Research Article

Fixed Points and Existence Theorems of Maximal Elements with Applications in *FC*-Spaces

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We present some fixed point theorems and existence theorems of maximal elements in *FC*-space from which we derive several coincidence theorems. Applications of these results to generalized equilibrium problems and minimax theory will be given. Our results improve and generalize some recent results.

1. Introduction and Preliminaries

It is well known that many fixed point theorems, coincidence theorems, and maximal elements have been established in topological vector spaces, *H*-spaces, and *G*-convex spaces by many authors (see [1–11]). In most of the known existence results of maximal elements and fixed point theorems, the convexity assumptions play a crucial role which strictly restricts the applicable area of these results. In fact, what we meet may not have convexity structure. Hence it is quite reasonable and valuable to study fixed point theorems, coincidence theorems, and maximal elements in general topological spaces without convexity structure.

In this paper, we will continue to study some problems in nonlinear analysis in *FC*-spaces. Some fixed point theorems and existence theorems of maximal elements are proved under noncompact setting of *FC*-spaces. We will apply these results to give some coincidence theorems. More specifically, the application of these results to generalized equilibrium problems and minimax theory will be given in *FC*-spaces. Our results improve and generalize the corresponding results in [1, 2, 12–14].

Let Y be a nonempty set. We denote by 2^Y and $\langle Y \rangle$ the family of all subsets of Y and the family of all nonempty finite subsets of Y , respectively. For each $A \in \langle X \rangle$, we denote by $|A|$ the cardinality of A . Let Δ_n be the standard n -dimensional simplex with vertices $\{e_0, \dots, e_n\}$.

If J is a nonempty subset of $\{0, 1, \dots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$.

Let X and Y be two sets, and, $T : X \rightarrow 2^Y$ be a set-valued mapping. We will use the following notations in the following material:

- (i) $T(x) = \{y \in Y : y \in T(x)\},$
- (ii) $T(A) = \bigcup_{x \in A} T(x),$
- (iii) $T^{-1}(y) = \{x \in X : y \in T(x)\},$
- (iv) $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}.$

For topological spaces X and Y , a subset A of X is said to be compactly open (resp., compactly closed) if for each nonempty compact subset K of X , $A \cap K$ is open (resp., closed) in K . The compact closure of A and the compact interior of A (see [5]) are defined, respectively, by

$$\begin{aligned} c \text{ cl } A &= \bigcap \{B \subset X : A \subset B \text{ and } B \text{ is compactly closed in } X\}, \\ c \text{ int } A &= \bigcup \{B \subset X : B \subset A \text{ and } B \text{ is compactly open in } X\}. \end{aligned} \quad (1.1)$$

It is easy to see that $c \text{ cl}(X \setminus A) = X \setminus c \text{ int } A$, $\text{int } A \subset c \text{ int } A \subset A$, $A \subset c \text{ cl } A \subset \text{cl } A$, A is compactly open in X if and only if $A = c \text{ int } A$. For each nonempty compact subset K of X , $c \text{ cl } A \cap K = \text{cl}_K(A \cap K)$ and $c \text{ int } A \cap K = \text{int}_K(A \cap K)$, where $\text{cl}_K(A \cap K)$ (resp., $\text{int}_K(A \cap K)$) denotes the closure (resp., interior) of $A \cap K$ in K .

A set-valued mapping $T : X \rightarrow 2^Y$ is said to be transfer compactly open valued on X (see [5]) if for each $x \in X$ and $y \in T(x)$, there exists $x' \in X$ such that $y \in c \text{ int } T(x')$. It is clear that each transfer open valued correspondence is transfer compactly open valued. The inverse is not true in general.

Throughout this paper, all topological spaces are assumed to be Hausdorff.

Definition 1.1 (see [7]). (X, φ_N) is said to be an FC-space if X is a topological space, and for each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$, where some elements in N may be the same, there exists a continuous mapping $\varphi_N : \Delta_n \rightarrow X$. A subset D of (X, φ_N) is said to be an FC-subspace of X if for each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$ and for each $\{x_{i_0}, \dots, x_{i_k}\} \subset N \cap D$, $\varphi_N(\Delta_k) \subset D$, where $\Delta_k = \text{co}(\{e_{i_j} : j = 0, \dots, k\})$ and co denotes convex hull.

In many recent papers (see [9, 11, 15]), one studies one or more of the following generalized equilibrium problems

Let X be a convex subset of a topological vector space, let Z and, V be nonempty sets and let $F : X \times Z \rightarrow 2^V, C : X \rightarrow 2^V$ be set-valued mappings. Find $x_0 \in X$ such that one of the following situations occurs:

- $F(x_0, z) \subseteq C(x_0)$ for all $z \in Z$;
- $F(x_0, z) \cap C(x_0) \neq \emptyset$ for all $z \in Z$;
- $F(x_0, z) \not\subseteq C(x_0)$ for all $z \in Z$;
- $F(x_0, z) \cap C(x_0) = \emptyset$ for all $z \in Z$.

