa + Fb$ is a zero ℓ -ring. There are the following cases.

- (2a) If $ae_1 = a$, $e_1 b = b$, and $be_1 = b$, then $A \cong (F1 + Fa + Fb) \oplus F$, where $Fa + Fb$ is the zero ℓ -ring.
 (2b) If $ae_1 = a$, $e_2 b = b$, and $be_2 = b$, then $A \cong (F1 + Fa) \oplus (F1 + Fb)$, where $a^2 = b^2 = 0$.
 If $ae_2 = a$, then the multiplication table for $\{e_1, e_2, a\}$ is given below:

$$\begin{array}{c|ccc}
 & e_1 & e_2 & a \\
 \hline
 e_1 & e_1 & 0 & a \\
 e_2 & 0 & e_2 & 0 \\
 a & 0 & a & 0
 \end{array} \tag{4.7}$$

Clearly $Fe_1 + Fe_2 + Fa$ is isomorphic to $T_2(F)$, and hence there are the following cases depending on the products between e_1 , e_2 , and b .

- (2c) If $e_1 b = b$ and $be_1 = b$, then $A \cong T_2(F) + Fb$ with $e_{11}b = be_{11} = b$, $be_{12} = 0$, and $b^2 = 0$.
 (2d) If $e_1 b = b$ and $be_2 = b$, then $A \cong T_2(F) + Fb$ with $e_{11}b = b$, $be_{22} = b$, and $b^2 = 0$.
 (2e) If $e_2 b = b$ and $be_1 = b$, then $A \cong T_2(F) + Fb$ with $e_{22}b = b$, $be_{11} = b$, $e_{12}b = be_{12} = 0$, and $b^2 = 0$.
 (2f) If $e_2 b = b$ and $be_2 = b$, then $A \cong T_2(F) + Fb$ with $e_{22}b = be_{22} = b$, $e_{12}b = 0$, and $b^2 = 0$.

Two more cases we need to consider are (i) $ae_1 = a$, $e_1 b = b$, $be_1 = 0$ and (ii) $ae_1 = a$, $e_2 b = b$, $be_2 = 0$. But when we switch a and b , the case (i) becomes (2c), and when we switch a and b , and also switch e_1 and e_2 , the case (ii) becomes (2f). Thus every possible case when $\delta = 0$ is covered. The above discussion has produced (12)–(17) in Theorem 2.1.

4.2.2. a Is Nilpotent and $a^2 \neq 0$

As we have noticed before, in this case $T = \{e_1, e_2, a, a^2\}$ is a d -basis for the ℓ -algebra A over F . Since $a \wedge e_1 = 0$ implies that $a^3 \wedge a^2 = 0$ and $a^2 \wedge e_1 = 0$ implies that $a^3 \wedge a = 0$, $a^3 = \beta_1 e_1$, for some $\beta_1 \in F^+$. If $\beta_1 \neq 0$, then a cannot be nilpotent, so we must have $\beta_1 = 0$. Therefore $a^3 = 0$, and $A \cong (F1 + Fa + Fa^2) \oplus F$ with $a^3 = 0$. This is (18) in Theorem 2.1.

4.2.3. $a^2 = 0$ and b Is Not Nilpotent

Since b is not nilpotent, b^2 is basic, so $b^2 = \epsilon s$ for some $s \in S$ and $0 < \epsilon \in F$. Clearly $s \neq a$ or b . Thus either $b^2 = \epsilon e_1$ or $b^2 = \epsilon e_2$. In the first case, we have $b^2 a = \epsilon e_1 a = b(\gamma e_1 + \delta e_2) = \gamma b e_1$, so $\epsilon a = \gamma b$, which is a contradiction. Thus we must have $b^2 = \epsilon e_2$. If $a e_2 = a$, then $ab^2 = \epsilon a$, so $(\alpha e_1 + \beta e_2)b = \epsilon a$, which implies that $\beta b = \epsilon a$, a contradiction. Thus $a e_2 = 0$, and hence $a e_1 = a$, and hence $a e_2 = 0$. Then $ab = ba = 0$. Therefore we have the following multiplication table for the elements of S :

$$\begin{array}{c|cccc}
 & e_1 & a & e_2 & b \\
 \hline
 e_1 & e_1 & a & 0 & 0 \\
 a & a & 0 & 0 & 0 \\
 e_2 & 0 & 0 & e_2 & b \\
 b & 0 & 0 & b & \epsilon e_2
 \end{array} \tag{4.8}$$

