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Research Article

Analytical Solution for the Differential Equation Containing Generalized Fractional Derivative Operators and Mittag-Leffler-Type Function

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We discuss and derive the analytical solution for the fractional partial differential equation with generalized Riemann-Liouville fractional operator $D_{0,t}^{\alpha,\beta}$ of order α and β. Here, we derive the solution of the given differential equation with the help of Laplace and Hankel transform in terms of Fox's H-function as well as in terms of Fox-Wright function ${}_p\psi_q$.

1. Introduction, Definition, and Preliminaries

Applications of fractional calculus require fractional derivatives of different kinds [1–9]. Differentiation and integration of fractional order are traditionally defined by the right-sided Riemann-Liouville fractional integral operator I_{a+}^{P} and the left-sided Riemann-Liouville fractional integral operator I_{a-}^{P}, and the corresponding Riemann-Liouville fractional derivative operators D_{a+}^{P} and D_{a-}^{P}, as follows [10, 11]:

$$\left(I_{a+}^{\mu} f\right)(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}} dt \quad (x > a; R(\mu) > 0), \tag{1.1}$$

$$\left(I_{a-}^{\mu} f\right)(x) = \frac{1}{\Gamma(\mu)} \int_x^a \frac{f(t)}{(t-x)^{1-\mu}} dt \quad (x < a; R(\mu) > 0), \tag{1.2}$$

$$\left(D_{a\pm}^{\mu} f\right)(x) = \left(\pm \frac{d}{dx}\right)^n \left(I_{a\pm}^{n-\mu} f\right)(x) \quad (R(\mu) \geq 0; n = [R(\mu)] + 1), \tag{1.3}$$

where the function f is locally integrable, $R(\mu)$ denotes the real part of the complex number $\mu \in C$ and $[R(\mu)]$ means the greatest integer in $R(\mu)$.

Recently, a remarkable large family of generalized Riemann-Liouville fractional derivatives of order α $(0 < \alpha < 1)$ and type β $(0 \leq \beta \leq 1)$ was introduced as follows [1–3, 5, 6, 8].

Definition 1.1. The right-sided fractional derivative $D_{a+}^{\alpha,\beta}$ and the left-sided fractional derivative $D_{a-}^{\alpha,\beta}$ of order α $(0 < \alpha < 1)$ and type β $(0 \leq \beta \leq 1)$ with respect to x are defined by

$$\left(D_{a\pm}^{\alpha,\beta} f\right)(x) = \left(\pm I_{a\pm}^{\beta(1-\alpha)} \frac{d}{dx}\left(I_{a\pm}^{(1-\beta)(1-\alpha)} f\right)\right)(x), \tag{1.4}$$

whenever the second number of (1.4) exists. This generalization (1.4) yields the classical Riemann-Liouville fractional derivative operator when $\beta = 0$. Moreover, for $\beta = 1$, it gives the fractional derivative operator introduced by Liouville [12] which is often attributed to Caputo now-a-days and which should more appropriately be referred to as the Liouville-Caputo fractional derivative. Several authors [7, 9] called the general operators in (1.4) the Hilfer fractional derivative operators. Applications of $D_{a\pm}^{\alpha,\beta}$ are given [3].

Using the formulas (1.1) and (1.2) in conjunction with (1.3) when $n = 1$, the fractional derivative operator $D_{a\pm}^{\alpha,\beta}$ can be written in the following form:

$$\left(D_{a\pm}^{\alpha,\beta} f\right)(x) = \left(\pm I_{a\pm}^{\beta(1-\alpha)}\left(D_{a\pm}^{\alpha+\beta-\alpha\beta} f\right)\right)(x). \tag{1.5}$$

The difference between fractional derivatives of different types becomes apparent from their Laplace transformations. For example, it is found for $0 < \alpha < 1$ that [1, 2, 9]

$$L\left[\left(D_{0+}^{\alpha,\beta} f\right)(x)\right](s) = s^{\alpha} L\left[f(x)\right](s) - s^{\beta(\alpha-1)}\left(I_{0+}^{(1-\beta)(1-\alpha)} f\right)(0+) \quad (0 < \alpha < 1), \tag{1.6}$$

where $(I_{0+}^{(1-\beta)(1-\alpha)} f)(0+)$ is the Riemann-Liouville fractional integral of order $(1-\beta)(1-\alpha)$ evaluated in the limit as $t \to 0+$, it being understood (as usual) that [13],

$$L\left[f(x)\right](s) := \int_0^{\infty} e^{-sx} f(x)dx := F(s), \tag{1.7}$$

provided that the defining integral in (1.7) exists.

