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Research Article

Common Fixed Points Approximation for Asymptotically Nonexpansive Semi group in Banach Spaces

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Let E be a real Banach space satisfying local uniform Opial's condition, whose duality map is weakly sequentially continuous. Let $\mathcal{J} := \{T(t)t \geq 0\}$ be a uniformly asymptotically regular family of asymptotically nonexpansive semigroup of E with function $k : [0,\infty) \to [0,\infty)$. Let $\mathcal{F} := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$ and $f : E \to E$ be weakly contractive map. Let $G : E \to E$ be δ-strongly accretive and λ-strictly pseudocontractive map with $\delta + \lambda > 1$. Let $\{t_n\}$ be an increasing sequence in $[0,\infty)$and let$\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ satisfying some conditions. For some positive real number γ appropriately chosen, let $\{x_n\}$ be a sequence defined by $x_0 \in E$, $x_{n+1} = \beta_n x_n + (1-\beta_n)y_n$, $y_n = (I - \alpha_n G)T(t_n)x_n + \alpha_n \gamma f(x_n)$, $n \geq 0$. It is proved that $\{x_n\}$ converges strongly to a common fixed point q of the family $\mathcal{J}$ which is also the unique solution of the variational inequality $\langle (G-\gamma f)q, j(q-x)\rangle \geq 0$, for all $x \in \mathcal{F}$.

1. Introduction

Let E be a real Banach space and let E^* be the dual space of E. A mapping $\varphi : [0,\infty) \to [0,\infty)$ is called a gauge function if it is strictly increasing, continuous and $\varphi(0) = 0$. Let φ be a gauge function, a generalized duality mapping with respect to φ, $J_\varphi : E \to 2^{E^*}$ is defined by, $x \in E$,

$$J_\varphi x = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|),\ \|x^*\| = \varphi(\|x\|)\right\}, \tag{1.1}$$

where $\langle \cdot,\cdot\rangle$ denotes the duality pairing between element of E and that of E^*. If $\varphi(t) = t$, then J_φ is simply called the normalized duality mapping and is denoted by J. For any $x \in E$, an element of $J_\varphi x$ is denoted by $j_\varphi(x)$.

The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \epsilon = \|x-y\|\right\}, \tag{1.2}$$

and E is called *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$. A Banach space E is said to satisfy *Opial's condition* [1] if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ as $n \to \infty$ implies that

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|, \quad \forall y \in E,\ y \neq x. \tag{1.3}$$

All Hilbert spaces and l_p spaces, $1 \le p < \infty$ satisfy Opial's condition. However L_p, $p \neq 2$ do not satisfy this condition; see, for example, [2]. The space E is said to have weakly (sequentially) continuous duality map if there exists a gauge function φ such that J_φ is singled valued and (sequentially) continuous from E with weak topology to E^* with weak* topology. It is known that every Banach space with weakly sequentially continuous duality mapping satisfies Opial's condition (see [3]). Every l_p space, $(1 < p < \infty)$ has a weakly sequentially continuous duality map.

The space E is said to have *uniform Opial's condition* [4] if for each $c > 0$, there exists an $r > 0$ such that

$$1 + r \le \liminf_{n\to\infty}\|x + x_n\| \tag{1.4}$$

for each $x \in E$ with $\|x\| \ge c$ and each sequence $\{x_n\}$ satisfying $x_n \rightharpoonup 0$ as $n \to \infty$, and $\liminf_{n\to\infty}\|x_n\| \ge 1$.

E is said to satisfy the local uniform Opial's condition [5] if, for any weak null sequence $\{x_n\}$ in E with $\liminf_{n\to\infty}\|x_n\| \ge 1$ and any $c > 0$, there exists $r > 0$ such that

$$1 + r \le \liminf_{n\to\infty}\|x + x_n\| \tag{1.5}$$

for all $x \in E$ with $\|x\| \ge c$. Observe that uniform Opial's condition implies local uniform Opial's condition which in turn implies Opial's condition.

A self-mapping $T : E \to E$ is said to be *contraction* if $\|Tx - Ty\| \le \alpha\|x - y\|$, for all $x, y \in E$, where $\alpha \in [0,1)$ is a fixed constant. It is said to be weakly contractive if there exists a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ satisfying $\psi(t) = 0$ if and only if $t = 0$ and $\|Tx - Ty\| \le \|x - y\| - \psi(\|x - y\|)$, for all $x, y \in E$. It is known that the class of weakly contractive maps contain properly the class of contractive ones; see [6, 7]. The map T is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in E$ and *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} k_n = 0$ such that $\|T^n x - T^n y\| \le (1 + k_n)\|x - y\|$, for all $x, y \in E$ and $n \in \mathbb{N}$. The set of fixed point of T is defined as $F(T) := \{x \in E : Tx = x\}$.