From the above table, it is clear that if $\sqrt{\epsilon} \in F$, then $A \cong F[G] \oplus (F1 + Fa)$ with $a^2 = 0$, where G is a group of order two, and if $\sqrt{\epsilon} \notin F$, then $A \cong F[\sqrt{\epsilon}] \oplus (F1 + Fa)$ with $a^2 = 0$, where $F[\sqrt{\epsilon}]$ is a quadratic extension ℓ -field of F . Those are (19) and (20) in Theorem 2.1.

Now we have constructed all lattice orders on A such that 1 is a sum of two disjoint basic elements.

5. 1 Is a Basic Element

In this section, A has a d -basis $S = \{1, a, b, c\}$. We still first consider the ℓ -reduced case.

5.1. A Is ℓ -Reduced

Since a is not nilpotent, by Theorem 1.1, there exist positive integer n_a and $0 < \alpha_a \in F$ such that $a^{n_a} = \alpha_a 1$. We also assume that n_a is the smallest positive integer such that $a^{n_a} = \alpha_a 1$. Similarly $b^{n_b} = \alpha_b 1$ and $c^{n_c} = \alpha_c 1$, for some positive integers n_b, n_c , and $0 < \alpha_b, \alpha_c \in F$. By Theorem 1.1 again, each a^k is a basic element for $k \geq 1$.

Lemma 5.1. *The subset $\{1, a, a^2, \dots, a^{n_a-1}\}$ of A is a disjoint set of A .*

Proof. Let $1 \leq i < n_a$. Since a^i is a basic element, $a^i = \alpha s$ for some $s \in S$ and $0 < \alpha \in F$. Since $i < n_a$, $s \neq 1$, so $1 \wedge s = 0$ implies that $1 \wedge a^i = 0$. Now let $0 \leq u < v < n_a$, then $a^u \wedge a^v = a^u(1 \wedge a^{v-u}) = 0$ by the above argument. $\square$

Since A is four-dimensional, $n_a \leq 4$ by Lemma 5.1. We claim that $n_a \neq 3$. Supposing that $n_a = 3$, then we may assume that $\{1, a, a^2, b\}$ is a d -basis. By Theorem 1.1, ab is a basic element and $ab \wedge a = a(b \wedge 1) = 0$, $ab \wedge b = (a \wedge 1)b = 0$, $ab \wedge a^2 = a(b \wedge a) = 0$, so $ab = \alpha 1$ for some $0 < \alpha \in F$. Thus $a^3 b = \alpha a^2$, and hence $\alpha_a b = \alpha a^2$, which is a contradiction. Hence $n_a \neq 3$, so $n_a = 4$ or $n_a = 2$.

By the above analysis, we need to consider two cases: (i) one of the following: n_a, n_b, n_c , is 4, and we may assume that $n_a = 4$; (ii) $n_a = n_b = n_c = 2$.

5.1.1. $n_a = 4$ and Hence $\{1, a, a^2, a^3\}$ Is a Disjoint Subset of Basic Elements of A

In this case $T = \{1, a, a^2, a^3\}$ is a d -basis of A over F , and we have the following multiplication table for the elements in T :

$$\begin{array}{c|cccc}
 & 1 & a & a^2 & a^3 \\
 \hline
 1 & 1 & a & a^2 & a^3 \\
 a & a & a^2 & a^3 & \alpha_a \\
 a^2 & a^2 & a^3 & \alpha_a 1 & \alpha_a a \\
 a^3 & a^3 & \alpha_a 1 & \alpha_a a & \alpha_a a^2
 \end{array} \tag{5.1}$$

We have the following three cases to be considered.

