Research Article

Some Results for the Family $KKM(X, Y)$ and the Φ -Mapping in FC -Spaces

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We first establish a fixed point theorem for a k -set contraction map on the family $KKM(X, X)$, which does not need to be a compact map. Next, we present the KKM type theorems, matching theorems, coincidence theorems, and minimax theorems on the family $KKM(X, Y)$ and the Φ -mapping in FC -spaces. Our results improve and generalize some recent results.

1. Introduction

In 1929, Knaster et al. [1] first established the well-known KKM theorem in finite-dimensional spaces. In 1961, Fan [2] generalized the KKM theorem to infinite dimensional topological vector spaces and gave some applications in several directions. Later, Chang and Yen [3] introduced the family $KKM(X, Y)$ and got some results about fixed point theorems, coincidence theorems, and its applications on this family. Recently, Lin and Chen [4] studied the coincidence theorems for two families of multivalued functions, Chen and Chang [5] obtained some results for the family $KKM(X, Y)$ and the Φ -mapping in Hausdorff topological vector spaces. For the related results, the reader may consult [4–10]. In this paper, we first establish a fixed point theorem for a k -set contraction map on the family $KKM(X, X)$, which does not need to be a compact map. Next, we present the KKM type theorems, matching theorems, coincidence theorems, and minimax theorems on the family $KKM(X, Y)$ and the Φ -mapping in FC -spaces. Our results improve and generalize the corresponding results in [5, 8, 9].

2. Preliminaries

Let Y be a nonempty set. We denote by 2^Y and $\langle Y \rangle$ the family of all subsets of Y and the family of all nonempty finite subsets of Y , respectively. For each $A \in \langle X \rangle$, we denote by

$|A|$ the cardinality of A . Let Δ_n denote the standard n -dimensional simplex with the vertices $\{e_0, \dots, e_n\}$. If J is a nonempty subset of $\{0, 1, \dots, n\}$, we will denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$.

Let X and Y be two sets, and $T : X \rightarrow 2^Y$ be a set-valued mapping. We will use the following notations in the following material:

- (i) $T(x) = \{y \in Y : y \in T(x)\}$,
- (ii) $T(A) = \bigcup_{x \in A} T(x)$,
- (iii) $T^{-1}(y) = \{x \in X : y \in T(x)\}$,
- (iv) $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$,
- (v) $C(X, Y)$ denote the family of single-valued continuous mappings from X to Y .

For topological spaces X and Y , $T : X \rightarrow 2^Y$ is said to be closed if its graph $\mathcal{G}_T = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed. T is said to be compact if the image $T(X)$ of X under T is contained in a compact subset of Y . A subset D of X is said to be compactly open (resp., compactly closed) if for each nonempty compact subset K of X , $D \cap K$ is open (resp., closed) in K . The compact closure of A and the compact interior of A (see [5]) are defined, respectively, by

$$\begin{aligned} \text{ccl } A &= \bigcap \{B \subset X : A \subset B \text{ and } B \text{ is compactly closed in } X\}, \\ \text{cint } A &= \bigcup \{B \subset X : B \subset A \text{ and } B \text{ is compactly open in } X\}. \end{aligned} \quad (2.1)$$

It is easy to see that $\text{ccl}(X \setminus A) = X \setminus \text{cint } A$, $\text{int } A \subset \text{cint } A \subset A$, $A \subset \text{ccl } A \subset \text{cl } A$, A is compactly open (resp., compactly closed) in X if and only if $\text{cint } A = A$ (resp., $\text{ccl } A = A$). For each nonempty compact subset K of X , $\text{ccl } A \cap K = \text{cl}_K(A \cap K)$ and $\text{cint } A \cap K = \text{int}_K(A \cap K)$, where $\text{cl}_K(A \cap K)$ (resp., $\text{int}_K(A \cap K)$) denotes the closure (resp., interior) of $A \cap K$ in K .

A set-valued mapping $T : X \rightarrow 2^Y$ is said to be transfer compactly closed valued on X (see [5]) if for each $x \in X$ and $y \notin T(x)$, there exists $x' \in X$ such that $y \notin \text{ccl } T(x')$. T is said to be transfer compactly open valued on X if for each $x \in X$ and $y \in T(x)$, there exists $x' \in X$ such that $y \in \text{cint } T(x')$. T is said to have the compactly local intersection property on X if for each nonempty compact subset K of X and for each $x \in X$ with $T(x) \neq \emptyset$, there exists an open neighborhood $N(x)$ of x in X such that $\bigcap_{z \in N(x) \cap K} T(z) \neq \emptyset$.