The familiar Mittag-Leffler functions $E_{\mu}(z)$ and $E_{\mu,\nu}(z)$ are defined by the following series:

$$E_{\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + 1)} := E_{\mu,1}(z) \quad (z \in C; R(\mu) > 0), \tag{1.8}$$

$$E_{\mu,\nu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + \nu)} \quad (z, \nu \in C; R(\mu) > 0), \tag{1.9}$$

respectively. These functions are natural extensions of the exponential, hyperbolic, and trigonometric functions, since

$$E_1(z) = e^z, E_2\left(z^2\right) = \cosh z, E_2\left(-z^2\right) = \cos z,$$
$$E_{1,2}(z) = \frac{e^z - 1}{z}, E_{2,2}\left(z^2\right) = \frac{\sinh z}{z}. \tag{1.10}$$

For a detailed account of the various properties, generalizations, and applications of the Mittag-Leffler functions, the reader may refer to the recent works by, for example, Gorenflo et al. [14] and Kilbas et al. [15–17]. The Mittag-Leffler function (1.1) and some of its various generalizations have only recently been calculated numerically in the whole complex plane [18, 19]. By means of the series representation, a generalization of the Mittag-Leffler function $E_{\mu,\nu}(z)$ of (1.2) was introduced by Prabhakar [20] as follows:

$$E^{\lambda}_{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{\Gamma(\mu n + \nu)} \frac{z^n}{n!} \quad (z, \nu, \lambda \in C; R(\mu) > 0), \tag{1.11}$$

where $(\lambda)_\nu$ denotes the familiar Pochhammer symbol, defined (for $\lambda, \nu \in C$ and in terms of the familiar Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in C \setminus \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in N; \lambda \in C). \end{cases} \tag{1.12}$$

Clearly, we have the following special cases:

$$E^1_{\mu,\nu}(z) = E_{\mu,\nu}(z), \qquad E^1_{\mu,1}(z) = E_\mu(z). \tag{1.13}$$

Indeed, as already observed earlier by Srivastava and Saxena [21], the generalized Mittag-Leffler function $E^{\lambda}_{\mu,\nu}(z)$ itself is actually a very specialized case of a rather extensively investigated function ${}_p\Psi_q$ as indicated below [17]:

$$E^{\lambda}_{u,\nu}(z) = \frac{1}{\Gamma(\lambda)} {}_1\Psi_1 \left[\begin{matrix} (\lambda, 1); \\ (\nu, \mathrm{u}); \end{matrix} \; z \right]. \tag{1.14}$$

Here and in what follows, ${}_p\Psi_q$ denotes the Wright (or more appropriately, the Fox-Wright) generalized of the hypergeometric ${}_p\mathrm{F}_q$ function, which is defined as follows [12]:

$${}_p\Psi_q = \left[\begin{matrix} (a_1, A_1), \ldots, (a_p, A_p); \\ (b_1, B_1), \ldots, (b_q, B_q); \end{matrix} \; z \right] = \sum_{x=0}^{\infty} \frac{\Gamma(a_1 + A_1 k) \cdots \Gamma(a_p + A_p k) z^k}{\Gamma(b_1 + B_1 k) \cdots \Gamma(b_q + B_q k) k!}, \tag{1.15}$$

$$\left[R(A_j) > 0 \ (j = 1, \ldots, p); R(B_j) > 0 \ (j = 1, \ldots, q); 1 + R\left(\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \right) \geq 0 \right], \tag{1.16}$$

in which we assumed in general that

$$a_j, A_j \in C \ (j = 1, \ \ldots, p), \qquad b_j, B_j \in C \ (j = 1, \ \ldots, \ q). \tag{1.17}$$

In application of Mittag-Leffler function, it is useful to have the following Laplace inverse transform formula:

$$L^{-1}\left\{ \frac{S^{\gamma-\beta}}{(S^{\gamma} + A)^{k+1}} \right\} = \frac{1}{k!} t^{\gamma k+\beta-1} E^{k}_{\gamma,\beta}(-At^{\gamma}), \tag{1.18}$$

where $E^{j}_{\gamma,\beta}(z) = (d^j/dz^j)E_{\gamma,\beta}(z)$.

