Research Article

Fixed Point for Partial Metric Spaces

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We consider the partial metric on a set X , define ϵ -fixed point for maps, and obtain some sufficient and necessary conditions on that, also we obtain some sufficient and necessary theorems on ϵ -fixed point.

1. Introduction

The partial metric spaces were introduced in [1] as a part of the study of denotational semantics of dataflow networks. In particular, he established the precise relationship between partial metric spaces and the so-called weightable quasimetric spaces and proved a partial metric generalization of Banach contraction mapping theorem.

A partial metric [1] on a set X is a function $p : X \times X \rightarrow [0, \infty)$ such that, for all $x, y, z \in X$

- (1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
- (2) $p(x, x) \leq p(x, y)$;
- (3) $p(x, y) \leq p(y, x)$;
- (4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair (X, p) , where p is a partial metric on X . Each partial metric p on X induces a T_0 topology τ_p on X which has as a base the family of open balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

If p is a partial metric on X , then the function $p^s : X \times X \rightarrow [0, \infty)$ given by $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on X .

A sequence $\{x_n\}_{n \in \mathbb{N}}$ in a partial metric space (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m} p(x_n, x_m)$ [1, Definition 5.2].

Note that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s) [1, page 194].

A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ in X converges, with respect to τ_p , to a point $x \in X$ such that $p(x, x) = \lim_{n,m} p(x_n, x_m)$ [1, Definition 5.3].

It is well known and easy to see that a partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete.

In [2], S. J. O'Neill proposed one significant change to Matthews definition of the partial metrics, and that was to extend their range from R^+ to R . In the following, partial metrics in the O'Neill sense will be called dualistic partial metrics and a pair (X, p) such that X is a nonempty set and p is a dualistic partial metric on X will be called a dualistic partial metric space.

2. ϵ -Fixed Point

Our basic references for quasi-metric spaces are [3, 4]. In our context, by a quasi-metric on a set X we mean a nonnegative real-valued function d on $X \times X$ such that, for all $x, y, z \in X$:

- (i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$,
- (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A quasi-metric space is a pair (X, d) such that X is a (nonempty) set and d is a quasi-metric on X .

Each quasi-metric d on X generates a T_0 -topology $T(d)$ on X which has as a base the family of open d -balls $B_d(x, \epsilon) = \{x \in X : \epsilon > 0\}$, where $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon > 0\}$ for all $x \in X$ and $\epsilon > 0$.

If d is a quasi-metric on X , then the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ is a metric on X .

Lemma 2.1. *If (X, p) is a dualistic partial metric space, then the function $d_p : X \times X \rightarrow R^+$ defined by $d_p(x, y) = p(x, y) - p(x, x)$, is a quasi-metric on X such that $\tau(p) = \tau(d_p)$.*

Definition 2.2. Let (X, p) be a dualistic partial metric space and let $T : X \rightarrow X$ be a map. Then $x_0 \in X$ is ϵ -fixed point for T if

$$d_p(Tx_0, x_0) \leq \epsilon. \quad (2.1)$$

Theorem 2.3. *Let (X, p) be a dualistic partial metric space and let $T : X \rightarrow X$ be a map, $x_0 \in X$, and $\epsilon > 0$. If $d_p(T^n(x_0), T^{n+k}(x_0)) \rightarrow 0$ as $n \rightarrow \infty$ for some $k > 0$, then T^k has an ϵ -fixed point.*

Proof. Since $d_p(T^n(x_0), T^{n+k}(x_0)) \rightarrow 0$ as $n \rightarrow \infty$, $\epsilon > 0$,

$$\exists n_0 > 0 \quad \text{s.t. } \forall n \geq n_0 \quad d_p(T^n(x_0), T^{n+k}(x_0)) < \epsilon. \quad (2.2)$$

Then $d_p(T^{n_0}(x_0), T^k(T^{n_0}(x_0))) < \epsilon$. Therefore $T^{n_0}(x_0)$ is an ϵ -fixed point of T^k . $\square$

Theorem 2.4. Let (X, p) be a dualistic partial metric space and let $T : X \rightarrow X$ be a map also for all $x, y \in X$,

$$d_p(Tx, Ty) \leq c d_p(x, y) \quad 0 < c < 1, \quad (2.3)$$

then T has an ϵ -fixed point in partial metric. Moreover, if $x, y \in X$ are ϵ -fixed points of T , then $d_p(x, y) \leq 2\epsilon / (1 - c)$.

