Research Article

Analytical Solution for the Differential Equation Containing Generalized Fractional Derivative Operators and Mittag-Leffler-Type Function

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We discuss and derive the analytical solution for the fractional partial differential equation with generalized Riemann-Liouville fractional operator $D_{\alpha,\beta}^{\mu}$ of order $\alpha$ and $\beta$. Here, we derive the solution of the given differential equation with the help of Laplace and Hankel transform in terms of Fox’s $H$-function as well as in terms of Fox-Wright function $\psi$.

1. Introduction, Definition, and Preliminaries

Applications of fractional calculus require fractional derivatives of different kinds [1–9]. Differentiation and integration of fractional order are traditionally defined by the right-sided Riemann-Liouville fractional integral operator $I_{a+}^{\mu}$ and the left-sided Riemann-Liouville fractional integral operator $I_{a-}^{\mu}$, and the corresponding Riemann-Liouville fractional derivative operators $D_{a+}^{\mu}$ and $D_{a-}^{\mu}$, as follows [10, 11]:

$$\left( I_{a+}^{\mu}f \right)(x) = \frac{1}{\Gamma(\mu)} \int_{a}^{x} \frac{f(t)}{(x-t)^{1-\mu}} dt \quad (x > a; R(\mu) > 0), \quad (1.1)$$

$$\left( I_{a-}^{\mu}f \right)(x) = \frac{1}{\Gamma(\mu)} \int_{x}^{a} \frac{f(t)}{(t-x)^{1-\mu}} dt \quad (x < a; R(\mu) > 0), \quad (1.2)$$

$$\left( D_{a+}^{\mu}f \right)(x) = \left( \pm \frac{d}{dx} \right)^{n} \left( I_{a+}^{\mu-n}f \right)(x) \quad (R(\mu) \geq 0; n = [R(\mu)] + 1), \quad (1.3)$$
where the function \( f \) is locally integrable, \( R(\mu) \) denotes the real part of the complex number \( \mu \in \mathbb{C} \) and \( [R(\mu)] \) means the greatest integer in \( R(\mu) \).

Recently, a remarkable large family of generalized Riemann-Liouville fractional derivatives of order \( \alpha \) (\( 0 < \alpha < 1 \)) and type \( \beta \) (\( 0 \leq \beta \leq 1 \)) was introduced as follows [1–3, 5, 6, 8].

**Definition 1.1.** The right-sided fractional derivative \( D^a_{\alpha+} \) and the left-sided fractional derivative \( D^a_{\alpha-} \) of order \( \alpha \) (\( 0 < \alpha < 1 \)) and type \( \beta \) (\( 0 \leq \beta \leq 1 \)) with respect to \( x \) are defined by

\[
\left( D^a_{\alpha \pm} f \right)(x) = \left( \pm \int_{a \pm} f^{(1-\alpha)} \frac{d}{dx} f^{(1-\alpha)}(x) \right)(x),
\]

whenever the second number of (1.4) exists. This generalization (1.4) yields the classical Riemann-Liouville fractional derivative operator when \( \beta = 0 \). Moreover, for \( \beta = 1 \), it gives the fractional derivative operator introduced by Liouville [12] which is often attributed to Caputo now-a-days and which should more appropriately be referred to as the Liouville-Caputo fractional derivative. Several authors [7, 9] called the general operators in (1.4) the Hilfer fractional derivative operators. Applications of \( D^a_{\alpha \pm} \) are given [3].

Using the formulas (1.1) and (1.2) in conjunction with (1.3) when \( n = 1 \), the fractional derivative operator \( D^a_{\alpha \pm} \) can be written in the following form:

\[
\left( D^a_{\alpha \pm} f \right)(x) = \left( \pm \int_{a \pm} f^{(1-\alpha)} \left( D^a_{\alpha \pm - a\beta} f \right) \right)(x).
\]

The difference between fractional derivatives of different types becomes apparent from their Laplace transformations. For example, it is found for \( 0 < \alpha < 1 \) that [1, 2, 9]

\[
L\left( \left( D^a_{0+} f \right)(x) \right)(s) = s^{\alpha} L[f(x)](s) - s^{\beta(s-1)} \left( I_{0+} (1-\beta)(1-\alpha) f \right)(0+) \quad (0 < \alpha < 1),
\]

where \( (I_{0+}^{(1-\beta)(1-\alpha)} f)(0+) \) is the Riemann-Liouville fractional integral of order \( (1-\beta)(1-\alpha) \) evaluated in the limit as \( t \to 0+ \), it being understood (as usual) that [13],

\[
L[f(x)](s) := \int_0^\infty e^{-sx} f(x) dx := F(s),
\]

provided that the defining integral in (1.7) exists.