Let A_i ($i = 1, \dots, m$) is transfer compactly open valued, then $\bigcap_{i=1}^m \text{cint } A_i = \text{cint } \bigcap_{i=1}^m A_i$. It is clear that each transfer open valued correspondence is transfer compactly open valued. The inverse is not true in general.

Throughout this paper, all topological spaces are assumed to be Hausdorff. The following notion of a finitely continuous topological space (in short, *FC*-space) was introduced by Ding in [11].

Definition 2.1. (Y, φ_N) is said to be an *FC*-space if Y is a topological space and for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \rightarrow Y$. If A and B are two subsets of Y , B is said to be an *FC*-subspace of Y relative to A if for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ and for any $\{y_{i_0}, \dots, y_{i_k}\} \subset N \cap A$, $\varphi_N(\Delta_k) \subset B$, where $\Delta_k = \text{co}(\{e_{i_j} : j = 0, \dots, k\})$.

If $A = B$, then B is called an *FC*-subspace of Y . Clearly, each *FC*-subspace of Y is also an *FC*-space.

For a subset A of X , we can define the FC -hull of A (see [12]) as follows: $FC(A) = \bigcap \{B \subset X : A \subset B \text{ and } B \text{ is } FC\text{-subspace of } X\}$.

Definition 2.2 (see [8]). An FC -space (X, φ_N) is said to be a locally FC -space, denoted by $(X, \mathcal{U}, \varphi_N)$, if X is a uniform topological space with a uniform structure $\mathcal{U}$ having an open base β of symmetric entourages such that for each $x \in X$ and for each $U \in \mathcal{U}$, $U[x] = \{y \in X : (y, x) \in U\}$ is an FC -subspace of X .

Lemma 2.3 (see [5]). Let X and Y be two topological spaces, and let $F : X \rightarrow 2^Y$ be a set-valued mapping. Then the following conditions are equivalent:

- (1) F has the compactly local intersection property,
- (2) for each compact subset K of X and for each $y \in Y$, there exists an open subset O_y of X such that $O_y \cap K \subset F^{-1}(y)$ and $K = \bigcup_{y \in Y} (O_y \cap K)$,
- (3) for any compact subset K of X , there exists a set-valued mapping $P : X \rightarrow 2^Y$ such that $P(x) \subset F(x)$ for each $x \in X$, $P^{-1}(y)$ is open in X and $P^{-1}(y) \cap K \subset F^{-1}(y)$ for each $y \in Y$, and $K = \bigcup_{y \in Y} (P^{-1}(y) \cap K)$,
- (4) for each compact subset K of X and for each $x \in K$, there exists $y \in Y$ such that $x \in \text{cint } F^{-1}(y) \cap K$ and $K = \bigcup_{y \in Y} (\text{cint } F^{-1}(y) \cap K)$,
- (5) F^{-1} is transfer compactly open valued on Y ,
- (6) $X = \bigcup_{y \in Y} \text{cint } F^{-1}(y)$.

Now, we introduce the following Definitions 2.4 and 2.6.

Definition 2.4. Let Y be a topological space and (X, φ_N) be an FC -space. A set-valued mapping $T : Y \rightarrow 2^X$ is called a Φ -mapping if there exists a set-valued mapping $F : Y \rightarrow 2^X$ such that

- (i) for each $y \in Y$, $T(y)$ is a nonempty FC -subspace of X relative to $F(y)$,
- (ii) F satisfies one of the conditions (1)–(6) in Lemma 2.3.

The mapping F is said to be a companion mapping of T .

Remark 2.5. If $T : Y \rightarrow 2^X$ be a Φ -mapping, then for each nonempty subset Y_1 of Y , $T|_{Y_1} : Y_1 \rightarrow 2^X$ is also a Φ -mapping.