2. Fox's H-function

The Fox function, also referred as the Fox's H-function, generalizes the Mellin-Barnes function. The importance of the Fox function lies in the fact that it includes nearly all special functions occurring in applied mathematics and statistics as special cases. Fox H-function is defined as [22]

$$H^{1,p}_{p,q+1}\left[-x \middle| \begin{matrix} (1-a_1, A_1), \ldots, (1-a_p, A_p) \\ (0,1), (1-b_1, B_1), \ldots, (1-b_q, B_q) \end{matrix} \right] = \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + A_1 k) \cdots \Gamma(a_p + A_p k)}{k!\Gamma(b_1 + B_1 k) \cdots \Gamma(b_p + B_q k)} x^k. \tag{2.1}$$

We need this relation

$$\begin{aligned} E^{k}_{\alpha,\beta}(x) &= \sum_{n=k}^{\infty} \frac{n! x^{n-k}}{(n-k)!\Gamma(\alpha n + \beta)} = \sum_{j=0}^{\infty} \frac{\Gamma(j+k+1) x^j}{j!\Gamma(\alpha j + \alpha k + \beta)} \\ &= H^{1,1}_{1,2}\left[-x \middle| \begin{matrix} (-k, 1) \\ (0,1), (1-\alpha k - \beta, \alpha) \end{matrix} \right]. \end{aligned} \tag{2.2}$$

3. Finite Hankel Transform

If $f(r)$ satisfies Dirichlet conditions in closed interval $(0, a)$ and if its finite Hankel transform is defined to be [23]

$$H[f(r)] = \overline{f}(\lambda_n) = \int_0^a r f(r) J_0(r\lambda_n) dr, \tag{3.1}$$

where λ_n are the roots of the equation $J_0(r) = 0$. Then at each point of the interval at which $f(r)$ is continuous:

$$f(r) = \frac{2}{a^2} \sum_{n=1}^{\infty} \overline{f}(\lambda_n) \frac{J_0(\lambda_n r)}{J_1^2(\lambda_n a)}, \tag{3.2}$$

where the sum is taken over all positive roots of $J_0(r) = 0$, J_0 and J_1 are Bessel functions of first kind.

In application of the finite Hankel transform to physical problems, it is useful to have the following formula [23]

$$H\left[\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right] = -\lambda_n^2 \overline{f}(r) + a\lambda_n f(a) J_1(\lambda_n a). \tag{3.3}$$

Example 3.1. Solve the differential equation

$$D_{0,t}^{2\alpha,\beta} u(r,t) + a D_{0,t}^{\alpha,\beta} u(r,t) = d\left(\frac{\partial^2 u(r,t)}{\partial r^2} + \frac{1}{r} u(r,t)\right) + f(t), \tag{3.4}$$

where $0 < \alpha \leq 1/2$ and $0 \leq \beta \leq 1$ with initial condition

$$\begin{gathered} I_t^{(1-\beta)(1-2\alpha)} u(r,0) = \phi_1(r), \\ I_t^{(1-\beta)(1-\alpha)} u(r,0) = \phi_2(r), \\ u(r,t) = 0 \quad \text{everywhere for } t < 0, \\ u(r,t) = 0 \quad \text{for } r = 1,\ t > 0, \\ u(r,t) = \text{finite} \quad \text{at } r = 0,\ t > 0. \end{gathered} \tag{3.5}$$

Solution 1. Taking Laplace transform of (3.4), we get

$$s^{2\alpha}\tilde{u}(r,s) - s^{\beta(2\alpha-1)}\phi_1(r) + a s^{\alpha}\tilde{u}(r,s) - a s^{\beta(\alpha-1)}\phi_2(r) = d\left[\frac{\partial^2 \tilde{u}(r,s)}{\partial r^2} + \frac{1}{r}\tilde{u}(r,s)\right] + \tilde{f}(s). \tag{3.6}$$

Taking Hankel transform on both side of the above equation, we get

$$\begin{aligned} &s^{2\alpha}\tilde{\tilde{u}}(r,s) - s^{\beta(2\alpha-1)}\tilde{\phi}_1(r) + a s^{\alpha}\tilde{\tilde{u}}(r,s) - a s^{\beta(\alpha-1)}\tilde{\phi}_2(r) \\ &\quad = d\left[-\lambda_n^2 \tilde{\tilde{u}}(r,s)\right] + \tilde{f}(s)\frac{J_1(\lambda_n)}{\lambda_n}, \end{aligned} \tag{3.7}$$