The familiar Mittag-Leffler functions \( E_\mu(z) \) and \( E_{\mu,v}(z) \) are defined by the following series:

\[
E_\mu(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + 1)} := E_{\mu,1}(z) \quad (z \in \mathbb{C}; R(\mu) > 0), \quad (1.8)
\]

\[
E_{\mu,v}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + v)} \quad (z, v \in \mathbb{C}; R(\mu) > 0), \quad (1.9)
\]
respectively. These functions are natural extensions of the exponential, hyperbolic, and trigonometric functions, since

\[ E_1(z) = e^z, E_2(z^2) = \cosh z, E_2(-z^2) = \cos z, \]

\[ E_{1,2}(z) = \frac{e^z - 1}{z}, E_{2,2}(z^2) = \frac{\sinh z}{z}. \] (1.10)

For a detailed account of the various properties, generalizations, and applications of the Mittag-Leffler functions, the reader may refer to the recent works by, for example, Gorenflo et al. [15–17]. The Mittag-Leffler function (1.1) and some of its various generalizations have only recently been calculated numerically in the whole complex plane [18, 19]. By means of the series representation, a generalization of the Mittag-Leffler function \( E_{\mu,\nu}(z) \) of (1.2) was introduced by Prabhakar [20] as follows:

\[ E_{\mu,\nu}^\lambda(z) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{\Gamma(\mu n + \nu)} \frac{z^n}{n!} \quad (z, \nu, \lambda \in \mathbb{C}; R(\mu) > 0), \] (1.11)

where \((\lambda)_n\) denotes the familiar Pochhammer symbol, defined (for \(\lambda, \nu \in \mathbb{C}\) and in terms of the familiar Gamma function) by

\[ (\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1), & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}). \end{cases} \] (1.12)

Clearly, we have the following special cases:

\[ E_{1,\nu}^\lambda(z) = E_{\mu,\nu}(z), \quad E_{1,1}^\lambda(z) = E_{\mu}(z). \] (1.13)

Indeed, as already observed earlier by Srivastava and Saxena [21], the generalized Mittag-Leffler function \( E_{\mu,\nu}^\lambda(z) \) itself is actually a very specialized case of a rather extensively investigated function \( p \Psi_q \) as indicated below [17]:

\[ E_{\mu,\nu}^\lambda(z) = \frac{1}{\Gamma(\lambda)} \Psi_1 \begin{bmatrix} (\lambda, 1); \\
(\nu, u); \end{bmatrix} \left( \frac{z}{\nu} \right). \] (1.14)

Here and in what follows, \( p \Psi_q \) denotes the Wright (or more appropriately, the Fox-Wright) generalized of the hypergeometric \( p \Gamma_q \) function, which is defined as follows [12]:

\[ p \Psi_q = \begin{bmatrix} (a_1, A_1), \ldots, (a_p, A_p); \\
(b_1, B_1), \ldots, (b_q, B_q); \end{bmatrix} z = \sum_{k=0}^{\infty} \Gamma(a_1 + A_1 k) \cdots \Gamma(a_p + A_p k) z^k \Gamma(b_1 + B_1 k) \cdots \Gamma(b_q + B_q k) k! \quad \left( R(A_j) > 0 \ (j = 1, \ldots, p); R(B_j) > 0 \ (j = 1, \ldots, q); 1 + R \left( \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \right) \geq 0 \right), \] (1.15)
in which we assumed in general that

\[ a_j, A_j \in C \quad (j = 1, \ldots, p), \quad b_j, B_j \in C \quad (j = 1, \ldots, q). \tag{1.17} \]

In application of Mittag-Leffler function, it is useful to have the following Laplace inverse transform formula:

\[
L^{-1} \left\{ \frac{S^{\gamma - \beta}}{(S + A)^{k+1}} \right\} = \frac{1}{k!} t^{k+\beta-1} E_{1,\beta}^k(-A t), \tag{1.18}
\]

where \( E_{1,\beta}^k(z) = (d^k/dz^k)E_{1,\beta}(z) \).

