Research Article
On New Wilker-Type Inequalities

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Received 23 February 2011; Accepted 27 April 2011

1. Introduction

Wilker [1] proposed two open questions as the following statements.

Problem 1. If $0 < x < \pi/2$, then

\[
\left( \frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} > 2. \tag{1.1}
\]

Problem 2. There exists a largest constant $c$ such that

\[
\left( \frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} > 2 + c x^3 \tan x \tag{1.2}
\]

for $0 < x < \pi/2$.

Sumner et al. [2] affirmed the truth of the problems above and obtained a further result as follows.
Theorem 1.1. If $0 < x < \pi/2$, then

$$\frac{16}{\pi^3} x^3 \tan x < \left( \frac{\sin x}{x} \right)^2 + \frac{x}{\tan x} - 2 < \frac{8}{45} x^3 \tan x. \tag{1.3}$$

Furthermore, $16/\pi^4$ and $8/45$ are the best constants in (1.3).

Guo et al. [3] gave new proofs of inequalities (1.1) and (1.2). In recent years, Zhu [4] showed a new simple proof of inequality (1.1); Pinelis [5] got other proof of inequalities (1.3) by using L’Hospital rules for monotonicity; Zhang and Zhu [6] gave a new elementary proof of double inequalities (1.3).

Besides, in article [7], Wang offered another type of the inequality and proved it by the power series expansions of circular functions as follows

Theorem 1.2. Let $0 < x < \pi/2$. Then,

$$\frac{2}{45} x^3 \sin x < \left( \frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} - 2 < \frac{2}{\pi} - \frac{16}{\pi^3} \right)x^3 \sin x. \tag{1.4}$$

Furthermore, $2/45$ and $(2/\pi - 16/\pi^3)$ are the best constants in (1.4).

Zhu [8] established a new Wilker-type inequality involving hyperbolic functions as follows:

$$\left( \frac{\sinh x}{x} \right)^2 + \frac{x}{\tanh x} - 2 > \frac{8}{45} x^3 \tanh x, \quad x > 0, \tag{1.5}$$

where the constant $8/45$ is the optimum constant in (1.5).

In fact, we can show a new Wilker-type inequality involving hyperbolic functions referring to the result above.

Theorem 1.3. Let $x > 0$. Then

$$\left( \frac{x}{\sinh x} \right)^2 + \frac{x}{\tanh x} - 2 < \frac{2}{45} x^3 \sinh x. \tag{1.6}$$

Furthermore, $2/45$ is the best constant in (1.6).

The purpose of this paper is to present a concise proof of inequality (1.6) by using the power series expansion of hyperbolic functions and to give an elementary proof of inequality (1.4) in another way.

2. The Proof of Theorem 1.3

Let $((x/\sinh x)^2 + x/\tanh x - 2)/(x^3 \sinh x) = (x^2 + x \sinh x \cosh x - 2(\sinh x)^2)/(x \sinh x)^3 = A(x)/B(x)$, where $A(x) = x^2 + x \sinh x \cosh x - 2(\sinh x)^2$, and $B(x) = (x \sinh x)^3$. We get that
the existence of Theorem 1.3 can be ensured when proving the following two statements: the first is $A(x)/B(x) \leq 2/45$ or $2B(x) \geq 45A(x)$, and the second is $\lim_{x \rightarrow 0^+} (A(x)/B(x)) = 2/45$. Since

$$A(x) = x^2 + \frac{1}{2}x \sinh x - (\cosh 2x - 1) = x^2 + \frac{1}{2}x \sum_{n=0}^{\infty} \frac{(2x)^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!} + 1$$

$$= x^2 + \frac{1}{2} \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+2} - \sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!} + 1$$

$$= \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n+1)!} x^{2n+2} - \sum_{n=1}^{\infty} \frac{2^{2n+2}}{(2n+2)!} x^{2n+2}$$

$$= \sum_{n=2}^{\infty} \left( \frac{2^{2n}}{(2n+1)!} - \frac{2^{2n+2}}{(2n+2)!} \right) x^{2n+2} = \sum_{n=2}^{\infty} \left( \frac{2^{2n}}{(2n+1)!} - \frac{2^{2n+2}}{(2n+2)!} \right) x^{2n+2}$$

