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Research Article

Robustness of Krasnoselski-Mann's Algorithm for Asymptotically Nonexpansive Mappings

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Iterative approximation of fixed points of nonexpansive mapping is a very active theme in many aspects of mathematical and engineering areas, in particular, in image recovery and signal processing. Because the errors usually occur in few places, it is necessary to show that whether the iterative algorithm is robust or not. In the present work, we prove that Krasnoselski-Mann's algorithm is robust for asymptotically nonexpansive mapping in a Banach space setting. Our results generalize the corresponding results existing in the literature.

1. Introduction

Many practical problems can be formulated as the fixed point problem of $x = Tx$, where T is a nonexpansive mapping. Iterative methods as a powerful tool are often used to approximate the fixed points of such mapping. It has been show that the methods used to find fixed points of nonexpansive mapping covered a widely applied mathematics problems, such as the convex feasibility problem [1–3] and the split feasibility problem [4–6]. It is recommended for interested reader to [7] for an extensive study on the theory about iterative fixed point theory.

Let X be a real Banach space. $T : X \to X$ is called a nonexpansive mapping if for any $x, y \in X$, $\|Tx - Ty\| \leq \|x - y\|$. Krasnoselski-Mann's iteration method for finding fixed points of T is defined by

$$\text{for any initial } x_0 \in X, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

In 2001, Combettes [8] considered a parallel projection method algorithm in signal synthesis problems in a real Hilbert space H as follows:

$$x_{n+1} = x_n + \lambda_n \left(\sum_{i=1}^{m} \omega_i (P_i x_n + c_{i,n}) - x_n \right), \tag{1.2}$$

where $\{\lambda_n\} \subseteq (0,2)$, $\{\omega_i\}_{i=1}^m$ are positive weights such that $\sum_{i=1}^m \omega_i = 1$, P_i is the projection of a signal $x \in H$ onto a closed convex subset S_i of H, and $c_{i,n}$ stands for the error made in computing the projection onto S_i at each iteration n. He firstly proved that the sequence $\{x_n\}$ generated by (1.2) converges weakly to a point in G, where $G := \bigcap_{i=1}^m S_i$.

Kim and Xu [9] generalized the results of Combettes [8] from Hilbert spaces to uniformly convex Banach spaces and obtained its equivalent form as follows:

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n (T x_n + e_n), \quad n \geq 0, \tag{1.3}$$

where $\alpha_n := \lambda_n / 2 \in (0,1)$, $e_n := 2 \sum_{i=1}^m \omega_i c_{i,n}$, and T is nonexpansive. They proved that the weak convergence of the (1.3) in a uniformly convex Banach space. More precisely, they proved that the following main theorems.

Theorem 1.1 (see [9]). *Assume that X is a uniformly convex Banach space. Assume, in addition, that either X^* has the Kadec-Klee property or X satisfies Opial's property. Let $T : X \to X$ be a nonexpansive mapping such that $F(T) \neq \emptyset$ ($F(T)$ denotes the set of fixed points of T, that is, $F(T) = \{x \in X : Tx = x\}$). Given an initial guess $x_0 \in X$. Let $\{x_n\}$ be generated by* (1.3) *and satisfy the following properties:*

(i) $\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n \|e_n\| < \infty$.

Then the sequence $\{x_n\}$ converges weakly to a fixed point of T.

Theorem 1.2 (see [9]). *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Given an initial guess $x_0 \in X$. Let $\{x_n\}$ be generated by either*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_C (T x_n + e_n), \quad n \geq 0, \tag{1.4}$$

or

$$x_{n+1} = P_C [(1 - \alpha_n) x_n + \alpha_n (T x_n + e_n)], \quad n \geq 0, \tag{1.5}$$

where the sequences $\{\alpha_n\}$ and $\{e_n\}$ are such that

(i) $\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n \|e_n\| < \infty$.

Then $\{x_n\}$ converges weakly to a fixed point of T.

Very recently, Ceng et al. [10] extended the algorithm (1.3) of Kim and Xu [9] to Krasnoselski-Mann's algorithm with perturbed mapping defined by the following:

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)(Tx_n + e_n) - \lambda_n \mu_n F(x_n), \quad n \geq 0, \tag{1.6}$$

where $\lambda_n, \mu_n \in [0,1]$, and F is a strongly accretive and strictly pseudocontractive mapping.