- (1) If $\sqrt[4]{\alpha_a} \in F$, then $A \cong F[G]$, where G is a cyclic group of order four.
- (2) If the polynomial $f(x) = x^4 - \alpha_a \in F[x]$ is irreducible, then $A \cong F[\sqrt[4]{\alpha_a}]$, which is an ℓ -field extension of F of order four.
- (3) The polynomial $f(x) = x^4 - \alpha_a \in F[x]$ is not irreducible and $\sqrt[4]{\alpha_a} \notin F$.

Suppose that $x^4 - \alpha_a = (x^2 + a_1x + b_1)(x^2 + a_2x + b_2)$. Then we have $a_1 + a_2 = 0$, $b_1 + b_2 + a_1a_2 = 0$, $b_1a_2 + b_2a_1 = 0$, and $b_1b_2 = -\alpha_a < 0$. Thus $a_2 = -a_1$ and $0 = b_1a_2 + b_2a_1 = a_1(-b_1 + b_2)$. If $a_1 \neq 0$, then $-b_1 + b_2 = 0$, and hence $b_1 = b_2$, which implies that $b_1b_2 = b_1^2 = -\alpha_a < 0$, a contradiction. So we must have $a_1 = 0$, and hence $a_2 = 0$. Then $b_1 + b_2 = 0$ and $b_1b_2 = -\alpha_a$ implies that $b_2 = -b_1$ and $b_1^2 = b_2^2 = \alpha_a$. We may assume that $b_1 > 0$. It follows from $\sqrt[4]{\alpha_a} \notin F$ that $\sqrt{b_1} \notin F$. Let $c = a^2/b_1$. Then $V = \{1, c, a, ac\}$ is also a d -basis of A over F and the multiplication table for the elements in V is given below:

$$\begin{array}{c|cccc}
 & 1 & c & a & ac \\
 \hline
 1 & 1 & c & a & ac \\
 c & c & 1 & ac & a \\
 a & a & ac & b_1c & b_11 \\
 ac & ac & a & b_11 & b_1c
 \end{array} \tag{5.2}$$

Then $A \cong F[G] + F[G]a$, where $G = \{e, c\}$ is a group of order two and $a^2 = \alpha c$, $0 < \alpha \in F$, and $\sqrt{\alpha} \notin F$. These are (21), (22), and (23) in Theorem 2.1.

5.1.2. $n_a = n_b = n_c = 2$

Then $S = \{1, a, b, c\}$ is a d -basis and $a^2 = \alpha_a 1$, $b^2 = \alpha_b 1$, and $c^2 = \alpha_c 1$. Since $ab \wedge a = ab \wedge b = 0$ and ab is basic, $ab = \alpha 1$ or $ab = \alpha c$ for some $0 < \alpha \in F$. In the first case, $ab^2 = ab$ implies that $\alpha_b a = ab$, which is a contradiction. Thus $ab = \alpha c$ for some $0 < \alpha \in F$. Then we may replace c by ab to get a d -basis $T = \{1, a, b, ab\}$.

Since $ba \wedge a = ba \wedge b = 0$ and ba is basic, $ba = \beta 1$ or $ba = \beta(ab)$ for some $0 < \beta \in F$. If $ba = \beta 1$, then $\alpha_a b = \beta a$, which is a contradiction. Thus $ba = \beta(ab)$, for some $0 < \beta \in F$, and hence

$$\alpha_b \alpha_a 1 = b(ba)a = \beta b(ab)a = \beta^2(ab)(ba) = \beta^2 \alpha_b \alpha_a 1. \tag{5.3}$$

So $\beta = 1$ and the multiplication table for the elements of T is given below:

$$\begin{array}{c|cccc}
 & 1 & a & b & ab \\
 \hline
 1 & 1 & a & b & ab \\
 a & a & \alpha_a 1 & ab & \alpha_a b \\
 b & b & ab & \alpha_b 1 & \alpha_b a \\
 ab & ab & \alpha_a b & \alpha_b a & \alpha_a \alpha_b 1
 \end{array} \tag{5.4}$$

There are the following cases.