$$= \sum_{n=2}^{\infty} a_n x^{2n+2},$$

$$B(x) = \frac{1}{4} x^3 (\sinh 3x - 3 \sinh x) = \frac{1}{4} x^3 \sum_{n=0}^{\infty} \frac{3^{2n+1} - 3}{(2n+1)!} x^{2n+1} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{3^{2n+1} - 3}{(2n+1)!} x^{2n+1}$$

$$= \frac{1}{4} \sum_{n=1}^{\infty} \frac{3^{2n+1} - 3}{(2n+1)!} x^{2n+1} = \sum_{n=2}^{\infty} b_n x^{2n+2},$$

where $a_n = 2^{2n}/(2n+1)! - 2^{2n+2}/(2n+2)!$, $b_n = (3^{2n+1} - 3)/4(2n-1)!$, $n \geq 2$, and $n \in \mathbb{N}^+$, according to the analysis above, the proof of Theorem 1.3 can be completed when proving that

$$2b_n \geq 45a_n,$$  \hspace{1cm} (2.2)

that is,

$$n(n+1)(2n+1)(9^n - 9) \geq 135(n-1)4^n$$  \hspace{1cm} (2.3)

for $n \geq 2$. 

Let \( f(x) = x(x + 1)(2x + 1)(9^x - 9) - 135(x - 1)4^x \) for \( x \in [2, +\infty) \). We compute

\[
\begin{align*}
  f'(x) &= \left(6x^2 + 6x + 1\right)9^x - 9\left(6x^2 + 6x + 1\right) \\
  &\quad + \left(2x^3 + 3x^2 + x\right)9^x \log 9 - 135 \cdot 4^x - 135(x - 1)4^x \log 4,
\end{align*}
\]

\[
  f''(x) = (12x + 6)9^x + 2\left(6x^2 + 6x + 1\right)9^x \log 9 - 9(12x + 6) \\
  &\quad + \left(2x^3 + 3x^2 + x\right)9^x (\log 9)^2 - 270 \cdot 4^x \log 4 - 135(x - 1)4^x (\log 4)^2,
\]

(2.4)

\[
  f^{(3)}(x) = 12 \cdot 9^x + (36x + 18)9^x \log 9 + 3\left(6x^2 + 6x + 1\right)9^x (\log 9)^2 - 9 \cdot 12 \\
  &\quad + \left(2x^3 + 3x^2 + x\right)9^x (\log 9)^3 - 405 \cdot 4^x (\log 4)^2 - 135(x - 1)4^x (\log 4)^3,
\]

\[
  f^{(4)}(x) = \left(ax^3 + bx^2 + cx + d\right)9^x - (ex + m)4^x,
\]

where

\[
\begin{align*}
  a &= 2(\log 9)^4 > 0, \\
  b &= 3(\log 9)^4 + 24(\log 9)^3 > 0, \\
  c &= (\log 9)^4 + 24(\log 9)^3 + 72(\log 9)^2 > 0, \\
  d &= 4(\log 9)^3 + 36(\log 9)^2 + 48(\log 9) > 0, \\
  e &= 135(\log 4)^4 > 0, \\
  m &= 540(\log 4)^3 - 135(\log 4)^4 \approx 940.059 > 0.
\end{align*}
\]

Then let \( H(x) = \left(ax^3 + bx^2 + cx + d\right)(9/4)^x / (ex + m) \); we compute \( H'(x) = \left(l(x) / (ex + m)^2\right)(9/4)^x \), where

\[
\begin{align*}
  l(x) &= a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0, \\
  a_4 &= ae \log \frac{9}{4} > 0, \\
  a_3 &= 2ae + (eb + am) \log \frac{9}{4} > 0, \\
  a_2 &= be + 3am + (ec + mb) \log \frac{9}{4} > 0, \\
  a_1 &= 2bm + (mc + de) \log \frac{9}{4} > 0, \\
  a_0 &= m\left(c + d \log \frac{9}{4}\right) - ed \approx 671800 > 0.
\end{align*}
\]
As we can see, all coefficients of the polynomial \( l(x) \) are positive integers. When \( x > 2 \), we have \( l(x) > 0 \) and \( H'(x) > 0 \). In view of that \( H(2) \approx 8.4763 > 1, f^{(4)}(2) \approx 231740 > 0, f^{(3)}(2) \approx 67857 > 0, f''(2) \approx 16922 > 0, f'(2) \approx 2849.9 > 0 \), we have that \( H(x) > 1 \) and \( f^{(4)}(x) > 0 \) for \( x > 2 \). We can also conclude that \( f^{(i)}(x) (i = 3, 2, 1, 0) \) is increasing on \([2, +\infty)\), respectively. Since \( f(2) = 0 \), we have \( f(x) \geq 0 \) for \( x \geq 2 \). So the proof of inequality (2.3) is completed. Furthermore, \( \lim_{x \to 0^+} (A(x)/B(x)) = a_2/b_2 = (2^4/5! - 2^5/6!)/(3^3 - 3)/4 \cdot 3! = 2/45 \); the proof of Theorem 1.3 is completed.