In this paper, we will try a unified approach for all these problems considering a (binary) relation ρ on 2^V and looking for a point $x_0 \in X$ such that $F(x_0, z) \rho C(x_0)$ for all $z \in Z$.

Denote by ρ^c the complementary relation of ρ , that is, for any $A, B \subset V$, exactly one of the following assertions $A\rho B, A\rho^c B$ holds.

Since Fan [16] and Liu [17] extended the von Neumann-Sion principle to obtain two-function minimax inequalities, many such results involving two or more functions have appeared in the literature. We also obtain a very general minimax inequality of the following type:

$$\inf_{x \in X} h(x, x) \leq \sup_{x \in X} \inf_{z \in Z} f(x, z) + \sup_{z \in Z} \inf_{x \in X} g(x, z), \quad (1.2)$$

which is, to the best of our knowledge, different from all the minimax inequalities known in the literature.

2. Fixed Points and Maximal Elements

In order to prove our main results, we need the following definition and lemmas.

Definition 2.1. Let (X, φ_N) and (Z, φ_M^*) be two FC-spaces. A set-valued mapping $T : X \rightarrow 2^Z$ is said to be weakly-FC valued if for each $A = \{x_0, \dots, x_n\} \in \langle X \rangle$, $x \in \varphi_N(\Delta_n)$ and $B = \{z_0, \dots, z_n\} \in \langle Z \rangle$ with $z_i \in T(x_i)$ ($i = 0, \dots, n$), we have $T(x) \cap \varphi_M^*(\Delta'_n) \neq \emptyset$, where Δ'_n denotes the standard n-dimensional simplex corresponding, $B = \{z_0, \dots, z_n\} \in \langle Z \rangle$.

Lemma 2.2 (see [10]). *Let X and Y be two topological spaces, and let $G : X \rightarrow 2^Y$ be a set-valued mapping. Then the following statements are equivalent:*

- (i) $G^{-1} : Y \rightarrow 2^X$ is transfer compactly open valued, and for all $x \in X$, $G(x)$ is nonempty;
- (ii) $X = \bigcup_{y \in Y} c \text{ int } G^{-1}(y)$.

Theorem 2.3. *Let (X, φ_N) be an FC-space, K be a nonempty compact subset of X and let $S, Q : X \rightarrow 2^X$ be set-valued mappings satisfying the following conditions:*

- (i) for each $x \in X$ and $N \in \langle S(x) \rangle$, $\varphi_N(\Delta_k) \subseteq Q(x)$;
- (ii) $X = \bigcup_{y \in X} c \text{ int } S^{-1}(y)$;
- (iii) there exists a compact FC-subspace L_N of X containing N such that

$$L_N \setminus K \subseteq \bigcup \{ c \text{ int } S^{-1}(y) : y \in L_N \}. \quad (2.1)$$

Then there exists $\bar{x} \in X$ such that $\bar{x} \in Q(\bar{x})$.

Proof. Since $K \subseteq \bigcup_{y \in X} c \text{ int } S^{-1}(y)$ and K is a nonempty compact subset of X , there exists a finite set $N = \{y_1, \dots, y_n\} \in \langle X \rangle$ such that

$$K = \bigcup_{i=1}^n c \text{ int } S^{-1}(y_i). \quad (2.2)$$

By condition (iii), there exists a compact FC -subspace L_N of X containing N such that

$$L_N \setminus K \subseteq \bigcup \left\{ c \operatorname{int} S^{-1}(y) : y \in L_N \right\}. \quad (2.3)$$

It follows from (2.2) and $N \subset L_N$ that

$$L_N \cap K \subseteq \bigcup \left\{ c \operatorname{int} S^{-1}(y) : y \in N \right\} \subseteq \bigcup \left\{ c \operatorname{int} S^{-1}(y) : y \in L_N \right\}. \quad (2.4)$$

By (2.3) and (2.4), we have

$$L_N \subseteq \bigcup \left\{ c \operatorname{int} S^{-1}(y) : y \in L_N \right\}. \quad (2.5)$$

Since L_N is compact, there exists a finite set $M = \{z_0, \dots, z_m\} \in \langle L_N \rangle$ such that

$$L_N = \bigcup_{i=0}^m \left(c \operatorname{int} S^{-1}(z_i) \cap L_N \right). \quad (2.6)$$