Proof. Suppose $x \in X$, then

$$\begin{aligned} d_p(T^n(x), T^{n+1}(x)) &= d_p(T(T^{n-1}(x)), T(T^n(x))) \\ &\leq c d_p(T^{n-1}(x), T^n(x)) \\ &\leq \dots \\ &\leq c^{n-1} d_p(T(x), T^2(x)) \\ &\leq c^n d_p(x, Tx). \end{aligned} \quad (2.4)$$

Therefore $d_p(T^n(x), T^{n+1}(x)) \rightarrow 0$ as $n \rightarrow \infty$. From Theorem 2.3, T has an ϵ -fixed point and Since

$$d_p(x, y) \leq d_p(x, Tx) + d_p(Tx, Ty) + d_p(y, Ty) \leq 2\epsilon + c d_p(x, y). \quad (2.5)$$

Then $d_p(x, y) \leq 2\epsilon / (1 - c)$. □

Theorem 2.5. Let T be a mapping of a dualistic partial metric space (X, p) into itself such that

$$|p(Tx, Ty)| \leq c |p(x, y)| \quad 0 < c < 1 \quad (2.6)$$

for all $x, y \in X$. Then T^k has an ϵ -fixed point, for all k .

Proof. Fix $x \in X$, then it is clear that, for each $x \in N$,

$$\begin{aligned} |p(T^n x, T^n x)| &\leq c^n |p(x, x)|, \\ |p(T^n x, T^{n+1} x)| &\leq c^n |p(x, Tx)|, \end{aligned} \quad (2.7)$$

also

$$d_p(T^n x, T^{n+1} x) + p(T^n x, T^n x) = p(T^n x, T^{n+1} x). \quad (2.8)$$

We deduce that

$$d_p(T^n x, T^{n+1} x) + |p(T^n x, T^n x)| \leq c^n |p(x, Tx)|. \quad (2.9)$$

Hence

$$\begin{aligned}
 d_p(T^n(x), T^{n+1}(x)) &= c^n |p(x, Tx)| - |p(T^n x, T^n x)| \\
 &\leq c^n |p(x, Tx)| + |p(T^n x, T^n x)| \\
 &\leq c^n (|p(x, Tx)| - |p(x, x)|).
 \end{aligned} \tag{2.10}$$

Therefore, for $k, n \in N$,

$$\begin{aligned}
 d_p(T^n(x), T^{n+k}(x)) &\leq d_p(T^n(x), T^{n+1}(x)) + \dots + d_p(T^{n+k-1}(x), T^{n+k}(x)) \\
 &\leq (c^n + \dots + c^{n+k-1}) (|p(x, Tx)| + |p(x, x)|) \\
 &\leq \frac{c^n}{1-c} (|p(x, Tx)| + |p(x, x)|).
 \end{aligned} \tag{2.11}$$

Similarly, we obtain that

$$d_p(T^{n+k}(x), T^n(x)) \leq \frac{c^n}{1-c} (|p(x, Tx)| + |p(x, x)|). \tag{2.12}$$

Then

$$\lim_{n \rightarrow \infty} d_p(T^{n+k}(x), T^n(x)) = 0. \tag{2.13}$$

Therefore, T^k has an ϵ -fixed point. □

Example 2.6. Let $X = (-\infty, -2]$, and let p be the dualistic metric on X given by

$$p(x, y) = x \vee y, \tag{2.14}$$

for all $x, y \in X$.

Let T be the mapping from X into itself defined by $T(x) = x + 1$, for all $x \in (-\infty, -2]$. It is immediate to see that

$$p(T(x), T(y)) \leq \frac{1}{2} p(x, y), \tag{2.15}$$

for all $x, y \in X$. However, T has no fixed point, of course. But by the Theorem 2.5, for every $\epsilon > 0$, T has an ϵ -fixed point. That is, there exists $x_0 \in X$ such that

$$d_p(T(x_0), x_0) \leq \epsilon, \quad \epsilon > 0, \tag{2.16}$$

since

$$\begin{aligned} d_p(T(x_0), x_0) &= p(T(x_0), x_0) - p(T(x_0), Tx_0) \\ &= p(x_0 + 1, x_0) - p(x_0 + 1, x_0 + 1) \\ &= x_0 + 1 - (x_0 + 1) = 0 \leq \epsilon. \end{aligned} \quad (2.17)$$

Theorem 2.7. Let T be a mapping of a dualistic partial metric space (X, p) into itself such that $d_p(Tx, Ty) \leq \beta(d_p(x, Tx) + d_p(y, Ty))$ where $2\beta < 1$.