- (1) $\sqrt{\alpha_a}$ and $\sqrt{\alpha_b} \in F$.
 Replacing a and b by $a/\sqrt{\alpha_a}$ and $b/\sqrt{\alpha_b}$, respectively, we have $a^2 = b^2 = 1$.
 Therefore $A \cong F[G]$, where G is the Klein four group.
- (2) $\sqrt{\alpha_a} \notin F$ and $\sqrt{\alpha_b} \in F$ (or $\sqrt{\alpha_a} \in F$ and $\sqrt{\alpha_b} \notin F$).
 Replacing b by $b/\sqrt{\alpha_b}$, we may assume that $b^2 = 1$. Therefore $A \cong F[\sqrt{\alpha_a}][G]$,
 where $F[\sqrt{\alpha_a}]$ is the quadratic extension ℓ -field of F and G is a group of order two.
- (3) $\sqrt{\alpha_a} \notin F$ and $\sqrt{\alpha_b} \notin F$.
 If $\sqrt{\alpha_b} \in F[\sqrt{\alpha_a}]$, then $A \cong F[\sqrt{\alpha_a}][G]$, where G is a group of order two. We omit
 the checking and leave it to the reader.
 If $\sqrt{\alpha_b} \notin F[\sqrt{\alpha_a}]$, then $A \cong F[\sqrt{\alpha_a}][\sqrt{\alpha_b}]$, where $F[\sqrt{\alpha_a}]$ is a quadratic extension
 ℓ -field of F and $F[\sqrt{\alpha_a}][\sqrt{\alpha_b}]$ is a quadratic extension ℓ -field of $F[\sqrt{\alpha_a}]$. Those are
 (24), (25), and (26) in Theorem 2.1.

5.2. A Is Not ℓ -Reduced

Let $S = \{1, a, b, c\}$ be a d -basis of A over F . Then some elements of a, b, c must be nilpotent. We first observe the following result.

Lemma 5.2. *Let $x, y \in \{1, a, b, c\}$. If either x or y is nilpotent and the other one is not nilpotent, then xy is nilpotent.*

Proof. Let x be nilpotent and let y be not. By Theorem 1.1, there exist a positive integer n_y and $0 < \alpha_y \in F$ such that $y^{n_y} = \alpha_y 1$. If $xy = 0$, then $xy^{n_y} = 0$ implies that $\alpha_y x = 0$, which is a contradiction. Thus $xy \neq 0$, so xy is basic by Theorem 1.1. If xy is not nilpotent, then $xy = \gamma z$ for some $0 < \gamma \in F$ and $z \in \{1, a, b, c\}$ and z is not nilpotent. By Theorem 1.1, there exist a positive integer n_z , $0 < \alpha_z \in F$ such that $z^{n_z} = \alpha_z 1$. Let $n > 1$ be the smallest positive integer such that $x^n = 0$. Then $0 = x^n y = x^{n-1}(xy) = \gamma x^{n-1} z$, and hence $0 = \gamma x^{n-1} z$ implies that $0 = \gamma x^{n-1} z^{n_z} = \alpha_z \gamma x^{n-1}$. Thus $x^{n-1} = 0$, which is a contradiction. Therefore xy is nilpotent. $\square$

By Lemma 5.2, $\{a, b, c\}$ cannot have exactly one nilpotent element. Suppose not, let a be the unique nilpotent element. Since $ab \wedge a = 0$, ab cannot be a positive scalar multiple of a , so ab cannot be a nilpotent element, which is a contradiction by Lemma 5.2. So there are following two cases.

5.2.1. a, b , and c Are All Nilpotent

- (1) Either a, b , or c has nonzero square.

We may assume that $a^2 \neq 0$. Since $a^2 \wedge a = a(a \wedge 1) = 0$, $a^2 = \alpha 1 + \beta b + \gamma c$, where $\alpha, \beta, \gamma \in F^+$. Let k be the smallest positive integer such that $a^k = 0$. Then $0 = a^k = a^{k-2}a^2 = \alpha a^{k-2} + \beta a^{k-2}b + \gamma a^{k-2}c$ implies $\alpha a^{k-2} = 0$, so $\alpha = 0$. Thus $a^2 = \beta b + \gamma c$.

- (1a) $\beta > 0$ and $\gamma > 0$.