Since L_N is also compact FC -subspace of X , there exists a continuous mapping $\varphi_M : \Delta_m \rightarrow X$ such that

$$\varphi_M(\Delta_J) \subset L_N, \quad \forall J \subset \{0, \dots, m\}. \quad (2.7)$$

We may assume that $\{\varphi_i\}_{i=0}^m$ is the continuous partition of unity subordinated to the open covering $\{c \operatorname{int} S^{-1}(z_i) \cap L_N\}_{i=0}^m$ such that

- (1) for each $i = 0, \dots, m$, $\varphi_i : L_N \rightarrow [0, 1]$ is continuous;
- (2) for each $i = 0, \dots, m$ and $x \in L_N$,

$$\varphi_i(x) \neq 0 \implies x \in c \operatorname{int} S^{-1}(z_i) \cap L_N \subset S^{-1}(z_i) \implies (z_i) \in S(x), \quad (2.8)$$

- (3) for each $x \in L_N$, $\sum_{i=0}^m \varphi_i(x) = 1$.

Define a mapping $\psi : L_N \rightarrow \Delta_m$ by

$$\psi(x) = \sum_{i=0}^m \varphi_i(x) e_i, \quad \forall x \in L_N. \quad (2.9)$$

Hence ψ is continuous, and for each $x \in L_N$,

$$\psi(x) = \sum_{j \in J(x)} \varphi_j(x) e_j \in \Delta_{J(x)}, \quad \forall x \in L_N, \quad (2.10)$$

where $J(x) = \{j \in \{0, \dots, m\} : \varphi_j(x) \neq 0\}$. It follows from (2.8) that $\{z_i : i \in J(x)\} \subset S(x)$. By condition (i)

$$\varphi_M \circ \psi(x) \in \varphi_M(\Delta_{J(x)}) \subseteq Q(x), \quad (2.11)$$

$N \subset L_N \subset X$, and (2.7), we have, $\psi \circ \varphi_M : \Delta_m \rightarrow 2^{\Delta_m}$ has a fixed point $z \in \Delta_m$, that is, $z = \psi \circ \varphi_M(z)$. Let $\bar{x} = \varphi_M(z)$. Then

$$\bar{x} = \varphi_M(z) = \varphi_M \circ (\psi \circ \varphi_M(z)) = \varphi_M \circ \psi(\bar{x}) \subset Q(\bar{x}). \quad (2.12)$$

This completes the proof. $\square$

Lemma 2.4. *Let (X, φ_N) and (Z, φ_M^*) be two FC-spaces. let $P, T : X \rightarrow 2^Z$ be set-valued mappings. If for each $x \in X$, $P(x)$ is FC-subspace of Z and T being weakly-FC valued, then for each $x \in X$, $\{y \in X : P(x) \cap T(y) \neq \emptyset\}$ is FC-subspace of X .*

Proof. Let $x \in X$, $\{y_1, \dots, y_n\} \in \langle X \rangle$ with $P(x) \cap T(y_i) \neq \emptyset$ for each $i \in \{1, \dots, n\}$, and $y \in \varphi_N(\Delta_n)$. For each $i \in \{1, \dots, n\}$, let $z_i \in P(x) \cap T(y_i)$. It follows from T is weakly-FC valued that $T(y) \cap \varphi_M^*(\Delta'_n) \neq \emptyset$. Since $x \in X$, $P(x)$ is FC-subspace of Z , we have $\emptyset \neq T(y) \cap \varphi_M^*(\Delta'_n) \subset T(y) \cap P(x)$. Hence, for each $x \in X$, $\{y \in X : P(x) \cap T(y) \neq \emptyset\}$ is FC-subspace of X . This completes the proof. $\square$

Theorem 2.5. *Let (X, φ_N) be an FC-space, let Z be a nonempty set, and, let $P, T : X \rightarrow 2^Z$, $Q : X \rightarrow 2^X$ be three set-valued mappings satisfying the following conditions:*

- (i) *for each $x \in X$, $Q(x)$ is FC-subspace of X ;*
- (ii) *for each $x \in X$, $\{y \in X : P(x) \cap T(y) \neq \emptyset\} \subseteq Q(x)$;*
- (iii) $X = \bigcup_{z \in T(X)} c \text{ int } P^{-1}(z)$;
- (iv) *there exists a compact subset K of X such that for each $N \in \langle X \rangle$, there exists a compact FC-subspace L_N of X containing N such that*

$$L_N \setminus K \subseteq \bigcup_{y \in L_N} c \text{ int } \{x \in X : P(x) \cap T(y) \neq \emptyset\}. \quad (2.13)$$

Then there exists $\bar{x} \in X$ such that $\bar{x} \in Q(\bar{x})$.