The class KKM was introduced by Ding [8]. Let (X, φ_N) be an FC -space and let Y be a topological space. If $F, T : X \rightarrow 2^Y$ are two set-valued mappings such that for each $N \in \langle X \rangle$ and for each $\{e_{i_0}, \dots, e_{i_k}\} \subset \{e_0, \dots, e_n\}$, $T(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^k F(x_{i_j})$, then F is said to be a generalized F - KKM mapping with respect to T . Let $T : X \rightarrow 2^Y$ be a set-valued mapping such that if $F : X \rightarrow 2^Y$ is a generalized F - KKM mapping with respect to T , then the family $\{\overline{Fx} : x \in X\}$ has the finite intersection property, where $\overline{Fx}$ denotes the closure of Fx , then T is said to have the KKM property. Write

$$KKM(X, Y) = \left\{ T : X \rightarrow 2^Y : T \text{ has the } KKM \text{ property} \right\}. \quad (2.2)$$

Let (X, φ_N) be an FC -space and $B(X)$ be the family of nonempty bounded subsets. Let $\rho = \{P : P \text{ is a family of seminorms which determines the topology on } X\}$. Let $\mathcal{R}^+$ be the

set of all nonnegative real numbers. If $\{p \in P : P \in \mathcal{P}\}$ is a family of seminorms which determines the topology on X , then for each $P \in \mathcal{P}$ and $\Omega \in X$, we define the set-measure of noncompactness $\alpha_p : 2^X \rightarrow \mathcal{R}^+$ by

$$\alpha_p(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ can be covered by a finite number of sets} \quad (2.3)$$

$$\text{and each } p\text{-diameter of the sets is less than } \varepsilon\},$$

where p -diameter of a set $D = \sup\{p(x - y) : x, y \in D\}$.

Definition 2.6. Let (X, φ_N) be an FC-space, and a mapping $T : X \rightarrow 2^X$ is said to be a k -set contraction map, if there exists $P \in \mathcal{P}$ such that for each $p \in P$, $\alpha_p(T(\Omega)) \leq k\alpha_p(\Omega)$ with $0 < k < 1$ for each bounded subset Ω of X and $T(X)$ is bounded.

3. Main Results

In order to prove our main results, we need the following Lemmas. The following results are [8, Lemma 3.1(i) and Theorem 3.1].

Lemma 3.1. Let (X, φ_N) be an FC-space and let Y be a topological space. Then we have $T \in KKM(X, Y)$ if and only if $T|_D \in KKM(D, Y)$ for each nonempty subset D of X .

Lemma 3.2. Let $(X, \mathcal{M}, \varphi_N)$ be a locally FC-space. If $T \in KKM(X, Y)$ is a compact mapping, then for each open entourage $U_i \in \mathcal{M}$ there exists $x_i \in X$ such that $T(x_i) \cap U_i[x_i] \neq \emptyset$.

Lemma 3.3. Let Y be a compact topological space and (X, φ_N) be an FC-space. Let $T : Y \rightarrow 2^X$ be a Φ -mapping. Then there exists a continuous function $f : Y \rightarrow X$ such that for each $y \in Y$, $f(y) \in T(y)$; that is, T has a continuous selection.

Proof. Since T is a Φ -mapping, there exists a companion mapping $F : Y \rightarrow 2^X$ such that

- (i) for each $y \in Y$, for each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$, and for any $\{x_{i_0}, \dots, x_{i_k}\} \in N \cap F(y)$, $\varphi_N(\Delta_k) \subset T(y)$;
- (ii) $Y = \bigcup_{x \in X} \text{cint } F^{-1}(x)$.

Since Y be a compact, there exists $M = \{x_0, \dots, x_m\} \in \langle X \rangle$ such that $Y = \bigcup_{i=0}^m \text{cint } F^{-1}(x_i)$. Let $\{\psi_i\}_{i=0}^m$ be the continuous partition of unity subordinated to the open covering $\{\text{cint } F^{-1}(x_i)\}_{i=0}^m$ of Y , then for each $i \in \{0, 1, \dots, m\}$ and $y \in Y$, we have

$$\psi_i(y) \neq 0 \iff y \in \text{cint } F^{-1}(x_i) \subset F^{-1}(x_i) \implies x_i \in F(y). \quad (3.1)$$

Define a mapping $\psi : Y \rightarrow \Delta_n$ by $\psi(y) = \sum_{i=0}^m \psi_i(y)e_i$, then ψ is continuous and for each $y \in Y$, $\psi(y) = \sum_{j \in J(y)} \psi_j(y)e_j \in \Delta_{J(y)}$, where $J(y) = \{j \in \{0, 1, \dots, m\} : \psi_j(y) \neq 0\}$. By (3.1), we have $\{x_j : j \in J(y)\} \in \langle F(y) \rangle \cap \{x_0, \dots, x_m\}$. From the definition of Φ -mapping, we obtain for each $y \in Y$, $\varphi_N(\Delta_{J(y)}) \subset T(y)$. It follows that

$$f(y) = \varphi_N \circ \psi(y) \in \varphi_N(\Delta_{J(y)}) \subset T(y). \quad (3.2)$$