A one parameter family $\mathcal{J} = \{T(t) : t \ge 0\}$ of self-mapping of E is called *nonexpansive semigroup* if the following conditions are satisfied:

(i) $T(0)x = x$ for all $x \in E$;

(ii) $T(t + s) = T(t) \circ T(s)$ for all $t, s \ge 0$;

(iii) for each $x \in E$, the mapping $t \to T(t)x$ is continuous;

(iv) for $x, y \in E$ and $t \ge 0$, $\|T(t)x - T(t)y\| \le \|x - y\|$.

The family $\mathcal{J}$ is said to be asymptotically nonexpansive semigroup if conditions (i)–(iii) are satisfied and, in addition, there exists a function $k : [0,\infty) \to [0,\infty)$ satisfying $\lim_{t\to\infty} k(t) = 0$ and $\|T(t)x - T(t)y\| \le (1 + k(t))\|x - y\|$ for all $x, y \in E$.

The family $\mathcal{J} = \{T(t) : t \geq 0\}$ is said to be asymptotically regular if

$$\lim_{s \to \infty} \|T(t+s)x - T(s)x\| = 0, \tag{1.6}$$

for all $t \in (0, \infty)$ and $x \in K$. It is said to be uniformly asymptotically regular if, for any $t \geq 0$ and for any bounded subset C of K,

$$\lim_{s \to \infty} \sup_{x \in C} \|T(t+s)x - T(s)x\| = 0. \tag{1.7}$$

For some positive real numbers δ and λ, the mapping $G : E \to E$ is said to be δ-strongly accretive if for any $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Gx - Gy, j(x-y) \rangle \geq \delta \|x-y\|^2 \tag{1.8}$$

and it is called λ-strictly pseudocontractive if

$$\langle Gx - Gy, j(x-y) \rangle \leq \|x-y\|^2 - \lambda \|(I-G)x - (I-G)y\|^2. \tag{1.9}$$

Let C be a nonempty closed convex subset of E and $T : E \to E$ be a map. Then, a variational inequality problem with respect to C and T is find $x^* \in C$ such that

$$\langle Tx^*, j(y - x^*) \rangle \geq 0, \quad \forall y \in C,\ j(y - x^*) \in J(y - x^*). \tag{1.10}$$

The problem of solving a variational inequality of the form (1.10) has been intensively studied by numerous authors due to its various applications in several physical problems, such as in operations research, economics, and engineering design; see, for example, [8–10] and the references therein. Iterative methods for approximating fixed points of nonexpansive mappings, nonexpansive semigroups, and their generalizations which solves some variational inequalities problems have been studied by a number of authors (see, e.g., [11–17] and the references therein).

A typical problem is to minimize a quadratic function over the set of the fixed points of some nonexpansive mapping in a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle. \tag{1.11}$$

Here, C is the fixed point set of a nonexpansive mapping T of H, b is a point in H, and A is some bounded, linear, and strongly positive operator on H, where a map $A : H \to H$ is said to be strongly positive if there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \text{for all } x \in H. \tag{1.12}$$

For a strongly positive bonded linear operator A and any $x, y \in H$, we have

$$\langle Ax - Ay, x - y \rangle \geq \overline{\gamma} \|x - y\|^2. \tag{1.13}$$

This implies that A is $\overline{\gamma}$-strongly accretive (or in particular $\overline{\gamma}$-strongly monotone). On the other hand, by simple calculation, the following relation also holds:

$$\langle Ax - Ay, x - y \rangle \leq \frac{\left(1 + \|A\|^2\right)}{2} \|x - y\|^2 - \frac{1}{2} \|(I - A)x - (I - A)y\|^2. \tag{1.14}$$

This implies that $A/\|A\|$ is $1/2$-strictly pseudocontractive.

Let H be a real Hilbert space. In 2003, Xu [18] proved that the sequence $\{x_n\}$ defined by $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b, \quad n \geq 0, \tag{1.15}$$

converges strongly to the unique solution of the minimization problem (1.11) provided that the sequence $\{\alpha_n\}$ satisfies certain control conditions.

In 2000, Moudafi [12] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on H. Starting with an arbitrary initial point $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n), \quad n \geq 0, \tag{1.16}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. It was proved in [12] that, under certain appropriate conditions impose on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.16) converges strongly to the unique solution $x^* \in C$ of the variational inequality:

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad x \in C. \tag{1.17}$$

For a strongly positive linear bounded map A on H with coefficient γ, Marino and Xu [11] combined the iterative method (1.15) with the viscosity approximation method (1.16) and studied the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \tag{1.18}$$

They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.18) converges strongly to the unique solution of the variational inequality:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in C, \tag{1.19}$$

which is also the optimality condition for the minimization problem $\min_{x \in C}(1/2)\langle Ax, x \rangle - h\langle x \rangle$, where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$, for $x \in H$).