Proof. Define a mapping $S : X \rightarrow 2^X$ by

$$S(x) = \{y \in X : P(x) \cap T(y) \neq \emptyset\}. \quad (2.14)$$

It is easy to see that for a family $\{A_i\}_{i \in I}$ of subsets of a topological space, $\bigcup_{i \in I} c \text{ int } A_i \subseteq c \text{ int}(\bigcup_{i \in I} A_i)$. Thus, we have

$$\bigcup_{y \in X} c \text{ int } S^{-1}(y) = \bigcup_{y \in X} c \text{ int} \left(\bigcup_{z \in T(y)} P^{-1}(z) \right) \supseteq \bigcup_{y \in X} \bigcup_{z \in T(y)} c \text{ int } P^{-1}(z) = \bigcup_{z \in T(X)} c \text{ int } P^{-1}(z). \quad (2.15)$$

By condition (iii), it follows that $X = \bigcup_{y \in X} c \text{ int } S^{-1}(y)$. Clearly condition (iv) of Theorem 2.5 implies condition (iii) of Theorem 2.3. By condition (i) and (ii), for each $x \in X$ and $N \in \langle S(x) \rangle$, we have $\varphi_N(\Delta_k) \subseteq Q(x)$. Then all conditions of Theorem 2.3 are satisfied. By Theorem 2.3, there exists $\bar{x} \in X$ such that $\bar{x} \in Q(\bar{x})$. This completes the proof. $\square$

Particularly, when $Q(x) = \{y \in X : P(x) \cap T(y) \neq \emptyset\}$, Theorem 2.5 reduces to the following result.

Corollary 2.6. *Let (X, φ_N) be an FC-space, let Z be a nonempty set, and let $P, T : X \rightarrow 2^Z$ be two set-valued mappings satisfying the following conditions:*

- (i) $X = \bigcup_{z \in T(X)} c \text{ int } P^{-1}(z)$;
- (ii) for each $x \in X$, $Q(x)$ is FC-subspace of X ;
- (iii) there exists a compact subset K of X such that for each $N \in \langle X \rangle$, there exists a compact FC-subspace L_N of X containing N such that

$$L_N \setminus K \subseteq \bigcup_{y \in L_N} c \text{ int} \{x \in X : P(x) \cap T(y) \neq \emptyset\}. \quad (2.16)$$

Then there exists $\bar{x} \in X$ such that $P(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

Remark 2.7. Theorem 2.3 generalizes Theorem 4 in [14]. Theorem 2.5 generalizes Theorem 1 in [2]. Corollary 2.6 generalizes Theorem 8 in [12], Theorem 3.6 in [13], and Theorem 2 in [2].

Corollary 2.8. *Let (X, φ_N) be an FC-space, Z be a nonempty set and $P, T : X \rightarrow 2^Z$ be two set-valued mappings satisfying the following conditions:*

- (i) P^{-1} is transfer compactly open valued;
- (ii) for each $x \in X$, $\{y \in X : P(x) \cap T(y) \neq \emptyset\}$ is FC-subspace of X ;
- (iii) there exists a compact subset K of X such that for each $N \in \langle X \rangle$, there exists a compact FC-subspace L_N of X containing N such that

$$L_N \setminus K \subseteq \bigcup_{y \in L_N} c \text{ int} \{x \in X : P(x) \cap T(y) \neq \emptyset\}, \quad (2.17)$$

- (iv) for each $x \in X$, $P(x) \cap T(x) = \emptyset$.

Then at least one of the following assertions holds:

- (a) there exists $\bar{x} \in X$ such that $P(\bar{x}) = \emptyset$,
- (b) there exists $\bar{z} \in Z$ such that $T^{-1}(\bar{z}) = \emptyset$.

Proof. By means of contradiction, suppose that for each $x \in X, P(x) \neq \emptyset$, and for each $z \in Z, T^{-1}(z) \neq \emptyset$ (that is, $T(X) = Z$). Then, by Lemma 2.2, we have

$$X = \bigcup_{z \in Z} c \operatorname{int} P^{-1}(z) = \bigcup_{z \in T(X)} c \operatorname{int} P^{-1}(z). \quad (2.18)$$

Then all conditions of Corollary 2.6 are satisfied, and it follows that there exists $\bar{x} \in X$ such that $P(\bar{x}) \cap T(\bar{x}) \neq \emptyset$, but this contradicts condition (iv). This completes the proof. $\square$

Corollary 2.9. *In Corollary 2.8, if one assumes further that $T(X) = Z$, then there exists $\bar{x} \in X$ such that $P(\bar{x}) = \emptyset$.*