An important generalization of the class of nonexpansive mapping is asymptotically nonexpansive mapping (i.e., for $T : C \to C$, if there exists a sequence $\{u_n\} \subset [0,+\infty)$, $\lim_{n\to\infty} u_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1+u_n)\|x-y\|, \tag{1.7}$$

for all $x, y \in C$ and $n \geq 0$), which was introduced by Goebel and Kirk [11]; they proved that if C is a nonempty closed, convex, and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping has a fixed point. The class of asymptotically nonexpansive mapping has been studied by many authors and some recent results can be found in [12–17] and references cited therein.

Inspired and motivated by the above works, the purpose of this paper is to extend the results of Kim and Xu [9] from nonexpansive mapping to asymptotically nonexpansive mapping. We prove that the Krasnoselski-Mann iterative sequence converges weakly to the fixed point of asymptotically nonexpansive mapping.

2. Preliminaries

In this section, we collect some useful results which will be used in the following section.

We use the following notations:

(i) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence,

(ii) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

It is well known that a Hilbert space H satisfies Opial's condition [18]; that is, for each sequence $\{x_n\}$ in H which converges weakly to a point $x \in H$, one has

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \tag{2.1}$$

for all $y \in H$, $y \neq x$.

Recall that given a closed convex subset of C of a real Hilbert space H, the nearest point projection P_C from H onto C assigns to each $x \in C$ its nearest point denoted by $P_C x$ in C from x to C; that is, $P_C x$ is the unique point in X with the property

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.2}$$

A Banach space X is said to have the Kadec-Klee property [19] if for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply that $x_n \to x$.

A mapping T is said to be demiclosed at zero if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{Tx_n\}$ converges strongly to zero, then $Tx = 0$.

Lemma 2.1 (see [20]). *Let X be a real uniformly convex Banach space, let C be a nonempty closed convex subset of X, and let $T : C \to X$ be an asymptotically nonexpansive mapping with a sequence $\{u_n\} \subset [0,\infty)$ and $\lim_{n\to\infty} u_n = 0$; then $(I - T)$ is demiclosed at zero.*

Lemma 2.2 (see [21]). *Given a number $r > 0$, a real Banach space is uniformly convex if and only if there exists a continuous strictly increasing function $\phi : [0,\infty) \to [0,\infty)$, $\phi(0) = 0$, such that*

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\phi(\|x-y\|), \tag{2.3}$$

for all $\lambda \in [0,1]$ and $x, y \in X$ such that $\|x\| \le r$ and $\|y\| \le r$.

Lemma 2.3 (see [22]). *Let X be a real uniformly convex Banach space such that its dual X^* has Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in X and $q_1, q_2 \in \omega_w(\{x_n\})$. Suppose that that*

$$\lim_{n\to\infty} \|\alpha x_n + (1-\alpha)q_1 - q_2\| \tag{2.4}$$

exists for all $\alpha \in [0,1]$. Then $q_1 = q_2$.

Lemma 2.4 (see [23]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1+c_n)a_n + b_n, \quad n \ge 1. \tag{2.5}$$

If $\sum_{n=1}^{\infty} c_n < +\infty$, $\sum_{n=1}^{\infty} b_n < +\infty$, then (i) $\lim_{n\to\infty} a_n$ exists. (ii) In particular, if $\liminf_{n\to\infty} a_n = 0$, one has $\lim_{n\to\infty} a_n = 0$.

3. Main Results

We state our first theorem as follows.

Theorem 3.1. *Suppose that X is a uniformly convex Banach space, and X^* has the Kadec-Klee property or X satisfies Opial's property. Let $T : X \to X$ be an asymptotically nonexpansive mapping with $\sum_{n=0}^{\infty} u_n < \infty$. For any $x_0 \in X$, the sequence $\{x_n\}$ is generated by the following Krasnoselski-Mann's algorithm:*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(T^n x_n + e_n), \quad n \ge 0, \tag{3.1}$$

where $\{\alpha_n\}$ and $\{e_n\}$ satisfy the following conditions:

(i) $0 < a < \alpha_n < b < 1$, *for some $a, b \in (0,1)$ and for all $n \ge 0$;*

(ii) $\sum_{n=0}^{\infty} \alpha_n \|e_n\| < \infty$.