then we get

$$\tilde{\tilde{u}}(r,s) = \frac{s^{\beta(2\alpha-1)}\tilde{\phi}_1(r)}{\left(s^{2\alpha}+as^{\alpha}+d\lambda_n^2\right)} + \frac{as^{\beta(\alpha-1)}\tilde{\phi}_2(r)}{\left(s^{2\alpha}+as^{\alpha}+d\lambda_n^2\right)} + \frac{\tilde{f}(s)}{\left(s^{2\alpha}+as^{\alpha}+d\lambda_n^2\right)}\frac{J_1(\lambda_n)}{\lambda_n}, \tag{3.8}$$

$$\tilde{\tilde{u}}(r,s) = \tilde{\tilde{G}}_1\tilde{\phi}_1(r) + a\tilde{\tilde{G}}_2\tilde{\phi}_2(r) + \tilde{\tilde{G}}_3\tilde{f}(s)\frac{J_1(\lambda_n)}{\lambda_n}, \tag{3.9}$$

where

$$\tilde{\tilde{G}}_1 = \frac{s^{\beta(2\alpha-1)}}{\left(s^{2\alpha}+as^{\alpha}+d\lambda_n^2\right)}, \tag{3.10}$$

$$\tilde{\tilde{G}}_2 = \frac{s^{\beta(\alpha-1)}}{\left(s^{2\alpha}+as^{\alpha}+d\lambda_n^2\right)}, \tag{3.11}$$

$$\tilde{\tilde{G}}_3 = \frac{1}{\left(s^{2\alpha}+as^{\alpha}+d\lambda_n^2\right)}. \tag{3.12}$$

On taking Laplace inverse of (3.10), (3.11), and (3.12), respectively,

$$L^{-1}\left\{\frac{s^{\beta(2\alpha-1)}}{\left(s^{2\alpha}+as^{\alpha}+d\lambda_n^2\right)}\right\} = \sum_{m=0}^{\infty}\frac{(-1)^m}{a^{m+1}}\frac{t^{\alpha+\beta-2\alpha\beta-m\alpha-1}}{m!}E^m_{\alpha,\alpha+\beta-2\alpha\beta-2m\alpha}\left(\frac{-d\lambda_n^2}{a}t^{\alpha}\right), \tag{3.13}$$

$$L^{-1}\left\{\frac{s^{\beta(\alpha-1)}}{\left(s^{2\alpha}+as^{\alpha}+d\lambda_n^2\right)}\right\} = \sum_{m=0}^{\infty}\frac{(-1)^m}{a^{m+1}}\frac{t^{\alpha+\beta-\alpha\beta-m\alpha-1}}{m!}E^m_{\alpha,\alpha+\beta-\alpha\beta-2m\alpha}\left(\frac{-d\lambda_n^2}{a}t^{\alpha}\right), \tag{3.14}$$

$$L^{-1}\left\{\frac{1}{\left(s^{2\alpha}+as^{\alpha}+d\lambda_n^2\right)}\right\} = \sum_{m=0}^{\infty}\frac{(-1)^m}{a^{m+1}}\frac{t^{\alpha-m\alpha-1}}{m!}E^m_{\alpha,\alpha-2m\alpha}\left(\frac{-d\lambda_n^2}{a}t^{\alpha}\right). \tag{3.15}$$

After taking Inverse Laplace and Hankel transform of (3.9) put the value (3.13) through (3.15) in (3.9), we get

$$\begin{aligned} u(r,t) = {} & 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{a^{m+1}}\frac{J_0(\lambda_n r)}{m!J_1^2(\lambda_n)}\phi_1(r)t^{-2\alpha\beta-m\alpha+\alpha+\beta-1}\sum_{j=0}^{\infty}\frac{(j+m+1)!\left(-d\lambda_n^2t^{\alpha}/a\right)^j}{j!\Gamma\left(j\alpha+\alpha+\beta-2\alpha\beta-2m\alpha\right)} \\ & + 2a\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{a^{m+1}}\frac{J_0(\lambda_n r)}{J_1^2(\lambda_n)}\phi_2(r)t^{-\alpha\beta-m\alpha+\alpha+\beta-1}\sum_{j=0}^{\infty}\frac{(j+m+1)!\left(-d\lambda_n^2t^{\alpha}/a\right)^j}{j!\Gamma\left(j\alpha+\alpha+\beta-\alpha\beta-m\alpha\right)} \\ & + 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{a^{m+1}}\frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)}\int_0^t u^{\alpha-m\alpha-1}\sum_{j=0}^{\infty}\frac{(j+m+1)!\left(-d\lambda_n^2u^{\alpha}/a\right)^j}{j!\Gamma\left(\alpha j+\alpha-m\alpha\right)}f(t-u)du. \end{aligned} \tag{3.16}$$