3. Another Proof of Theorem 1.2

We simplify the double inequality (1.4) into another form:

\[
\frac{2}{45} < \frac{(x/\sin x)^2 + x/\tan x - 2}{x^3 \sin x} < \frac{2}{\pi} - \frac{16}{\pi^3}, \quad 0 < x < \frac{\pi}{2}.
\]

(3.1)

Here we will discuss the monotonicity of the function \( g(x) = ((x/\sin x)^2 + x/\tan x - 2)/x^3 \sin x = ((1/\sin x)(x/\sin x)^2 + x \cos x/\sin^2 x - 2/\sin x)/x^3 \). If the function \( g(x) \) is increasing for \( 0 < x < \pi/2 \) and both \( \lim_{x \to 0^+} g(x) = 2/45 \) and \( \lim_{x \to (\pi/2)^-} g(x) = 2/\pi - 16/\pi^3 \) are right, the proof of equality (1.4) is completed.

From [9] or [10, page 75], we know that for \( 0 < |x| < \pi \) the equality

\[
\frac{1}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n-1} \quad (3.2)
\]

holds. So we can also know that for \( 0 < |x| < \pi \)

\[
\frac{\cos x}{\sin^2 x} = \left( -\frac{1}{\sin x} \right)' = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| (2n - 1) x^{2n-2},
\]

\[
\frac{1}{\sin^3 x} = \left( -\frac{\cos x}{2\sin^2 x} \right)' = \frac{1}{x^3} + \sum_{n=2}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| (2n - 1)(2n - 2) x^{2n-3}
\]

\[
+ \frac{1}{2x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n-1}
\]

(3.3)
hold. Then we compute \( g(x) = \sum_{n=2}^{\infty} a_n x^{2n-4} \) by using the results above, where

\[
a_n = \frac{1}{2} \frac{2^{2n-2}}{2(2n)!} |B_{2n}| (2n - 1)(2n - 2) + \frac{1}{2} \frac{2^{2n-2}}{2(2n - 2)!} |B_{2n-2}|
- \frac{2^{2n-2}}{2(2n)!} |B_{2n}| (2n - 1) - 2 \frac{2^{2n-2}}{2(2n - 2)!} |B_{2n}|
= \frac{1}{2} \frac{2^{2n-2}}{2(2n - 2)!} |B_{2n-2}| + \frac{(2n-1)(2n-2) - 2(2n-1) - 4}{2(2n)!} \left( 2^{2n-2} - 2 \right) |B_{2n}|
= \frac{1}{2} \frac{2^{2n-2}}{2(2n - 2)!} |B_{2n-2}| + \frac{2n(2n-5)}{2(2n)!} \left( 2^{2n-2} - 2 \right) |B_{2n}|.
\]

When \( n = 2 \), we can compute \( a_2 = (1/2)(2/2!)[B_2] + (-4/2) \cdot 4 \cdot |B_4| = (1/2) \cdot (1/6) - (7/6) \cdot (1/30) = 2/45 > 0 \) by using \( |B_2| = 1/6 \) and \( |B_4| = 1/30 \). Furthermore, there is an obvious fact that \( a_n > 0 \) for \( n \geq 3 \). So far, we can demonstrate that \( a_n > 0 \) for all \( n \geq 2 \). So the function \( g(x) \) is increasing on \((0, \pi/2)\). Evidently, \( \lim_{x \to 0^+} g(x) = 2/45 \) and \( \lim_{x \to (\pi/2)^-} g(x) = 2/\pi - 16/\pi^3 \) are true. So the proof of the double inequality of (1.4) is completed.

**Remark 3.1.** Wilker’s inequalities (1.1) and (1.2) have been further refined by many scholars in the past few years; the readers can refer to [11–16].

**References**