Corollary 2.10. *Let (X, φ_N) be an FC-space, and let $P : X \rightarrow 2^X$ be a set-valued mapping satisfying the following conditions:*

- (i) P^{-1} is transfer compactly open valued;
- (ii) for each $x \in X, P(x)$ is FC-subspace of X ;
- (iii) for each $x \in X, x \notin P(x)$;
- (iv) there exists a compact subset K of X such that for each $N \in \langle X \rangle$, there exists a compact FC-subspace L_N of X containing N such that

$$L_N \setminus K \subseteq \bigcup_{y \in L_N} c \operatorname{int} P^{-1}(y). \quad (2.19)$$

Then there exists $\bar{x} \in X$ such that $P(\bar{x}) = \emptyset$.

Proof. Define a mapping $T : X \rightarrow 2^X$ by $T(x) = \{x\}$ for all $x \in X$. Then Corollary 2.10 follows from Corollary 2.8. $\square$

Remark 2.11. Corollary 2.8 generalizes Theorem 3 in [2]. Corollary 2.9 and Corollary 2.10 improve Corollary 1 and Corollary 2 in [2], respectively. If the FC-space X is compact, condition (iii) in Theorem 2.3 is satisfied trivially. Consequently, condition (iv) of Theorem 2.5, condition (iv) of Corollary 2.10, condition (iii) of Corollary 2.6, condition (iii) of Corollary 2.8 and condition (iii) of Corollary 2.9 are all satisfied trivially.

Corollary 2.12. *Let (X, φ_N) and (Z, φ_M^*) be two FC-spaces and let $P, T : X \rightarrow 2^Z$ be two set-valued mappings satisfying conditions (i) and (iii) in Corollary 2.6. If for each $x \in X, P(x)$ is FC-subspace of Z and T is weakly-FC valued, then there exists $\bar{x} \in X$ such that $P(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.*

Proof. By Lemma 2.4 and Corollary 2.6, we have that the conclusion of Corollary 2.12 holds. $\square$

Corollary 2.13. Let (X, φ_N) and (Z, φ_M^*) be two FC-spaces and $P, T : X \rightarrow 2^Z$ be set-valued mappings satisfying conditions (i), (iii) and (iv) in Corollary 2.8. If for each $x \in X$, $P(x)$ is FC-subspace of Z and T is weakly-FC valued, then at least one of the following assertions holds:

- (a) there exists $\bar{x} \in X$ such that $P(\bar{x}) = \emptyset$;
- (b) there exists $\bar{z} \in Z$ such that $T^{-1}(\bar{z}) = \emptyset$.

Proof. By Lemma 2.4 and Corollary 2.8, we have that the conclusion of Corollary 2.13 holds. $\square$

Remark 2.14. Corollary 2.12 generalizes Theorem 4 in [2]. Corollary 2.13 generalizes Theorem 5 in [2].

3. Generalized Equilibrium Theorems and Minimax Inequality

Inspired by Balaj and Lin [2], we have the following concept.

Definition 3.1. Let (X, φ_N) be an FC-space, let Z and V be nonempty sets, let ρ be a relation on 2^V , and let $F : X \times Z \rightarrow 2^V, C : X \rightarrow 2^V$ be set-valued mappings. We say that F is (C, ρ) -transfer compactly continuous in the first variable if for any $x \in X, z \in \{u \in Z : F(x, u) \rho C(x)\}$, there exists $\bar{x} \in X$ such that $z \in c \text{ int}\{u \in Z : F(\bar{x}, u) \rho C(\bar{x})\}$.

The following concept generalizes the corresponding notion in Ding [6, 8].

Definition 3.2. Let (X, φ_N) be an FC-space and Y be topological space. For each $y \in Y, f : Y \times X \rightarrow R$ is said to be FC-quasiconvex (resp., FC-quasiconcave) in the first variable, if for each $\lambda \in R$, the set $\{x \in X : f(y, x) < \lambda\}$ (resp., $\{x \in X : f(x, y) > \lambda\}$) is an FC-subspace of X .

The following concept generalizes the corresponding of notion in Chen and Chang [5].

Definition 3.3. Let X and Y be two topological spaces, and let $f : Y \times X \rightarrow R \cup \{-\infty, +\infty\}$ be a function. Then f is said to be α -transfer compactly lower semicontinuous (in short, α -transfer compactly l.s.c) in the first variable if for each $y \in Y$ with $x \in \{u \in X : f(y, u) > \alpha\}$, there exists $\bar{y} \in Y$ such that $x \in c \text{ int}\{u \in X : f(\bar{y}, u) > \alpha\}$. f is said to be α -transfer compactly u.s.c in the first variable if and only if $-f$ is α -transfer compactly l.s.c in the first variable. If f is α -transfer compactly l.s.c in the first variable for each $\alpha \in R$, we say that f is transfer compactly l.s.c in the first variable.