2. Fox’s \( H \)-function

The Fox function, also referred as the Fox’s \( H \)-function, generalizes the Mellin-Barnes function. The importance of the Fox function lies in the fact that it includes nearly all special functions occurring in applied mathematics and statistics as special cases. Fox \( H \)-function is defined as

\[
H_{p,q}^{1,p} \left[ -x \begin{matrix} (1-a_1, A_1), \ldots, (1-a_p, A_p) \\ (0,1), (1-b_1, B_1), \ldots, (1-b_q, B_q) \end{matrix} \right] = \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + A_1 k) \cdots \Gamma(a_p + A_p k)}{k! \Gamma(b_1 + B_1 k) \cdots \Gamma(b_q + B_q k)} x^k. \tag{2.1}
\]

We need this relation

\[
E_{\alpha,\beta}^k(x) = \sum_{n=0}^{\infty} \frac{n! x^{n-k}}{(n-k)! \Gamma(an+\beta)} = \sum_{j=0}^{\infty} \frac{\Gamma(j+k+1) x^j}{j! \Gamma(aj+ak+\beta)}. \tag{2.2}
\]

3. Finite Hankel Transform

If \( f(r) \) satisfies Dirichlet conditions in closed interval \((0, a)\) and if its finite Hankel transform is defined to be

\[
H[f(r)] = \tilde{f}(\lambda_n) = \int_0^a f(r) J_0(r \lambda_n) dr, \tag{3.1}
\]

where \( \lambda_n \) are the roots of the equation \( J_0(r) = 0 \). Then at each point of the interval at which \( f(r) \) is continuous:

\[
f(r) = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{f(\lambda_n) J_0(\lambda_n r)}{J_1^2(\lambda_n a)}, \tag{3.2}
\]
where the sum is taken over all positive roots of \( J_0(r) = 0 \), \( J_0 \) and \( J_1 \) are Bessel functions of first kind.

In application of the finite Hankel transform to physical problems, it is useful to have the following formula [23]

\[
H \left[ \frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} \right] = -\lambda_n^2 f(r) + a\lambda_n f(a) J_1(\lambda_n a). \tag{3.3}
\]

**Example 3.1.** Solve the differential equation

\[
D_{0,\frac{\alpha}{2}}^{2\alpha,\beta} u(r, t) + aD_{0,\frac{\beta}{2}}^{2\beta,\alpha} u(r, t) = d \left( \frac{\partial^2 u(r, t)}{\partial r^2} + \frac{1}{r} u(r, t) \right) + f(t), \tag{3.4}
\]

where \( 0 < \alpha \leq 1/2 \) and \( 0 \leq \beta \leq 1 \) with initial condition

\[
I_{t}^{(1-\beta)(1-2\alpha)} u(r, 0) = \phi_1(r),
\]

\[
I_{t}^{(1-\beta)(1-\alpha)} u(r, 0) = \phi_2(r),
\]

\[
u(r, t) = 0 \quad \text{everywhere for } t < 0,
\]

\[
u(r, t) = 0 \quad \text{for } r = 1, \ t > 0,
\]

\[
u(r, t) = \text{finite} \quad \text{at } r = 0, \ t > 0.
\]

**Solution 1.** Taking Laplace transform of (3.4), we get

\[
s^{2\alpha} \tilde{u}(r, s) - s^{\beta(2\alpha-1)} \tilde{\phi}_1(r) + as^\alpha \tilde{u}(r, s) - as^{\beta(\alpha-1)} \tilde{\phi}_2(r) = d \left[ \frac{\partial^2 \tilde{u}(r, s)}{\partial r^2} + \frac{1}{r} \tilde{u}(r, s) \right] + \tilde{f}(s). \tag{3.6}
\]