Yao et al. [19] proved that the iterative scheme defined by

$$\begin{aligned} x_0 &\in H, \\ x_{n+1} &= \beta_n x_n + (1-\beta_n) y_n, \\ y_n &= (1-\alpha_n A) T x_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \end{aligned} \tag{1.20}$$

where $\{\beta_n\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$ satisfying some control conditions, converges to a fixed point of a nonexpansive mapping T which solves the variational inequality (1.19).

Acedo and Suzuki [20], recently, proved the strong convergence of the Browder's implicit scheme, $x_0, u \in C$,

$$x_n = \alpha_n u + (1-\alpha_n) T(t_n) x_n, \quad n \geq 0, \tag{1.21}$$

to a common fixed point of a uniformly asymptotically regular family $\{T(t) : t \geq 0\}$ of nonexpansive semigroup in the framework of a real Hilbert space.

Let S be a semigroup and let $B(S)$ be the subspace of all bounded real valued functions defined on S with supremum norm. For each $s \in S$, the left translator operator $l(s)$ on $B(S)$ is defined by $(l(s)f)(t) = f(st)$ for each $t \in S$ and $f \in B(S)$. Let X be a subspace of $B(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. Let X be l_s invariant; that is, $l_s(X) \subset X$ for each $s \in S$. A mean μ on X is said to be left invariant if $\mu(l_s f) = \mu(f)$ for each $s \in S$ and $f \in X$.

Recently, Saeidi and Naseri [14] studied the problem of approximating common fixed point of a family of nonexpansive semigroup and solution of some variational inequality problem and proved the following theorem.

Theorem 1.1 (Saeidi and Naseri [14]). *Let $\mathcal{J} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on a real Hilbert space H such that $F(\mathcal{J}) \neq \emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \to \langle T(t)x, y\rangle$ is an element of X for each $x, y \in H$. Let $f : E \to E$ be a contraction with constant α and let $G : H \to H$ be strongly positive map with constant $\overline{\gamma} > 0$. Let $\{\mu_n\}$ be a left regular sequence of means on X and let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that* (i) $\lim \alpha_n = 0$ *and* (ii) $\sum \alpha_n = \infty$. *Let $\gamma \in (0, \overline{\gamma}/\alpha)$ and $\{x_n\}$ be a sequence generated by $x_0 \in H$*

$$x_{n+1} = (I - \alpha_n G) T(\mu_n) x_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \tag{1.22}$$

Then, $\{x_n\}$ converges strongly to a common fixed point of the family $\mathcal{J}$ which is the unique solution of the variational inequality $\langle (G - \gamma f)x^*, x - x^*\rangle \geq 0$ for all $x \in F(\mathcal{J})$. Equivalently one has $P_{F(\mathcal{J})}(I - G + \gamma f)x^* = x^*$.

More recently, as commented by Golkarmanesh and Naseri [21], Piri and Vaezi [13] gave a minor variation of Theorem 1.1 as follows.

Theorem 1.2 (Piri and Vaezi [13]). *Let $\mathcal{J} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on a real Hilbert space H such that $F(\mathcal{J}) \neq \emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \to \langle T(t)x, y\rangle$ is an element of X for each $x, y \in H$. Let $f : E \to E$ be a contraction and let $G : H \to H$ be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Let $\{\mu_n\}$*

be a left regular sequence of means on X *and let* $\{\alpha_n\}$ *be a sequence in* $(0,1)$ *such that* (i) $\lim \alpha_n = 0$ *and* (ii) $\sum \alpha_n = \infty$. *Let* $\{x_n\}$ *be generated by* $x_0 \in H$:

$$x_{n+1} = (I - \alpha_n G)T(\mu_n)x_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.23}$$

where $0 < \gamma < (1 - \sqrt{(1-\delta)/\lambda})/\alpha$. *Then,* $\{x_n\}$ *converges strongly to a common fixed point of the family* $\mathcal{J}$ *which is the unique solution of the variational inequality* $\langle (G-\gamma f)x^*, x-x^* \rangle \geq 0$ *for all* $x \in F(\mathcal{J})$. *Equivalently one has* $P_{F(\mathcal{J})}(I - G + \gamma f)x^* = x^*$.