This shows that for each $y \in Y$, $f(y) \in T(y)$, that is, T has a continuous selection. This completes the proof. $\square$

Lemma 3.4. *Let (X, φ_N) be an FC-space and let Y be a topological space. If $T : X \rightarrow 2^Y$ is a Φ -mapping, then $T \in KKM(X, Y)$.*

Proof. Since T is a Φ -mapping, we have that for each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$ and $\varphi_N(\Delta_n)$ in X , $T|_{\varphi_N(\Delta_n)} : \varphi_N(\Delta_n) \rightarrow Y$ is also a Φ -mapping. By Lemma 3.3, $T|_{\varphi_N(\Delta_n)}$ has a continuous selection function. Then $T|_{\varphi_N(\Delta_n)} \in KKM(\varphi_N(\Delta_n), Y)$. It follows from Lemma 3.1 that $T \in KKM(X, Y)$. This completes the proof. $\square$

Lemma 3.5. *Let Y and Z be two topological spaces and (X, φ_N) be an FC-space. If $T \in KKM(X, Y)$ and $f \in C(Y, Z)$, then $fT \in KKM(X, Z)$.*

Proof. Let $F : X \rightarrow 2^Z$ be a generalized F -KKM mapping with respect to fT such that Fx is closed in Z for each $x \in X$. Take $N = \{x_0, \dots, x_n\} \in \langle X \rangle$ and for any $\{e_{i_0}, \dots, e_{i_k}\} \subset \{e_0, \dots, e_n\}$, we have $fT(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^k F(x_{i_j})$, and hence $T(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^k f^{-1}F(x_{i_j})$. So $f^{-1}F$ is a generalized F -KKM mapping with respect to T . Since $T \in KKM(X, Y)$, then $\{f^{-1}Fx : x \in X\}$ has the finite intersection property, and hence $\{Fx : x \in X\}$ has also the finite intersection property and $fT \in KKM(X, Z)$. This completes the proof. $\square$

Remark 3.6. Lemmas 3.3, 3.4, and 3.5 generalize Lemmas 2, 3, and 5 in [5], respectively.

Corollary 3.7. *Let $(X, \mathcal{U}, \varphi_N)$ be a locally FC-space and $T \in KKM(X, Y)$ is compact and closed. Then T has a fixed point in X .*

Proof. Since T is compact, we have $K = \overline{T(X)}$ is compact in X . Without loss of generality, assume that $\{V_i\}_{i \in I}$ be a basis of symmetric open entourages for the uniformity $\mathcal{U}$. It follows from the Lemma 3.2 that for each $V \in \{V_i\}_{i \in I}$ there exists $x_i \in X$ such that $T(x_i) \cap V[x_i] \neq \emptyset$. Hence, for each $V \in \{V_i\}_{i \in I}$ there exists $y_i \in T(x_i)$ and $y_i \in V[x_i]$. Since $\{y_i\} \in K$ and K is compact, we may assume that $\{y_i\}$ converges to some $\bar{y} \in K$, and then x_i also converges to $\bar{y}$. Since T is closed, we have $\bar{y} \in T(\bar{y})$. This completes the proof. $\square$

Theorem 3.8. *Let (X, φ_N) be a bounded FC-space. Assume that $T : X \rightarrow 2^X$ be a k -set contraction map, $0 < k < 1$. Then X contains a precompact FC-subspace.*

Proof. Since T is a k -set contraction map, $0 < k < 1$, there exists $P \in \mathcal{P}$ such that for each $p \in P$, we have $\alpha_p(T(A)) \leq k\alpha_p(A)$ for each subset A of X . Take $y \in X$. Let

$$\begin{aligned} X_0 &= X, & X_1 &= FC\left(T(X_0) \bigcup \{y\}\right), \\ X_{n+1} &= FC\left(T(X_n) \bigcup \{y\}\right) \quad \text{for each } n \in N. \end{aligned} \tag{3.3}$$

Then

- (1) X_n is FC-subspace of X for each $n \in N$,
- (2) $X_{n+1} \subset X_n$ for each $n \in N$,

- (3) $T(X_n) \subset X_{n+1}$ for each $n \in N$,
 (4) $\alpha_p(X_{n+1}) \leq \alpha_p(T(X_n)) \leq k\alpha_p(X_n) \leq \dots \leq k^{n+1}\alpha_p(X_0)$ for each $n \in N$.