Since $a^2 \wedge ab = a(a \wedge b) = 0$, $b \wedge ab = c \wedge ab = 0$. Also $a \wedge ab = a(1 \wedge b) = 0$. Thus $ab = \delta 1$. Then we have $0 = a^k b = \delta a^{k-1}$, so $\delta = 0$ and $ab = 0$. Similarly $ba = 0$, $ac = 0$, and $ca = 0$. Since $a^2 = \beta b + \gamma c$, $0 = a^2 b = \beta b^2 + \gamma cb$ implies that $b^2 = cb = 0$. Similarly $c^2 = bc = 0$.

Replacing b and c by βb and γc , respectively, we may assume that $a^2 = b + c$. Therefore we have the following multiplication table:

	1	a	b	c
1	1	a	b	c
a	a	$b + c$	0	0
b	b	0	0	0
c	c	0	0	0

(5.5)

Then $A \cong F1 + Fa + Fb + Fc$, where $a^2 = b + c$ and the product of any other two elements of a, b, c is zero. This is (27) in Theorem 2.1. We want to point out that this case provides an example in which the product of two basic elements may not be basic.

- (1b) $\beta = 0$ or $\gamma = 0$.

We may assume that $\gamma = 0$, so $\beta > 0$. Then $a^2 = \beta b$, and $a(aa) = (aa)a$ implies that $\beta(ab) = \beta(ba)$, so $ab = ba$.

- (i) $ab = ba \neq 0$.

Then $ab = \delta c$ for some $0 < \delta \in F$. Thus $a^3 = aa^2 = \beta(ab) = \beta\delta c$, and hence we may replace b by a^2 and c by a^3 to get another d -basis $T = \{1, a, a^2, a^3\}$. Therefore $A \cong F1 + Fa + Fa^2 + Fa^3$ with $a^4 = 0$. This is (28) in Theorem 2.1.

- (ii) $ab = ba = 0$.

Then $\beta b^2 = a^2 b = 0$, so $b^2 = 0$. Since $ac = \mu b$ for some $0 \leq \mu \in F$, $a^2 \wedge ac = 0$ and $a^2 = \beta b$ implies that $\mu = 0$, so $ac = 0$. Similarly $ca = 0$. $0 = a^2 c = \beta(bc)$, so $bc = 0$. Similarly, $cb = 0$. Now replacing b by a^2 , we obtain another d -basis $T = \{1, a, a^2, c\}$.

If $c^2 \neq 0$, then $c^2 \wedge c = c^2 \wedge 1 = 0$ implies that $c^2 = \alpha_1 a + \alpha_2 a^2$. Then it follows from $ac = 0$ and $a^3 = 0$ that $\alpha_1 a^2 = 0$, so $\alpha_1 = 0$. Thus $c^2 = \alpha_2 a^2$ with $0 < \alpha_2 \in F$. Hence $A \cong F1 + Fa + Fa^2 + Fc$ with $c^2 = \alpha a^2$ for some $0 < \alpha \in F$ and $xy = 0$ for any two elements x and y in $\{a, a^2, c\}$ not both a or c . This is (29) in Theorem 2.1.

If $c^2 = 0$, then $A \cong F1 + Fa + Fa^2 + Fc$ with $xy = 0$ for any two elements x and y in $\{a, a^2, c\}$ which are not both a . This is (30) in Theorem 2.1.

We provide some familiar examples for the above two cases. Consider 4×4 matrix algebra over F . Let $1 = e_{11} + e_{22} + e_{33} + e_{44}$, $a = e_{12} + e_{23}$, and $c = e_{14} + 2e_{43}$. Then $a^2 = e_{13}$ and $c^2 = 2a^2$. Clearly $F1 + Fa + Fa^2 + Fc$ is a four-dimensional

ℓ -algebra over F with coordinatewise order. If $c = e_{13} + e_{43}$, then $c^2 = 0$ and $F1 + Fa + Fa^2 + Fc$ is a four-dimensional ℓ -algebra over F with $c^2 = 0$.

$$(2) a^2 = b^2 = c^2 = 0.$$

(2a) $xy = 0$ for any two elements in $\{a, b, c\}$.

Then $A \cong F1 + Fa + Fb + Fc$, where $Fa + Fb + Fc$ is zero ring. This is (31) in Theorem 2.1.