$$u(r,t) = 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{a^{m+1}}\frac{J_0(\lambda_n r)}{J_1^2(\lambda_n)}\phi_1(r)t^{-2\alpha\beta-m\alpha+\alpha+\beta-1}$$
$$\cdot H_{1,2}^{1,1}\left[\frac{d\lambda_n^2 t^\alpha}{a}\middle|\begin{matrix}(-m-1,1)\\(0,1),(1-\alpha-\beta+2\alpha\beta+2m\alpha,\alpha)\end{matrix}\right]$$
$$+2a\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{a^{m+1}}\frac{J_0(\lambda_n r)}{J_1^2(\lambda_n)}\phi_2(r)t^{-\alpha\beta-m\alpha+\alpha+\beta-1}$$
$$\cdot H_{1,2}^{1,1}\left[\frac{d\lambda_n^2 t^\alpha}{a}\middle|\begin{matrix}(-m-1,1)\\(0,1),(1-\alpha-\beta+\alpha\beta+m\alpha,\alpha)\end{matrix}\right]$$
$$+2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^n\frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)}\int_0^t u^{\alpha-m\alpha-1}H_{1,2}^{1,1}\left[\frac{d\lambda_n^2 u^\alpha}{a}\middle|\begin{matrix}(-m-1,1)\\(0,1),(1-\alpha+m\alpha,\alpha)\end{matrix}\right]f(t-u)du. \tag{3.17}$$

Example 3.2. Solve the differential equation (3.4) with initial condition

$$\begin{gathered}
I_t^{(1-\beta)(1-2\alpha)}u(r,0) = 0,\\
I_t^{(1-\beta)(1-\alpha)}u(r,0) = 0,\\
u(r,t) = 0 \quad \text{everywhere for } t \le 0,\\
u(r,t) = 0 \quad \text{for } r = 1,\ t > 0,\\
u(r,t) = \text{finite} \quad \text{at } r = 0,\ t > 0.
\end{gathered} \tag{3.18}$$

Solution 2. Taking Laplace and Hankel transform of (3.4), we get

$$\tilde{\tilde{u}}(r,s) = \frac{J_1(\lambda_n)}{\lambda_n}\frac{\tilde{f}(s)}{(s^{2\alpha}+as^\alpha+d\lambda_n^2)}, \tag{3.19}$$

on taking Inverse Laplace transform of equation (3.19), we get

$$\tilde{u}(r,t) = L^{-1}\left\{\tilde{f}(s)\frac{J_1(\lambda_n)}{\lambda_n}\right\}L^{-1}\left\{\left(\frac{1}{s^{2\alpha}+as^\alpha+d\lambda_n^2}\right)\right\}. \tag{3.20}$$

By using convolution theorem for Laplace transform and taking inverse Hankel transform, we get

$$u(r,t) = 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{a^{m+1}}\frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)}\int_0^t u^{\alpha-m\alpha-1}E_{\alpha,\alpha-2m\alpha}^m\left(-\frac{d\lambda_n^2 u^\alpha}{a}\right)f(t-u)du, \tag{3.21}$$

or

$$u(r,t) = 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{a^{m+1}}\frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)}\int_0^t u^{\alpha-m\alpha-1}\sum_{j=0}^{\infty}\frac{(j+m+1)!}{(j)!}\frac{\left(-d\lambda_n^2 u^\alpha/a\right)^j}{\Gamma(\alpha j+\alpha-m\alpha)}. \tag{3.22}$$

By using the relation (2.2)

$$u(r,t) = 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{a^{m+1}}\frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)}\int_0^t u^{\alpha-m\alpha-1}H_{1,2}^{1,1}\left[\frac{d\lambda_n^2 u^\alpha}{a}\middle|\begin{matrix}(-m-1,1)\\(0,1),(1-\alpha+m\alpha,\alpha)\end{matrix}\right]f(t-u)du, \tag{3.23}$$

or

$$u(r,t) = 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{a^{m+1}}\frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)}\int_0^t u^{\alpha-m\alpha-1}\frac{1}{\Gamma(m)}{}_1\Psi_1\left[\begin{matrix}(m,1);\\(\alpha-2m\alpha,\alpha);\end{matrix}\quad -\frac{d\lambda_n^2 u^\alpha}{a}\right], \tag{3.24}$$

which is the required solution.

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