Theorem 3.4. Let (X, φ_N) be an FC-space, let Z and V be nonempty sets, and let ρ be a relation on 2^V . Let $F : X \times Z \rightarrow 2^V, C : X \rightarrow 2^V$, and $T : X \rightarrow 2^Z$ be set-valued mappings satisfying the following conditions:

- (i) $T(X) = Z$;
- (ii) F is (C, ρ^c) -transfer compactly continuous in the first variable;
- (iii) there exists a compact subset K of X such that for each $N \in \langle X \rangle$, there exists a compact FC-subspace L_N of X containing N such that

$$L_N \bigcap \bigcap_{y \in L_N} c \text{ cl}\{x \in X : F(x, z) \rho C(x), \forall z \in T(y)\} \subseteq K, \quad (3.1)$$

- (iv) for each $x \in X$, the set $\{y \in X : F(x, z)\rho^c C(x), \text{ for some } z \in T(y)\}$ is FC-subspace of X ;
 (v) for all $x \in X$ and any $z \in T(x)$, $F(x, z)\rho^c C(x)$.

Then there exists $\bar{x} \in X$ such that $F(\bar{x}, z)\rho^c C(\bar{x})$ for all $z \in Z$.

Proof. Let $P : X \times Z \rightarrow 2^Z$ be the set-valued mapping defined by

$$P(x) = \{z \in Z : F(x, z)\rho^c C(x)\}. \quad (3.2)$$

We prove that the mappings satisfy the conditions of Theorem 2.5. It is clear that condition (ii) is equivalent to the fact that the mapping P^{-1} is transfer compactly open valued. By condition (iv), for each $x \in X$ the set $\{y \in X : P(x) \cap T(y) \neq \emptyset\}$ is FC-subspace of X . Let $N \in \langle X \rangle$ and for all $x \in L_N \setminus K$. Then, by condition (iii) there exists $y \in L_N$ such that

$$\begin{aligned} x &\in X \setminus c \text{ cl} \{x' \in X : F(x', z)\rho^c C(x'), \forall z \in T(y)\} \\ &= c \text{ int} \{x' \in X : F(x', z)\rho^c C(x'), \text{ for some } z \in T(y)\} \\ &= c \text{ int} \{x' \in X : P(x') \cap T(y) \neq \emptyset\}. \end{aligned} \quad (3.3)$$

Thus all the conditions of Theorem 2.5 are fulfilled, and by condition (i), $T^{-1}(z) \neq \emptyset$ for all $z \in Z$. Hence, there exists $\bar{x} \in X$ such that $P(\bar{x}) = \emptyset$, that is, $F(\bar{x}, z)\rho^c C(\bar{x})$ for all $z \in Z$. This completes the proof. $\square$

Remark 3.5. Theorem 3.4 generalizes Theorem 6 in [2] in the following several aspects: (a) from G -convex space to FC-space without linear structure; (b) from (C, ρ^c) -transfer continuous to (C, ρ^c) -transfer compactly continuous; (c) conditions (iii) and (iv) of Theorem 3.4 are weaker than conditions (iii) and (iv) of Theorem 6 in [2].

Corollary 3.6. Let (X, φ_N) and (Z, φ_M^*) be two FC-spaces, let V be nonempty set, and let ρ be a relation on 2^V . Let $F : X \times Z \rightarrow 2^V$, $C : X \rightarrow 2^V$, and $T : X \rightarrow 2^Z$ be set-valued mappings satisfying the following conditions: $P, T : X \rightarrow 2^Z$ are set-valued mappings satisfying conditions (i), (ii), (iii), and (v) in Theorem 3.4 and if for each $x \in X$, $\{z \in Z : F(x, z)\rho^c C(x)\}$ is FC-subspace of Z and T is weakly-FC valued, then there exists $\bar{x} \in X$ such that $F(\bar{x}, z)\rho^c C(\bar{x})$ for all $z \in Z$.

Proof. By Lemma 2.4 and Corollary 2.12, we have that the conclusion of Corollary 3.6 holds. $\square$

Remark 3.7. Corollary 3.6 generalizes Theorem 7 in [2] in the following several aspects: (a) from G -convex space to FC-space without linear structure; (b) from (C, ρ^c) -transfer continuous to (C, ρ^c) -transfer compactly continuous; (c) from G -weakly convex to weakly-FC valued; (c) conditions (iii) and “for each $x \in X$, $\{z \in Z : F(x, z)\rho^c C(x)\}$ is FC-subspace of Z ” of Corollary 3.6 are weaker than conditions (iii) and (vii) of Theorem 7 in [2].