Taking Hankel transform on both side of the above equation, we get

\[
s^{2\alpha} \tilde{u}(r, s) - s^{\beta(2\alpha-1)} \tilde{\phi}_1(r) + as^\alpha \tilde{u}(r, s) - as^{\beta(\alpha-1)} \tilde{\phi}_2(r) = d \left[ -\lambda_n^2 \tilde{u}(r, s) \right] + \tilde{f}(s) \frac{I_1(\lambda_n)}{\lambda_n}, \tag{3.7}
\]
then we get

\[ \tilde{u}(r, s) = \frac{s^{\beta(2a-1)}\tilde{\phi}_1(r)}{(s^{2\alpha} + as^\alpha + d\lambda_n^2)} + \frac{as^{\beta(a-1)}\tilde{\phi}_2(r)}{(s^{2\alpha} + as^\alpha + d\lambda_n^2)} + \frac{f(s)}{(s^{2\alpha} + as^\alpha + d\lambda_n^2)} \frac{J_1(\lambda_n)}{\lambda_n}, \]  
(3.8)

\[ \tilde{u}(r, s) = \tilde{G}_1\tilde{\phi}_1(r) + a\tilde{G}_2\tilde{\phi}_2(r) + \tilde{G}_3\tilde{f}(s) \frac{J_1(\lambda_n)}{\lambda_n}, \]  
(3.9)

where

\[ \tilde{G}_1 = \frac{s^{\beta(2a-1)}}{(s^{2\alpha} + as^\alpha + d\lambda_n^2)}, \]  
(3.10)

\[ \tilde{G}_2 = \frac{s^{\beta(a-1)}}{(s^{2\alpha} + as^\alpha + d\lambda_n^2)}, \]  
(3.11)

\[ \tilde{G}_3 = \frac{1}{(s^{2\alpha} + as^\alpha + d\lambda_n^2)}. \]  
(3.12)

On taking Laplace inverse of (3.10), (3.11), and (3.12), respectively,

\[ L^{-1}\left\{ \frac{s^{\beta(2a-1)}}{(s^{2\alpha} + as^\alpha + d\lambda_n^2)} \right\} = \sum_{m=0}^{\infty} \frac{(-1)^m t^{\alpha+2a\beta-m-1}}{m!} E_{\alpha+\beta-m-2a}^m \left( \frac{-d\lambda_n^2}{a} t^\alpha \right), \]  
(3.13)

\[ L^{-1}\left\{ \frac{s^{\beta(a-1)}}{(s^{2\alpha} + as^\alpha + d\lambda_n^2)} \right\} = \sum_{m=0}^{\infty} \frac{(-1)^m t^{\alpha-a\beta-m-1}}{m!} E_{\alpha+\beta-m-2a}^m \left( \frac{-d\lambda_n^2}{a} t^\alpha \right), \]  
(3.14)

\[ L^{-1}\left\{ \frac{1}{(s^{2\alpha} + as^\alpha + d\lambda_n^2)} \right\} = \sum_{m=0}^{\infty} \frac{(-1)^m t^{\alpha-ma-1}}{m!} E_{\alpha-ma}^m \left( \frac{-d\lambda_n^2}{a} t^\alpha \right). \]  
(3.15)

After taking Inverse Laplace and Hankel transform of (3.9) put the value (3.13) through (3.15) in (3.9), we get

\[ u(r, t) = 2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{m+1} \lambda_n^{m+1}} J_0(\lambda_n r) \phi_1(r) t^{-2a\beta-ma+\alpha+\beta-1} \sum_{j=0}^{\infty} \frac{(j + m + 1)!(d\lambda_n^2 t^\alpha/a)^j}{j! \Gamma(j + \alpha + \beta - 2a\beta - 2ma)}, \]  
(3.16)

\[ + 2a \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{m+1} \lambda_n^{m+1}} J_0(\lambda_n r) \phi_2(r) t^{-a\beta-ma+\alpha+\beta-1} \sum_{j=0}^{\infty} \frac{(j + m + 1)!(d\lambda_n^2 t^\alpha/a)^j}{j! \Gamma(j + \alpha + \beta - \alpha\beta + ma)}, \]  
(3.17)