(2b) At least one product of two different elements in $\{a, b, c\}$ is nonzero.

Without loss of generality, we may assume that $ab \neq 0$. Then $ab = ac$ for some $0 < \alpha \in F$, and we may replace c by ab to form another d -basis $T = \{1, a, b, ab\}$. If $ba = 0$, then we have the following multiplication table:

		1	a	b	ab
1	1	1	a	b	ab
a	a	0	ab	0	
b	b	0	0	0	
ab	ab	0	0	0	

(5.6)

Thus $A \cong F1 + Fa + Fb + F(ab)$ with $a^2 = b^2 = ba = 0$. This is (32) in Theorem 2.1. If $ba \neq 0$, then $ba = \beta(ab)$ for some $0 < \beta \in F$ and we have the following multiplication table:

		1	a	b	ab
1	1	1	a	b	ab
a	a	0	ab	0	
b	b	$\beta(ab)$	0	0	
ab	ab	0	0	0	

(5.7)

Thus $A \cong F1 + Fa + Fb + F(ab)$ with $a^2 = b^2 = 0$ and $ba = \beta(ab)$ for some $0 < \beta \in F$. This is (33) in Theorem 2.1.

5.2.2. Two Elements in $\{a, b, c\}$ Are Nilpotent

We may assume that a and b are nilpotent elements. Then we must have $c^2 = \alpha 1$ for some $0 < \alpha \in F$. By Lemma 5.2, $ac = \beta b$ for some $0 < \beta \in F$. If $a^2 \neq 0$, then $a^2 \wedge a = a(a \wedge 1) = 0$ and that 1 and c are not nilpotent implies that $a^2 = \gamma b$ for some $0 < \gamma \in F$. Thus $ac \wedge a^2 \neq 0$, which is a contradiction, so $a^2 = 0$. Similarly, $b^2 = 0$. Replacing a by a/β and then replacing b by ac , we have another d -basis $\{1, c, a, ac\}$ with the following multiplication table:

		1	c	a	ac
1	1	1	c	a	ac
c	c	$\alpha 1$	ac	αa	
a	a	ac	0	0	
ac	ac	αa	0	0	

(5.8)

If $\sqrt{a} \in F$, then replacing c by $c/\sqrt{a}$, we have that $A \cong F[G] + F[G]a$ is a commutative ℓ -algebra, where G is a group of order two and $a^2 = 0$.

If $\sqrt{a} \notin F$, then $A \cong F[\sqrt{a}] + F[\sqrt{a}]a$ is a commutative ℓ -algebra over F , where $F[\sqrt{a}]$ is a quadratic extension ℓ -field of F and $a^2 = 0$. Those are (34) and (35) in Theorem 2.1.

This completes the proof of Theorem 2.1.

6. An Application

For an ℓ -ring R , its ℓ -radical $\ell\text{-}N(R)$ is defined as

$$\ell\text{-}N(R) = \{a \in R : \exists n = n(a) \text{ such that } x_0|a|x_1|a|\cdots|a|x_n = 0, \forall x_0, \dots, x_n \in R\}, \quad (6.1)$$

(see [1, Definition, page 45]). $\ell\text{-}N(R)$ is the sum of all of the nilpotent ℓ -ideals of R [1, Theorem 5]. It is clear that for an ℓ -algebra A , $\ell\text{-}N(A)$ is an ℓ -ideal of A .

A *positive derivation* on an ℓ -ring R is a function $\delta : R \rightarrow R$ such that for any $a, b \in R$, $\delta(a + b) = \delta(a) + \delta(b)$, $\delta(ab) = a\delta(b) + \delta(a)b$, and $\delta(R^+) \subseteq R^+$. For an Archimedean f -ring R , if δ is a positive derivation on R , then $\delta(R) \subseteq \ell\text{-}N(R)$ [2, 3]. Although we believe this result is true for a unital ℓ -algebra with a d -basis, we lack ability to prove it in general. In the following, we prove this result for unital four-dimensional ℓ -algebras with a d -basis based on the classifications obtained in Theorem 2.1.