Theorem 3.8. Let (X, φ_N) be a compact FC-space, let Z be nonempty set, let $f, g : Z \times X \rightarrow \bar{R}$, $h : X \times X \rightarrow \bar{R}$ be three functions, and let α, β, γ be three real numbers satisfying the following conditions:

- (i) for each $x \in X$, the set $\{y \in X : h(x, y) < \gamma\}$ is FC-subspace of X ;
- (ii) f is α -transfer compactly u.s.c in the first variable;
- (iii) for each $x, y \in X, z \in Z$, $f(z, x) < \alpha$ and $g(z, y) < \beta \Rightarrow h(x, y) < \gamma$;
- (iv) $h(x, x) \geq \gamma$ for all $x \in X$.

Then, at least one of the following assertions holds:

- (a) there exists $\bar{x} \in X$ such that $f(z, \bar{x}) \geq \alpha$ for all $z \in Z$;
- (b) there exists $\bar{z} \in Z$ such that $g(\bar{z}, x) \geq \beta$ for all $x \in X$.

Proof. Let $P, T : X \rightarrow 2^Z$, $Q : X \rightarrow 2^X$ be the set-valued mappings defined by

$$\begin{aligned} P(x) &= \{z \in Z : f(z, x) < \alpha\}, \\ T(x) &= \{z \in Z : g(z, x) < \beta\}, \\ Q(x) &= \{y \in X : h(x, y) < \gamma\} \end{aligned} \tag{3.4}$$

for all $x \in X$. By means of contradiction, suppose that both assertions (a) and (b) would be false; that is

- (1) for each $x \in X$, there exists $z \in Z$ such that $f(z, x) < \alpha$,
- (2) for each $z \in Z$, there exists $x \in X$ such that $g(z, x) < \beta$.

By (2), $T(X) = Z$. Arbitrarily choose $x \in X$. By (1) and condition (ii), it follows that $x \in \{u \in X : f(z, u) < \alpha\} = P^{-1}(z)$, and there exists $\bar{z} \in Z$ such that $x \in \text{int } P^{-1}(\bar{z})$. Hence, we infer that $X = \bigcup_{z \in Z} \text{int } P^{-1}(z) = \bigcup_{z \in T(X)} \text{int } P^{-1}(z)$. By condition (i), $Q(x)$ is FC-subspace of X , and by condition (iii), $\{y \in X : P(x) \cap T(y) \neq \emptyset\} \subseteq Q(x)$ for each $x \in X$. Taking into account Remark 2.11, all conditions of Theorem 2.3 are satisfied, and it follows that there exists $\bar{x} \in X$ such that $\bar{x} \in Q(\bar{x})$. Hence, we have $h(\bar{x}, \bar{x}) < \gamma$, which contradicts condition (iv). This completes the proof. $\square$

Remark 3.9. Theorem 3.8 generalizes Theorem 12 in [2] in the following several aspects: (a) from G -convex space to FC-space without linear structure; (b) from α -transfer u.s.c to α -transfer compactly u.s.c; (c) conditions (i) of Theorem 3.8 are weaker than conditions (i) of Theorem 12 in [2].

From Theorem 3.8, we may obtain the following minimax inequality.

Corollary 3.10. Let (X, φ_N) be a compact FC-space, let Z be nonempty set, and let $f, g : Z \times X \rightarrow \bar{R}$, $h : X \times X \rightarrow \bar{R}$ be three functions satisfying the following conditions:

- (i) $h(x, \cdot)$ is FC-quasiconvex, for each $x \in X$;
- (ii) f is transfer compactly u.s.c in the first variable;
- (iii) for each $x, y \in X, z \in Z$, $h(x, y) \leq f(z, x) + g(z, y)$.

Then

$$\inf_{x \in X} h(x, x) \leq \sup_{x \in X} \inf_{z \in Z} f(z, x) + \sup_{z \in Z} \inf_{x \in X} g(z, x), \quad (3.5)$$

with the convention $\infty + (-\infty) = \infty$.