\[ + 2a \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{m+1} \lambda_n^{m+1}} \frac{J_0(\lambda_n r)}{\lambda_n} J_1(\lambda_n) \int_0^t u^{-a-ma-1} \sum_{j=0}^{\infty} \frac{(j + m + 1)!(d\lambda_n^2 u^\alpha/a)^j}{j! \Gamma(j + \alpha - ma)} f(t-u)du. \]  
(3.18)
\[
\begin{align*}
    u(r, t) &= 2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{m+1}} \frac{J_0(\lambda_n r)}{f_1^2(\lambda_n)} \phi_1(r) t^{-2\alpha \beta + \alpha + \beta - 1} \\
    \cdot H_{1,2}^1 \left[ \frac{d\lambda_n^2 \mu^a}{a} \right] (0,1), (1 - \alpha - \beta + 2 \alpha \beta + 2 m \alpha, \alpha) \\
    + 2 a \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{m+1}} \frac{J_0(\lambda_n r)}{f_1^2(\lambda_n)} \phi_2(r) t^{-2\alpha \beta + \alpha + \beta - 1} \\
    \cdot H_{1,2}^1 \left[ \frac{d\lambda_n^2 \mu^a}{a} \right] (0,1), (1 - \alpha - \beta + \alpha \beta + m \alpha, \alpha) \\
    + 2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{J_0(\lambda_n r)}{\lambda_n f_1(\lambda_n)} \int_0^t u^{\alpha - m - 1} u^{2 m - 1} H_{1,2}^1 \left[ \frac{d\lambda_n^2 \mu^a}{a} \right] (0,1), (1 - \alpha - m \alpha, \alpha) f(t-u) \, du. \\
\end{align*}
\]}

(3.17)

**Example 3.2.** Solve the differential equation (3.4) with initial condition

\[
\begin{align*}
    i^{(1-\beta)(1-2\alpha)} u(r, 0) &= 0, \\
    i^{(1-\beta)(1-\alpha)} u(r, 0) &= 0, \\
    u(r,t) &= 0 \quad \text{everywhere for } t \leq 0, \\
    u(r,t) &= 0 \quad \text{for } r = 1, \ t > 0, \\
    u(r,t) &= \text{finite at } r = 0, t > 0. \\
\end{align*}
\]

(3.18)

**Solution 2.** Taking Laplace and Hankel transform of (3.4), we get

\[
\tilde{u}(r,s) = \frac{f_1(\lambda_n)}{\lambda_n} \frac{\tilde{f}(s)}{s^{2\alpha} + a s^\alpha + d \lambda_n^2}.
\]

(3.19)

on taking Inverse Laplace transform of equation (3.19), we get

\[
\tilde{u}(r,t) = L^{-1} \left\{ \tilde{f}(s) \frac{f_1(\lambda_n)}{\lambda_n} \right\} L^{-1} \left\{ \frac{1}{s^{2\alpha} + a s^\alpha + d \lambda_n^2} \right\}.
\]

(3.20)

By using convolution theorem for Laplace transform and taking inverse Hankel transform, we get

\[
\begin{align*}
    u(r,t) &= 2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{J_0(\lambda_n r)}{\lambda_n f_1(\lambda_n)} \int_0^t u^{\alpha - m - 1} u^{2 m - 1} E_{a,2^m a - 2 m a} \left( \frac{-d \lambda_n^2 \mu^a}{a} \right) f(t-u) \, du, \\
\end{align*}
\]}

(3.21)
or

\[
\begin{align*}
 u(r, t) &= 2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{m+1}} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)} \int_0^t u^{a-ma-1} \sum_{j=0}^{\infty} \frac{(j+m+1)!}{(j)!} \frac{(-d\lambda_n^2 u^a / a)^j}{\Gamma(a+j+a-ma)}. \\
&= 2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{m+1}} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)} \int_0^t u^{a-ma-1} \frac{d\lambda_n^2 u^a}{a} \left[ \psi_a(\alpha-2ma, \alpha); \psi_a(\alpha-1, 1) \right] f(t-u) du.
\end{align*}
\]

By using the relation (2.2)

\[
\begin{align*}
 u(r, t) &= 2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{m+1}} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)} \int_0^t u^{a-ma-1} H_{1,2}^{1,1} \left[ \frac{d\lambda_n^2 u^a}{a} \right] (0,1), (1-a+ma, a) \right] f(t-u) du, \\
&= 2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{m+1}} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)} \int_0^t u^{a-ma-1} \frac{1}{\Gamma(m)} \left[ \psi_a(\alpha-2ma, \alpha); \psi_a(\alpha-1, 1) \right] f(t-u) du.
\end{align*}
\]

which is the required solution.

References


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