Thus $\alpha_p(X_n) \rightarrow 0$, as $n \rightarrow \infty$, and hence $X_\infty = \bigcap_{n \geq 1} X_n$ is a nonempty precompact FC -subspace of X . This completes the proof. $\square$

Remark 3.9. Theorem 3.8 generalized [5, Theorem 1] from a nonempty bounded convex subset of a Hausdorff topological vector space to a bounded FC -space. In the process of the proof Theorem 3.8, we call the set X_∞ a precompact-inducing FC -subspace of X .

The following result is a fixed point theorem for a k -set contraction map on the family $KKM(X, Y)$, which does not need to be a compact map.

Theorem 3.10. *Let X be a bounded FC -subspace of a locally FC -space $(E, \mathcal{M}, \varphi_N)$, and let $T \in KKM(X, X)$ be a k -set contraction map, $0 < k < 1$ and closed with $\overline{T(X)} \subset X$. Then T has a fixed point in X .*

Proof. By the same process of the proof Theorem 3.8, we get a precompact-inducing FC -subspace X_∞ of X . Since $\overline{T(X)} \subset X$ and $T(X_{n+1}) \subset T(X_n) \subset T(X)$ for each $n \in N$, we have $\overline{T(X_{n+1})} \subset \overline{T(X_n)} \subset X$ for each $n \in N$. Since $\alpha_p(\overline{T(X_n)}) \rightarrow 0$ as $n \rightarrow \infty$, we have $\overline{T(X_\infty)} = \bigcap_{n \geq 1} \overline{T(X_n)}$ is a nonempty compact subset of X .

Since $T \in KKM(X, X)$ and X_∞ is a nonempty FC -subspace of X , by Lemma 3.1, we have $T|_{X_\infty} \in KKM(X_\infty, X)$.

Let $\{V_i\}_{i \in I}$ be a basis of symmetric open entourages for the uniformity $\mathcal{M}$, then there exists $V \in \{V_i\}_{i \in I}$ such that $V \in \mathcal{M}$. We now claim that for $U_i \in \mathcal{M}$ there exists $x_i \in X_\infty$ such that $T(x_i) \cap (x_i + U_i) \neq \emptyset$. If it is false, then there exists $U \in \mathcal{M}$ such that $T(x) \cap (x + U) = \emptyset$ for all $x \in X_\infty$. Let $K = \overline{T(X_\infty)}$. Define $F : X_\infty \rightarrow 2^K$ by

$$F(x) = K \setminus \left(x + \frac{1}{2}U \right) \quad \text{for each } x \in X_\infty, \quad (3.4)$$

then $F(x)$ is a compact for each $x \in X_\infty$. Next, let $V[x] = x + (1/2)U$, then $F(x) = K \setminus V[x]$, we prove that F is a generalized F - KKM mapping with respect to $T|_{X_\infty}$.

Suppose, on contrary, F is not a generalized F - KKM mapping with respect to $T|_{X_\infty}$. Then there exists $N = \{x_0, \dots, x_n\} \in \langle X_\infty \rangle$ and $\{e_{i_0}, \dots, e_{i_k}\} \subset \{e_0, \dots, e_n\}$ such that

$$T|_{X_\infty}(\varphi_N(\Delta_k)) \not\subset \bigcup_{j=0}^k F(x_{i_j}), \quad (3.5)$$

where $\Delta_k = \text{co}(\{e_{i_j} : j = 0, \dots, k\})$. Hence there exist $u \in \varphi_N(\Delta_k)$ and $v \in T|_{X_\infty}(u) \subset \overline{T(X_\infty)} = K$ such that $v \notin \bigcup_{j=0}^k F(x_{i_j})$. From the definition of F it follows that $v \in V[x_{i_j}]$ for all $j \in \{0, \dots, k\}$. Noting that $V \in \mathcal{M}$ is symmetric, we have $\{x_{i_j} : j = 0, \dots, k\} \subset V[v]$. Since $V[v]$ is FC -subspace of X , we have $u \in \varphi_N(\Delta_k) \subset V[v]$. By the symmetry of V , we obtain $v \in V[u]$ and $v \in T|_{X_\infty}(u) \cap V[u] \subset T|_{X_\infty}(u) \cap U[u] \subset T|_{X_\infty}(u) \cap (u + U)$ which contradicts the fact $T(x) \cap (x + U) = \emptyset$ for all $x_i \in X_\infty$. Therefore F is a generalized F - KKM mapping with respect to $T|_{X_\infty}$.