If $F(T) \ne \emptyset$, then the sequence $\{x_n\}$ converges weakly to a fixed point of T.

For the sake of convenience, we need the following lemmas.

Lemma 3.2. *Let X be a real normed linear space and let $T : X \to X$ be an asymptotically nonexpansive mapping with $\sum_{n=0}^{\infty} u_n < \infty$. Let $\{x_n\}$ be the sequence as defined in* (3.1) *and satisfy the conditions in Theorem 3.1. Suppose that $F(T) \neq \emptyset$; then the limit $\lim_{n\to\infty}\|x_n - p\|$ exists for $p \in F(T)$.*

Proof. By (3.1), one has

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p + e_n)\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|T^n x_n - p\| + \alpha_n\|e_n\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n(1+u_n)\|x_n - p\| + \alpha_n\|e_n\| \\ &\le (1+u_n)\|x_n - p\| + \alpha_n\|e_n\|.\end{aligned} \tag{3.2}$$

Since $\sum_{n=0}^{\infty} u_n < \infty$ and $\sum_{n=0}^{\infty} \alpha_n\|e_n\| < \infty$, we obtain from Lemma 2.4 that the limit $\lim_{n\to\infty}\|x_n - p\|$ exists. Furthermore, the sequence $\{x_n\}$ is bounded. □

Lemma 3.3. *Let X be a real uniformly convex Banach space and let $T : X \to X$ be an asymptotically nonexpansive mapping with $\sum_{n=0}^{\infty} u_n < \infty$. Let $\{x_n\}$ be the sequence as defined in* (3.1) *and satisfy the conditions in Theorem 3.1. Suppose that $F(T) \neq \emptyset$; then $\lim_{n\to\infty}\|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$ and $p, q \in F(T)$.*

Proof. Let $d_n(t) = \|tx_n + (1-t)p - q\|$; then $\lim_{n\to\infty} d_n(0) = \|p - q\|$ exists. It follows from Lemma 3.2 that $\lim_{n\to\infty} d_n(1) = \lim_{n\to\infty}\|x_n - q\|$ exists. Next, we show that $\lim_{n\to\infty} d_n(t)$ exists for any $t \in (0,1)$.

Let $T_n x := (1-\alpha_n)x + \alpha_n T^n x + \alpha_n e_n$, for all $x \in X$. For any $x, z \in X$, one has

$$\begin{aligned}\|T_n x - T_n z\| &= \|(1-\alpha_n)(x - z) + \alpha_n(T^n x - T^n z)\| \\ &\le (1-\alpha_n)\|x - z\| + \alpha_n\|T^n x - T^n z\| \\ &\le (1-\alpha_n)\|x - z\| + \alpha_n(1+u_n)\|x - z\| \\ &\le (1+u_n)\|x - z\|.\end{aligned} \tag{3.3}$$

Set $S_{n,m} = T_{n+m-1}T_{n+m-2}\cdots T_n$, $m \ge 1$. The rest of the proof is the same as Lemma 3.3 of [14, 16]. This completes the proof of Lemma 3.3. □

Now, we give the proof of Theorem 3.1.

Proof. Let $p \in F(T)$. With the help of Lemma 2.2 and the inequality $\|a+b\|^2 \le \|a\|^2 + 2\|a\| \cdot \|b\| + \|b\|^2$, one has