Theorem 6.1. *Let A be a unital four-dimensional ℓ -algebra with a d -basis. If δ is a positive derivation on A , then $\delta(A) \subseteq \ell\text{-}N(A)$.*

Proof. We notice first that since $1 > 0$, for any $a \in A$ and $\alpha \in F$, $\delta(\alpha a) = \alpha\delta(a)$ [4]; that is, δ is actually a linear transformation of A . Let $1 = e_1 + \cdots + e_n$, where $e_1, \dots, e_n$ are distinct elements from the d -basis and $n \leq 4$. Since $\delta(1) = \delta(1) + \delta(1)$ implies $\delta(1) = 0$, $0 = \delta(e_1) + \cdots + \delta(e_n)$. Then δ is positive implying that $\delta(e_i) = 0$ for each $i = 1, \dots, n$. Let a be an element in the d -basis, by Theorem 1.1, either a is nilpotent or $a^{n_a} = \alpha_a e_i$ for some $n_a \geq 1$, $0 < \alpha_a \in F$, and e_i . If a is nilpotent, then $\delta(a)$ is also nilpotent [4]. If $a^{n_a} = \alpha_a e_i$, then $\delta(a^{n_a}) = \delta(\alpha_a e_i) = \alpha_a \delta(e_i) = 0$ and δ is positive also implying that $\delta(a)$ is nilpotent. Thus for each element a in the d -basis, $\delta(a)$ is nilpotent, so if A is ℓ -reduced, then $\delta(a) = 0$ for each a in the d -basis. Therefore $\delta(A) = 0$.

We only need to consider those cases in Theorem 2.1 in which A is not ℓ -reduced. In cases (4)-(5), (12)-(20), and (27)-(35), the ℓ -radical consists of all nilpotent elements of the ℓ -algebra A . From the argument in the previous paragraph, for each element x in the d -basis, $\delta(x)$ is nilpotent, so $\delta(x) \in \ell\text{-}N(A)$. Thus $\delta(A) \subseteq \ell\text{-}N(A)$ for those cases.

The only case left is (11) in which $\ell\text{-}N(M_2(F)) = 0$, but $M_2(F)$ contains nonzero nilpotent elements e_{12} and e_{21} . We show that $\delta(e_{12}) = \delta(e_{21}) = 0$. Then together with $\delta(e_{11}) = \delta(e_{22}) = 0$, we have $\delta(M_2(F)) = 0$. Since $(e_{12} + e_{21})^2 = 1$,

$$(e_{12} + e_{21})\delta(e_{12} + e_{21}) + \delta(e_{12} + e_{21})(e_{12} + e_{21}) = 0, \quad (6.2)$$

so δ is positive implying that $(e_{12} + e_{21})\delta(e_{12} + e_{21}) = 0$. Multiplying this last equation by $(e_{12} + e_{21})$ from the left, we have $\delta(e_{12} + e_{21}) = 0$. Hence $\delta(e_{12}) = \delta(e_{21}) = 0$. Therefore $\delta(M_2(F)) = 0$.

This completes the proof. □

References

- [1] G. Birkhoff and R. S. Pierce, "Lattice-ordered rings," *Anais da Academia Brasileira de Ciências*, vol. 28, pp. 41–69, 1956.
- [2] P. Colville, G. Davis, and K. Keimel, "Positive derivations on f -rings," *Journal of the Australian Mathematical Society*, vol. 23, no. 3, pp. 371–375, 1977.
- [3] M. Henriksen and F. A. Smith, "Some properties of positive derivations on f -rings," in *Ordered Fields and Real Algebraic Geometry*, vol. 8 of *Contemporary Mathematics*, pp. 175–184, American Mathematical Society, Providence, RI, USA, 1982.
- [4] J. Ma and R. H. Redfield, "Positive derivations on Archimedean lattice-ordered rings," *Positivity*, vol. 13, no. 1, pp. 165–191, 2009.
- [5] J. Ma, "Finite-dimensional lattice-ordered algebras with d -elements," *Journal of Algebra*, vol. 280, no. 1, pp. 232–243, 2004.
- [6] J. Ma, "Lattice-ordered algebras with a d -basis," *Journal of Algebra*, vol. 299, no. 2, pp. 731–746, 2006.