Motivated by these results, it is our purpose in this paper to continue the study of this problem and prove new strong convergence theorem for common fixed point of family of asymptotically nonexpansive semigroup and solution of some variational inequality problem in the framework of a real Banach space much more general than Hilbert. Our theorem, proved for more general classes of maps, is applicable in l_p spaces, $1 < p < \infty$.

2. Preliminaries

In the sequel, we will make use of the following lemmas.

Lemma 2.1. *Let* E *be a real normed linear space. Then, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in E,\ j(x+y) \in J(x+y). \tag{2.1}$$

Lemma 2.2 (Suzuki [22]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. *Suppose that* $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ *for all integers* $n \geq 1$ *and* $\limsup(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim \|y_n - x_n\| = 0$.

Lemma 2.3 (Kim et al. [23]). *Let* E *be a real Banach space satisfying the local uniform Opial's condition and* C *a nonempty weakly compact convex subset of* E. *If* $\{T(t) : t \geq 0\}$ *is asymptotically nonexpansive semigroup on* C, *then* $(I - T(t))$ *is demiclosed at zero.*

Lemma 2.4 (Acedo and Suzuki, [20]). *Let* C *be a set of a separated topological vector space* E. *Let* $\{T(t) : t \geq 0\}$ *be a family of mappings on* C *such that* $T(s) \circ T(t) = T(s+t)$ *for all* $s, t \in [0,\infty)$. *Assume that* $\{T(t) : t \geq 0\}$ *is asymptotically regular, then* $F(T(t)) = \bigcap_{s\geq 0} F(T(s))$ *holds for all* $t \in (0,\infty)$.

Lemma 2.5 (Xu [24]). *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 0, \tag{2.2}$$

where, (i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$, (ii) $\limsup \sigma_n \leq 0$, (iii) $\gamma_n \geq 0$ *and* $\sum \gamma_n < \infty$. *Then,* $a_n \to 0$ *as* $n \to \infty$.

Let E be a real Banach space and δ, λ, and τ positive real numbers satisfying $\delta + \lambda > 1$ and $\tau \in (0,1)$. Let $G : E \to E$ be a δ-strongly accretive and λ-strictly pseudocontractive then, as shown in [13], $(I - G)$ and $(I - \tau G)$ are strict contractions. In fact, for $x, y \in E$,

$$\begin{aligned} \lambda \|(I-G)x - (I-G)y\|^2 &\le \|x-y\|^2 - \langle Gx - Gy, j(x-y)\rangle \\ &\le (1-\delta)\|x-y\|^2, \end{aligned} \tag{2.3}$$

which implies

$$\|(I-G)x - (I-G)y\| \le \sqrt{\frac{1-\delta}{\lambda}}\|x-y\|. \tag{2.4}$$

Also, for $\tau \in (0,1)$,

$$\begin{aligned} \|(I-\tau G)x - (I-\tau G)y\| &\le \|(1-\tau)(x-y) + \tau((I-G)x - (I-G)y)\| \\ &\le (1-\tau)\|(x-y)\| + \tau\sqrt{\frac{1-\delta}{\lambda}}\|x-y\| \\ &= 1 - \tau\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\|(x-y)\|. \end{aligned} \tag{2.5}$$

3. Main Results

Theorem 3.1. *Let E be a real Banach space with local uniform Opial's property whose duality mapping is sequentially continuous. Let $\mathcal{J} = \{T(t) : t \ge 0\}$ be uniformly asymptotically regular family of asymptotically nonexpansive semigroup of E, with function $k : [0,\infty) \to [0,\infty)$ and $\mathcal{F} := F(\mathcal{J}) = \bigcap_{t\ge 0} F(T(t)) \ne \emptyset$. Let $f : E \to E$ be weakly contractive and let $G : E \to E$ be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Let $\eta := (1 - \sqrt{(1-\delta)/\lambda})$ and $\gamma \in (0, \min\{\eta/2, \delta\})$. Let $\{\beta_n\}$ and $\{\alpha_n\}$ be sequences in $[0,1]$ and let $\{t_n\}$ be an increasing sequence in $[0,\infty)$ satisfying the following conditions:*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \lim_{n\to\infty} \frac{k(t_n)}{\alpha_n} = 0, \qquad \sum \alpha_n = \infty, \qquad 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1. \tag{3.1}$$

Define a sequence $\{x_n\}$ by $x_0 \in E$,

$$\begin{aligned} x_{n+1} &= \beta_n x_n + (1-\beta_n) y_n, \\ y_n &= (I - \alpha_n G)T(t_n)x_n + \alpha_n \gamma f(x_n), \quad n \ge 0. \end{aligned} \tag{3.2}$$

Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\mathcal{J}$ which solves the variational inequality:

$$\langle (G - \gamma f)q, j(x-q)\rangle \ge 0, \quad \forall x \in \mathcal{F}. \tag{3.3}$$

Proof. We start by showing that the solution of the variational inequality (3.3) in $\mathcal{F}$ is unique. Assume $q, p \in \mathcal{F}$ are solutions of the variational inequality (3.3), then

$$\langle (G-\gamma f)p, j(q-p)\rangle \geq 0, \qquad \langle (G-\gamma f)q, j(p-q)\rangle \geq 0. \tag{3.4}$$

Adding these two relations, we get

$$\langle (G-\gamma f)p-(G-\gamma f)q, j(p-q)\rangle \leq 0. \tag{3.5}$$

Therefore,

$$\begin{aligned}
0 &\geq \langle (G-\gamma f)p-(G-\gamma f)q, j(p-q)\rangle \\
&= \langle Gp-Gq, j(p-q)\rangle - \gamma\langle f(p)-f(q), j(p-q)\rangle \\
&\geq \delta\|p-q\|^2 - \gamma\|f(p)-f(q)\|\|q-p\| \\
&\geq \delta\|p-q\|^2 + \gamma\psi(\|p-q\|)\|q-p\| - \gamma\|q-p\|^2 \\
&= (\delta-\gamma)\|q-p\|^2 + \gamma\psi(\|p-q\|)\|q-p\|.
\end{aligned} \tag{3.6}$$

Since $\delta > \gamma$, we obtain that $p = q$ and so the solution is unique in $\mathcal{F}$.

Now, let $q \in \mathcal{F}$, since $(1-\alpha_n\gamma)(k(t_n)/\alpha_n) \to 0$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $(1-\alpha_n\gamma)(k(t_n)/\alpha_n) < (\eta-\gamma)/2$, for all $n \geq n_0$. Hence, for $n \geq n_0$, we have the following:

$$\begin{aligned}
\|y_n - q\| &= \|(I-\alpha_n G)T(t_n)x_n - (I-\alpha_n G)q + \alpha_n\gamma f(x_n) - \alpha_n\gamma f(q) + \alpha_n\gamma f(q) - \alpha_n G(q)\| \\
&\leq (1-\alpha_n\eta)(1+k(t_n))\|x_n - q\| + \alpha_n\gamma\|f(x_n)-f(q)\| + \alpha_n\|\gamma f(q) - G(q)\| \\
&\leq [1-\alpha_n(\eta-\gamma) + (1-\alpha_n\eta)k(t_n)]\|x_n - q\| + \alpha_n\|\gamma f(q) - G(q)\|,
\end{aligned} \tag{3.7}$$

so that

$$\begin{aligned}
\|x_{n+1} - q\| &\leq \beta_n\|x_n - q\| + (1-\beta_n)\|y_n - q\| \\
&\leq [\beta_n + (1-\beta_n)[1-\alpha_n(\eta-\gamma) + (1-\alpha_n\eta)k(t_n)]]\|x_n - q\| \\
&\quad + \alpha_n(1-\beta_n)\|\gamma f(q) - G(q)\| \\
&= \left[1-\alpha_n(1-\beta_n)\left((\eta-\gamma) - (1-\alpha_n\gamma)\frac{k(t_n)}{\alpha_n}\right)\right]\|x_n - q\| \\
&\quad + \alpha_n(1-\beta_n)\|\gamma f(q) - G(q)\| \\
&\leq \left[1-\alpha_n(1-\beta_n)\left((\eta-\gamma) - (1-\alpha_n\gamma)\frac{k(t_n)}{\alpha_n}\right)\right]\|x_n - q\| \\
&\quad + \alpha_n(1-\beta_n)\left((\eta-\gamma) - (1-\alpha_n\gamma)\frac{k(t_n)}{\alpha_n}\right)\frac{2\|\gamma f(q) - G(q)\|}{(\eta-\gamma)} \\
&\leq \max\left\{\|x_n - q\|, \frac{2\|\gamma f(q) - G(q)\|}{\eta-\gamma}\right\}.
\end{aligned} \tag{3.8}$$

By induction, we have

$$\|x_n - q\| \le \max\left\{\|x_{n_0} - q\|, \frac{2\|\gamma f(q) - G(q)\|}{\eta - \gamma}\right\}. \tag{3.9}$$