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p) + \alpha_n e_n\|^2 \\ &\le \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\|^2 \\ &\quad + 2\alpha_n\|e_n\| \cdot \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\| + \alpha_n^2\|e_n\|^2 \\ &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T^n x_n - p\|^2 - \alpha_n(1-\alpha_n)\phi(\|x_n - T^n x_n\|)\end{aligned}$$

$$
\begin{aligned}
&+ 2\alpha_n\|e_n\|\left((1-\alpha_n)\|x_n - p\| + \alpha_n(1+u_n)\|x_n - p\|\right) + \alpha_n^2\|e_n\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n(1+u_n)^2\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\phi(\|x_n - T^n x_n\|) \\
&\quad + 2\alpha_n\|e_n\|(1+u_n)\|x_n - p\| + \alpha_n^2\|e_n\|^2 \\
&\le (1+u_n)^2\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\phi(\|x_n - T^n x_n\|) \\
&\quad + 2\alpha_n\|e_n\|(1+u_n)\|x_n - p\| + \alpha_n^2\|e_n\|^2,
\end{aligned}
\tag{3.4}
$$

which follows that

$$
\begin{aligned}
a(1-b)\phi(\|x_n - T^n x_n\|) &\le \alpha_n(1-\alpha_n)\phi(\|x_n - T^n x_n\|) \\
&\le (1+u_n)^2\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2\alpha_n\|e_n\|(1+u_n)\|x_n - p\| + \alpha_n^2\|e_n\|^2.
\end{aligned}
\tag{3.5}
$$

This implies that

$$
\sum_{n=0}^{\infty}\phi(\|x_n - T^n x_n\|) < \infty. \tag{3.6}
$$

Therefore $\lim_{n\to\infty}\phi(\|x_n - T^n x_n\|) = 0$. Since ϕ is strictly increasing and continuous function with $\phi(0) = 0$, then $\lim_{n\to\infty}\|x_n - T^n x_n\| = 0$. Also, one has the following inequalities:

$$
\|x_{n+1} - x_n\| \le \alpha_n\|T^n x_n - x_n\| + \alpha_n\|e_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.7}
$$

$$
\begin{aligned}
\|T^n x_{n+1} - x_{n+1}\| &= \|T^n x_{n+1} - (1-\alpha_n)x_n - \alpha_n(T^n x_n + e_n)\| \\
&= \|(T^n x_{n+1} - T^n x_n) + (1-\alpha_n)(T^n x_n - x_n) - \alpha_n e_n\| \\
&\le \|T^n x_{n+1} - T^n x_n\| + (1-\alpha_n)\|T^n x_n - x_n\| + \alpha_n\|e_n\| \\
&\le (1+u_n)\|x_{n+1} - x_n\| + (1-\alpha_n)\|T^n x_n - x_n\| + \alpha_n\|e_n\| \\
&= (1+u_n)\|\alpha_n(T^n x_n - x_n) + \alpha_n e_n\| + (1-\alpha_n)\|T^n x_n - x_n\| + \alpha_n\|e_n\| \\
&\le (1+u_n)\alpha_n\|T^n x_n - x_n\| + (1-\alpha_n)\|T^n x_n - x_n\| + (1+u_n)\alpha_n\|e_n\| + \alpha_n\|e_n\| \\
&\le (1+u_n)\|T^n x_n - x_n\| + (1+u_n)\alpha_n\|e_n\| + \alpha_n\|e_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned}
\tag{3.8}
$$

On the other hand,

$$
\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\le \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + \left\|T^{n+1}x_{n+1} - Tx_{n+1}\right\| \\
&\le \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + (1+u_1)\|T^n x_{n+1} - x_{n+1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned}
\tag{3.9}
$$

By (3.7)–(3.9), we obtain

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_{n+1}\| + \|Tx_{n+1} - Tx_n\| \\ &\le (2+u_1)\|x_n - x_{n+1}\| + \|x_{n+1} - Tx_{n+1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{3.10}$$

that is, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

From Lemma 3.2, we know that $\{x_n\}$ is bounded. Since X is a uniformly convex Banach space, $\{x_n\}$ has a convergent subsequence $\{x_{n_j}\}$. By the demiclosedness principle of $I - T$, we obtain $\omega_w \subseteq F(T)$. The rest of proof is followed by the standard argument in Theorem 3.3 of Kim and Xu [9]. This completes the proof. □

Theorem 3.4. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping with $\sum_{n=0}^{\infty} u_n < \infty$. For any $x_0 \in X$, the sequence $\{x_n\}$ is generated by either*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C(T^n x_n + e_n), \quad n \ge 0, \tag{3.11}$$

or

$$x_{n+1} = P_C[(1-\alpha_n)x_n + \alpha_n(T^n x_n + e_n)], \quad n \ge 0, \tag{3.12}$$

where the sequences $\{\alpha_n\}$ and $\{e_n\}$ are such that

(i) $0 < a < \alpha_n < b < 1$, *for some $a, b \in (0,1)$ and for all $n \ge 0$;*

(ii) $\sum_{n=0}^{\infty} \alpha_n \|e_n\| < \infty$.