Proof. We may suppose that

$$\inf_{x \in X} h(x, x) > -\infty, \quad \sup_{x \in X} \inf_{z \in Z} f(z, x) < \infty, \quad \sup_{z \in Z} \inf_{x \in X} g(z, x) < \infty. \quad (3.6)$$

By means of contradiction, suppose that

$$\inf_{x \in X} h(x, x) > \sup_{x \in X} \inf_{z \in Z} f(z, x) + \sup_{z \in Z} \inf_{x \in X} g(z, x) \quad (3.7)$$

and choose $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\sup_{x \in X} \inf_{z \in Z} f(z, x) < \alpha, \quad \sup_{z \in Z} \inf_{x \in X} g(z, x) < \beta, \quad \gamma < \inf_{x \in X} h(x, x), \quad \alpha + \beta < \gamma. \quad (3.8)$$

It is easy to see that functions f, g , and h satisfy all the conditions of Theorem 3.8. We prove that neither assertion (a) nor (b) of the conclusion of Theorem 3.8 can take place.

If (a) happens, then

$$\alpha \leq \inf_{z \in Z} f(z, \bar{x}) \leq \sup_{x \in X} \inf_{z \in Z} f(z, x), \quad (3.9)$$

a contradiction.

If (b) happens, then

$$\sup_{z \in Z} \inf_{x \in X} g(z, x) \geq \inf_{x \in X} g(\bar{z}, x) \geq \beta, \quad (3.10)$$

a contradiction again.

This completes the proof. $\square$

Remark 3.11. Corollary 3.10 generalizes Theorem 13 in [2] in the following several aspects: (a) from G -convex space to FC -space without linear structure; (b) from G -quasiconvex to FC -quasiconvex; (c) from transfer u.s.c to transfer compactly u.s.c.

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References

- [1] M. Balaj, "Coincidence and maximal element theorems and their applications to generalized equilibrium problems and minimax inequalities," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 12, pp. 3962–3971, 2008.
- [2] M. Balaj and L. J. Lin, "Fixed points, coincidence points and maximal elements with applications to generalized equilibrium problems and minimax theory," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 1, pp. 393–403, 2009.
- [3] H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano, and J. V. Llinares, "Abstract convexity and fixed points," *Journal of Mathematical Analysis and Applications*, vol. 222, no. 1, pp. 138–150, 1998.
- [4] T. H. Chang and C. L. Yen, "KKM property and fixed point theorems," *Journal of Mathematical Analysis and Applications*, vol. 203, no. 1, pp. 224–235, 1996.
- [5] C. M. Chen and T. H. Chang, "Some results for the family $KKM(X, Y)$ and the Φ -mapping," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 92–101, 2007.
- [6] X. P. Ding, "Maximal element principles on generalized convex spaces and their applications," in *Set Valued Mappings with Applications in Nonlinear Analysis*, vol. 4 of *Series in Mathematical Analysis and Applications*, pp. 149–174, Taylor & Francis, London, UK, 2002.
- [7] X. P. Ding, "Maximal element theorems in product FC -spaces and generalized games," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 29–42, 2005.
- [8] X. P. Ding, "System of coincidence theorems in product topological spaces and applications. II," *Applied Mathematics and Mechanics*, vol. 26, no. 12, pp. 1556–1563, 2005.
- [9] L. J. Lin and W. P. Wan, "KKM type theorems and coincidence theorems with applications to the existence of equilibria," *Journal of Optimization Theory and Applications*, vol. 123, no. 1, pp. 105–122, 2004.
- [10] L. J. Lin and Q. H. Ansari, "Collective fixed points and maximal elements with applications to abstract economies," *Journal of Mathematical Analysis and Applications*, vol. 296, no. 2, pp. 455–472, 2004.
- [11] L. J. Lin and H. L. Chen, "The study of KKM theorems with applications to vector equilibrium problems with implicit vector variational inequalities problems," *Journal of Global Optimization*, vol. 32, no. 1, pp. 135–157, 2005.
- [12] M. Balaj, "Coincidence and maximal element theorems in generalized convex spaces," *Nonlinear Analysis Forum*, vol. 7, no. 1, pp. 123–130, 2002.
- [13] H. Kim and S. Park, "Remarks on the KKM property for open-valued multimaps on generalized convex spaces," *Journal of the Korean Mathematical Society*, vol. 42, no. 1, pp. 101–110, 2005.
- [14] Z.-T. Yu and L. J. Lin, "Continuous selection and fixed point theorems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 52, no. 2, pp. 445–455, 2003.
- [15] X. P. Ding and T. M. Ding, "KKM type theorems and generalized vector equilibrium problems in noncompact FC -spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 1230–1245, 2007.
- [16] K. Fan, "Sur un théorème minimax," *Comptes Rendus de l'Académie des Sciences. Série I*, vol. 259, pp. 3925–3928, 1964.
- [17] F. C. Liu, "A note on the von Neumann-Sion minimax principle," *Bulletin of the Institute of Mathematics: Academia Sinica*, vol. 6, no. 2, pp. 517–524, 1978.