Thus, $\{x_n\}$ is bounded and so are $\{T(t_n)x_n\}$, $\{GT(t_n)x_n\}$, $\{y_n\}$, and $\{f(x_n)\}$. Observe that

$$\begin{aligned} y_{n+1} - y_n = {} & ((I - \alpha_{n+1}G)T(t_{n+1})x_{n+1} - (I - \alpha_{n+1}G)T(t_{n+1})x_n) \\ & + ((I - \alpha_{n+1}G)T(t_{n+1})x_n - (I - \alpha_n G)T(t_{n+1})x_n) \\ & + ((I - \alpha_n G)T(t_{n+1})x_n - (I - \alpha_n G)T(t_n)x_n) \\ & + (\alpha_{n+1}\gamma f(x_{n+1}) - \alpha_{n+1}\gamma f(x_n)) + (\alpha_{n+1}\gamma f(x_n) - \alpha_n\gamma f(x_n)), \end{aligned} \tag{3.10}$$

so that

$$\begin{aligned} \|y_{n+1} - y_n\| \le {} & (1 - \alpha_{n+1}\eta)(1 + k(t_{n+1}))\|x_{n+1} - x_n\| + |\alpha_n - \alpha_{n+1}|\|GT(t_{n+1})x_n\| \\ & + (1 - \alpha_n\eta)\|T(t_{n+1})x_n - T(t_n)x_n\| + \alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| \\ & + |\alpha_{n+1} - \alpha_n|\gamma\|f(x_n)\| \\ = {} & (1 - \alpha_{n+1}\eta)(1 + k(t_{n+1}))\|x_{n+1} - x_n\| + |\alpha_n - \alpha_{n+1}|\|GT(t_{n+1})x_n\| \\ & + (1 - \alpha_n\eta)\|T((t_{n+1} - t_n) + t_n)x_n - T(t_n)x_n\| \\ & + \alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| + |\alpha_{n+1} - \alpha_n|\gamma\|f(x_n)\| \\ \le {} & (1 - \alpha_{n+1}\eta)(1 + k(t_{n+1}))\|x_{n+1} - x_n\| + |\alpha_n - \alpha_{n+1}|\|GT(t_{n+1})x_n\| \\ & + (1 - \alpha_n\eta)\sup_{z\in\{x_n\}, s\in\mathbb{R}^+}\|T(s + t_n)z - T(t_n)z\| \\ & + \alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| + |\alpha_{n+1} - \alpha_n|\gamma\|f(x_n)\|. \end{aligned} \tag{3.11}$$

From this, we obtain that

$$\begin{aligned} \|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \le {} & [(1 - \alpha_{n+1}\eta)(1 + k(t_{n+1})) - 1]\|x_{n+1} - x_n\| \\ & + |\alpha_n - \alpha_{n+1}|\|GT(t_{n+1})x_n\| \\ & + (1 - \alpha_n\eta)\sup_{z\in\{x_n\}, s\in\mathbb{R}^+}\|T(s + t_n)z - T(t_n)z\| \\ & + \alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| + |\alpha_{n+1} - \alpha_n|\gamma\|f(x_n)\|, \end{aligned} \tag{3.12}$$

which implies

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0 \tag{3.13}$$

and by Lemma 2.2,

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.14}$$

Thus, $\|x_{n+1} - x_n\| = (1-\beta_n)\|y_n - x_n\| \to 0$ as $n \to \infty$.

Next, we show that $\lim_{n\to\infty}\|x_n - T(t)x_n\| = 0$ and $\lim_{n\to\infty}\|y_n - T(t)y_n\| = 0$, for all $t \geq 0$.

Since

$$\begin{aligned}\|x_n - T(t_n)x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T(t_n)x_n\| \\ &\leq \|x_n - x_{n+1}\| + \beta_n\|x_n - T(t_n)x_n\| + (1-\beta_n)\|y_n - T(t_n)x_n\|,\end{aligned} \tag{3.15}$$

we have

$$\begin{aligned}(1-\beta_n)\|x_n - T(t_n)x_n\| &\leq \|x_n - x_{n+1}\| + (1-\beta_n)\|y_n - T(t_n)x_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n(1-\beta_n)\|\gamma f(x_n) - GT(t_n)x_n\|.\end{aligned} \tag{3.16}$$

From $\alpha_n \to 0$ as $n \to \infty$, we obtain $\lim_{n\to\infty}\|x_n - T(t_n)x_n\| = 0$.