If $F(T) \ne \emptyset$, then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. Let $p \in F(T)$. By (3.11), one has

$$\begin{aligned}\|x_{n+1} - p\| &= \|[(1-\alpha_n)x_n + \alpha_n P_C(T^n x_n + e_n)] - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|P_C(T^n x_n + e_n) - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|T^n x_n + e_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n(1+u_n)\|x_n - p\| + \alpha_n\|e_n\| \\ &\le (1+u_n)\|x_n - p\| + \alpha_n\|e_n\|.\end{aligned} \tag{3.13}$$

Notice the condition (ii) and $\sum_{n=0}^{\infty} u_n < \infty$; by Lemma 2.4, $\lim_{n\to\infty}\|x_n - p\|$ exists. Hence, $\{x_n\}$ is bounded.

By the well-known inequality $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$, for all $x, y \in H$ and $t \in [0,1]$, we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\|[(1-\alpha_n)x_n + \alpha_n P_C(T^n x_n + e_n)] - p\right\|^2 \\
&= \left\|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p) + \alpha_n[P_C(T^n x_n + e_n) - T^n x_n]\right\|^2 \\
&\le \left\|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\right\|^2 + \|\alpha_n[P_C(T^n x_n + e_n) - T^n x_n]\|^2 \\
&\quad + 2\alpha_n\left\|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\right\| \cdot \|P_C(T^n x_n + e_n) - T^n x_n\| \\
&\le (1-\alpha_n)\left\|x_n - p\right\|^2 + \alpha_n\left\|T^n x_n - p\right\|^2 - \alpha_n(1-\alpha_n)\|x_n - T^n x_n\|^2 \\
&\quad + 2\alpha_n\left[(1-\alpha_n)\left\|x_n - p\right\| + \alpha_n(1+u_n)\left\|x_n - p\right\|\right]\|e_n\| \\
&\le (1+u_n)^2\left\|x_n - p\right\|^2 - \alpha_n(1-\alpha_n)\|x_n - T^n x_n\|^2 + 2(1+u_n)\alpha_n\|e_n\|\left\|x_n - p\right\|.
\end{aligned} \tag{3.14}$$

That is,

$$\begin{aligned}
a(1-b)\|x_n - T^n x_n\|^2 &\le \alpha_n(1-\alpha_n)\|x_n - T^n x_n\|^2 \\
&\le (1+u_n)^2\left\|x_n - p\right\|^2 - \left\|x_{n+1} - p\right\|^2 + 2(1+u_n)\alpha_n\|e_n\|\left\|x_n - p\right\|.
\end{aligned} \tag{3.15}$$

This implies that

$$\sum_{n=0}^{\infty}\|x_n - T^n x_n\|^2 < \infty. \tag{3.16}$$

Therefore $\lim_{n\to\infty}\|x_n - T^n x_n\| = 0$. We also have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(1-\alpha_n)x_n + \alpha_n P_C(T^n x_n + e_n) - x_n\| \\
&\le \alpha_n\|T^n x_n - x_n\| + \alpha_n\|e_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\
\|T^n x_{n+1} - x_{n+1}\| &= \|T^n x_{n+1} - (1-\alpha_n)x_n - \alpha_n P_C(T^n x_n + e_n)\| \\
&= \|(T^n x_{n+1} - T^n x_n) + (T^n x_n - x_n) + \alpha_n(x_n - P_C(T^n x_n + e_n))\| \\
&\le (1+u_n)\|x_{n+1} - x_n\| + (1+\alpha_n)\|T^n x_n - x_n\| + \alpha_n\|e_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.17}$$

It follows from (3.9) and (3.10) that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Since a Hilbert space H must be a uniformly convex Banach space and satisfy Opial's property, then the rest of proof is the same as Theorem 3.1. So it is omitted. □

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