If x_0 is an ϵ -fixed point for T , then Tx_0 is an ϵ -fixed point for T^2 .

Proof. We have

$$d_p(Tx, T^2x) \leq \beta(d_p(x, Tx) + d_p(Tx, T^2x)), \quad (2.18)$$

therefore

$$d_p(Tx, T^2x) \leq \frac{\beta}{1-\beta} d_p(x, Tx). \quad (2.19)$$

Since $2\beta < 1$,

$$d_p(Tx, T^2x) \leq d_p(x, Tx). \quad (2.20)$$

Since x_0 is an ϵ -fixed point for T , then $d_p(Tx_0, T^2x_0) \leq \epsilon$. So Tx_0 is an ϵ -fixed point for T^2 . $\square$

Theorem 2.8. Let (X, p) be a dualistic partial metric, and let $T : X \rightarrow X$ be a mapping and $\epsilon > 0$. If $d_p(Tx, Ty) \leq \alpha d_p(x, Tx) + \beta d_p(y, Ty)$ and $\alpha + \beta < 1$. then T has ϵ -fixed point. Moreover, if $x, y \in X$ are ϵ -fixed points of T , then $d_p(x, y) \leq (2 + \alpha + \beta)\epsilon$.

Proof. We have

$$d_p(Tx, T^2x) \leq \alpha d_p(x, Tx) + \beta d_p(Tx, T^2x). \quad (2.21)$$

Therefore

$$d_p(Tx, T^2x) \leq \frac{\alpha}{1-\beta} d_p(x, Tx), \quad (2.22)$$

also

$$d_p(T^2x, T^3x) \leq \alpha d_p(Tx, T^2x) + \beta d_p(T^2x, T^3x), \quad (2.23)$$

so

$$d_p(T^2x, T^3x) \leq \left(\frac{\alpha}{1-\beta}\right)^2 d_p(Tx, T^2x) \quad (2.24)$$

and for every $n \geq 1$, we have

$$d_p(T^n x, T^{n+1} x) \leq \left(\frac{\alpha}{1-\beta}\right)^n d_p(x, Tx), \quad \frac{\alpha}{1-\beta} < 1. \quad (2.25)$$

Thus, since $\alpha/(1-\beta) < 1$, $d_p(T^n x, T^{n+1} x) \rightarrow 0$ as $n \rightarrow \infty$. Now, by Theorem 2.3, T has an ϵ -fixed point and since

$$d_p(x, y) \leq d_p(x, Tx) + d_p(Tx, Ty) + d_p(y, Ty) \leq (1+\alpha)d_p(x, Tx) + (1+\beta)d_p(y, Ty). \quad (2.26)$$

Then $d_p(x, y) \leq (2+\alpha+\beta)\epsilon$. □

Corollary 2.9. *Let (X, p) be a dualistic partial metric, and let $T : X \rightarrow X$ be a mapping and $\epsilon > 0$. If $d_p(Tx, Ty) \leq \beta(d_p(x, Tx) + d_p(y, Ty))$ and $2\beta < 1$, then T has an ϵ -fixed point.*

References

- [1] S. G. Matthews, "Partial metric topology," in *Proceedings of the 11th Summer Conference on General Topology and Applications*, vol. 728, pp. 183–197, The New York Academy of Sciences, Gorham, Me, USA, Augusts 1995.
- [2] S. J. O'Neill, "Partial metrics, valuations and domain theory," in *Proceedings of the 11th Summer Conference on General Topology and Applications*, vol. 806, pp. 304–315, The New York Academy of Sciences, Gorham, Me, USA, Augusts 1995.
- [3] P. Fletcher and W. F. Lindgren, *Quasi-Uniform Spaces*, Marcel Dekker, New York, NY, USA, 1982.
- [4] H. P. A. Kunzi, "Nonsymmetric distances and their associated topologies: about the origins of basic ideas in the area of asymmetric topology," in *Handbook of the History of General Topology*, vol. 3, pp. 853–968, Kluwer Academic, Dordrecht, The Netherlands, 2001.