Now, for any $t \geq 0$, we have

$$\begin{aligned}\|T(t)x_n - x_n\| &\leq \|T(t)x_n - T(t)T(t_n)x_n\| + \|T(t)T(t_n)x_n - T(t_n)x_n\| + \|T(t_n)x_n - x_n\| \\ &\leq \|T(t+t_n)x_n - T(t_n)x_n\| + (2+k(t))\|T(t_n)x_n - x_n\| \\ &\leq \sup_{z\in\{x_n\}, s\in\mathbb{R}^+}\|T(s+t_n)z - T(t_n)z\| + (2+k(t))\|T(t_n)x_n - x_n\|.\end{aligned} \tag{3.17}$$

Using this and the uniform asymptotic regularity of $\mathcal{J}$, we get

$$\lim_{n\to\infty}\|x_n - T(t)x_n\| = 0, \quad \forall t \geq 0. \tag{3.18}$$

We also have

$$\begin{aligned}\|y_n - T(t)y_n\| &\leq \|y_n - x_n\| + \|x_n - T(t)x_n\| + \|T(t)x_n - T(t)y_n\| \\ &\leq (2+k(t))\|y_n - x_n\| + \|x_n - T(t)x_n\|.\end{aligned} \tag{3.19}$$

This implies that

$$\lim_{n\to\infty}\|y_n - T(t)y_n\| = 0, \quad \forall t \geq 0. \tag{3.20}$$

Let $\{y_{n_j}\}$ be a subsequence of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(q) - G(q), j(y_n - q)\rangle = \lim_{j\to\infty}\left\langle \gamma f(q) - G(q), j\left(y_{n_j} - q\right)\right\rangle, \tag{3.21}$$

and assume without loss of generality that $y_{n_j} \rightharpoonup z \in E$. By Lemma 2.3, $(I - T(t))$ is demiclosed at zero, so $z \in F(T(t))$ and, by Lemma 2.4, $z \in \mathcal{F}$.

Since the duality map of E is weakly sequentially continuous, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(q) - G(q), j(y_n - q)\rangle &= \lim_{j\to\infty}\left\langle \gamma f(q) - G(q), j\left(y_{n_j} - q\right)\right\rangle \\ &= \langle (\gamma f - G)q, j(z - q)\rangle \le 0.\end{aligned} \tag{3.22}$$

We now conclude by showing that $x_n \to q$ as $n \to \infty$. Since $\lim_{n\to\infty}(k(t_n)/\alpha_n) = 0$, if we denote by σ_n the value $2k(t_n) + k(t_n)^2$, then we clearly have that $\lim_{n\to\infty}(\sigma_n/\alpha_n) = 0$. Let $N_0 \in \mathbb{N}$ be large enough such that $(1 - \alpha_n\eta)(\sigma_n/\alpha_n) < (\eta - 2\delta)/2$, for all $n \ge N_0$, and let M be a positive real number such that $\|x_n - q\| \le M$ for all $n \ge 0$. Then, using the recursion formula (3.2) and for $n \ge N_0$, we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le \beta_n\|x_n - q\|^2 + (1-\beta_n)\|y_n - q\|^2 \\ &= \beta_n\|x_n - q\|^2 + (1-\beta_n) \\ &\quad \times\|\alpha_n(\gamma f(x_n) - G(q)) + (I - \alpha_n G)T(t_n)x_n - (I - \alpha_n G)(q)\|^2 \\ &\le \beta_n\|x_n - q\|^2 + (1-\beta_n)(1-\alpha_n\eta)^2\|T(t_n)x_n - q\|^2 \\ &\quad + 2\alpha_n(1-\beta_n)\langle \gamma f(x_n) - Gq, j(y_n - q)\rangle \\ &\le \beta_n\|x_n - q\|^2 + (1-\beta_n)(1-\alpha_n\eta)(1+k(t_n))^2\|x_n - q\|^2 \\ &\quad + 2\alpha_n(1-\beta_n)\langle \gamma f(x_n) - \gamma f(q) + \gamma f(q) - Gq, j(y_n - q)\rangle \\ &\le \beta_n\|x_n - q\|^2 + (1-\beta_n)(1-\alpha_n\eta)(1+\sigma_n)\|x_n - q\|^2 \\ &\quad + 2\alpha_n(1-\beta_n)\langle \gamma f(q) - Gq, j(y_n - q)\rangle - 2\alpha_n(1-\beta_n)\gamma\|y_n - q\|\psi(\|x_n - q\|) \\ &\quad + 2\alpha_n(1-\beta_n)\gamma\|x_n - q\|\|(y_n - x_n) + (x_n - q)\| \\ &\le [\beta_n + (1-\beta_n)(1-\alpha_n\eta + (1-\alpha_n\eta)\sigma_n)]\|x_n - q\|^2 \\ &\quad + 2\alpha_n(1-\beta_n)[\langle \gamma f(q) - Gq, j(y_n - q)\rangle + \gamma\|y_n - x_n\|\|x_n - q\|] \\ &\quad + 2\alpha_n(1-\beta_n)\gamma\|x_n - q\|^2 \\ &\le \left[1 - \alpha_n(1-\beta_n)\left((\eta - 2\gamma) - (1-\alpha_n\eta)\frac{\sigma_n}{\alpha_n}\right)\right]\|x_n - q\|^2 \\ &\quad + 2\alpha_n(1-\beta_n)[\langle \gamma f(q) - Gq, j(y_n - q)\rangle + \gamma\|y_n - x_n\|M] \\ &= \left[1 - \alpha_n(1-\beta_n)\left((\eta - 2\gamma) - (1-\alpha_n\eta)\frac{\sigma_n}{\alpha_n}\right)\right]\|x_n - q\|^2 \\ &\quad + \alpha_n(1-\beta_n)\left((\eta - 2\gamma) - (1-\alpha_n\eta)\frac{\sigma_n}{\alpha_n}\right) \\ &\quad \times \frac{2[\langle \gamma f(q) - Gq, j(y_n - q)\rangle + \gamma\|y_n - x_n\|M]}{((\eta - 2\gamma) - (1-\alpha_n\eta)(\sigma_n/\alpha_n))}.\end{aligned} \tag{3.23}$$

Observe that $\sum \alpha_n(1-\beta_n)[(\eta-2\gamma)-(1-\alpha_n\eta)(\sigma_n/\alpha_n)]=\infty$ and

$$\limsup\left(\frac{2[\langle \gamma f(q)-Gq, j(y_n-q)\rangle+\gamma\|y_n-x_n\|M]}{((\eta-2\gamma)-(1-\alpha_n\eta)(\sigma_n/\alpha_n))}\right)\leq 0. \tag{3.24}$$

Applying Lemma 2.5, we obtain $\|x_n-q\|\to 0$ as $n\to\infty$. This completes the proof. □

Since every Banach space whose duality map is weakly sequentially continuous satisfies Opial's condition (see [3]) and every uniformly convex Banach space satisfying Opial's condition also satisfies local uniform Opial's condition (see [5]), we have the following theorem.

Theorem 3.2. *Let E be a real uniformly convex Banach space with weakly sequentially continuous duality mapping. Let $\mathcal{J}=\{T(t):t\geq 0\}$, $f, G, \{\alpha_n\}, \{\beta_n\}, \{t_n\}$, and $\{x_n\}$ be as in Theorem 3.1. Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\mathcal{J}$ which solves the variational inequality* (3.3).

The following corollaries follow from Theorem 3.1

Corollary 3.3. *Let $E=H$ be a real Hilbert space. Let $\mathcal{J}=\{T(t):t\geq 0\}$, $f, G, \{\alpha_n\}, \{\beta_n\}, \{t_n\}$, and $\{x_n\}$ be as in Theorem 3.1, then the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\mathcal{J}$ which solves the variational inequality $\langle (G-\gamma f)q, x-q\rangle\geq 0$, for all $x\in\mathcal{F}$.*

Corollary 3.4. *Let $E, f, G, \{\alpha_n\}, \{\beta_n\}$, and $\{t_n\}$ be as in Theorem 3.1. Let $\mathcal{J}=\{T(t):t\geq 0\}$ be a family of nonexpansive semigroup of E with $\mathcal{F}:=F(\mathcal{J})\neq\emptyset$, and let $\{x_n\}$ be define by* (3.2). *Then, $\{x_n\}$ converges strongly to a common fixed point of the family $\mathcal{J}$ which solves the variational inequality* (3.3).

Corollary 3.5. *Let $E=l_p$ space, $1<p<\infty$. Let $\mathcal{J}=\{T(t):t\geq 0\}, f, G, \{\alpha_n\}, \{\beta_n\}, \{t_n\}$, and $\{x_n\}$ be as in Theorem 3.1, then the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\mathcal{J}$ which solves the variational inequality* (3.3).

Corollary 3.6. *Let H be a real Hilbert space. Let $\mathcal{J}=\{T(t):t\geq 0\}$ be uniformly asymptotically regular family of asymptotically nonexpansive semigroup of H, with function $k:[0,\infty)\to[0,\infty)$ and $\mathcal{F}:=F(\mathcal{J})=\bigcap_{t\geq 0}F(T(t))\neq\emptyset$. Let $f, \{\alpha_n\}, \{\beta_n\}$, and $\{t_n\}$ be as in Theorem 3.1. Let $G:E\to E$ be a strongly positive, bounded, and linear operator on H with coefficient $\delta\in(1/2,1)$ and $\|G\|=1$. For a fixed real number $\gamma\in(0,1-\sqrt{2(1-\delta)})$, let $\{x_n\}$ be generated by* (3.2). *Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\mathcal{J}$ which solves the variational inequality $\langle (G-\gamma f)q, x-q\rangle\geq 0$, for all $x\in\mathcal{F}$.*

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