Since $T|_{X_\infty} \in KKM(X_\infty, X)$ and F is a generalized F - KKM mapping with respect to $T|_{X_\infty}$, the family $\{F(x) : x \in X_\infty\}$ has the finite intersection property, and so we conclude that $\bigcap_{x \in X_\infty} F(x) \neq \emptyset$. Choose $\eta \in \bigcap_{x \in X_\infty} F(x)$, then $\eta \in K \setminus V[x]$ for all $x \in X_\infty$. Since $\eta \in \bigcap_{x \in X_\infty} F(x) \subset K = \overline{T(X_\infty)} \subset \overline{X_\infty} + (1/4)U$, hence there exists $x_0 \in X_\infty$ such that $\eta \in x_0 + (1/2)U = V[x_0]$. But $\eta \in K \setminus V[x_0]$, a contradiction. Therefore, we have prove that for each $U_i \in \{V_i\}_{i \in I}$, there exists $x_i \in X_\infty$ such that $T(x_i) \cap (x_i + U_i) \neq \emptyset$. Let $y_i \in T(x_i) \cap (x_i + U_i)$. Since $\{y_i\} \subset K$ and K is compact, we may assume that $\{y_i\}$ converges to some $\bar{y} \in K$, and then x_i also converges to $\bar{y}$. Since T is closed, we have $\bar{y} \in T(\bar{y})$. This completes the proof. $\square$

By Lemma 3.4 and Theorem 3.10, we can get the following result immediately.

Corollary 3.11. *Let X be a bounded FC -subspace of a locally FC -subspace $(E, \mathcal{U}, \varphi_N)$, and let $T : X \rightarrow 2^X$ be a Φ -mapping, k -set contraction, $0 < k < 1$ and closed with $\overline{T(X)} \subset X$. Then T has a fixed point in X .*

Remark 3.12. Theorem 3.10 generalizes [5, Theorem 2] from a nonempty bounded convex subset of a locally convex space to a bounded FC -subspace of a locally FC -space and [8, Theorem 3.10]. Corollary 3.11 generalizes [5, Corollary 1] in several aspects.

4. Applications

By Definition 4.3 of Ding [13], we have the following definition.

Definition 4.1. Let (X, φ_N) be an FC -space. Y be topological space. $f : X \times Y \rightarrow R$ is said to be FC -quasiconvex (resp., FC -quasiconcave) if for each $y \in Y$ and for each $\lambda \in R$, the set $\{x \in X : f(x, y) < \lambda\}$ (resp., $\{x \in X : f(x, y) > \lambda\}$) is an FC -subspace of X .

Definition 4.2 (see [5]). Let X and Y be two topological spaces, and let $f : X \times Y \rightarrow R \cup \{-\infty, +\infty\}$ be a function. Then f is said to be transfer compactly lower semicontinuous (in short, transfer compactly l.s.c) in y if for each $y \in Y$ and $\gamma \in R$ with $y \in \{u \in Y : f(x, u) > \gamma\}$, there exists $\bar{x} \in X$ such that $y \in \text{cint}\{u \in Y : f(\bar{x}, u) > \gamma\}$. f is said to be transfer compactly u.s.c in y if $-f$ is transfer compactly l.s.c in y .

Now, we establish the following KKM -type theorem for a k -set contraction map.

Theorem 4.3. *Let (Y, φ_N) be a bounded FC -space and let X be a topological space. If $T, F : Y \rightarrow 2^X$ are two set-valued mappings satisfying the following:*

- (i) $T \in KKM(Y, X)$ is a k -set contraction map, $0 < k < 1$, with $\overline{T(Y)} \subset X$,
- (ii) for any $y \in Y$, $F(y)$ is compactly closed in X ,
- (iii) F is a generalized F - KKM mapping with respect to T .

Then

$$\overline{T(Y_\infty)} \cap \left(\bigcap \{F(y) : y \in Y_\infty\} \right) \neq \emptyset, \quad (4.1)$$

where Y_∞ is the precompact-inducing FC -subspace of Y .

Proof. Let $\{V_i\}_{i \in I}$ be a basis of symmetric open entourages for the uniformity $\mathcal{U}$. By the same process of the proof Theorem 3.10, we get a compact subset $\overline{T(Y_\infty)}$ of Y , and $T|_{Y_\infty} \in KKM(Y_\infty, X)$, since $T \in KKM(Y, X)$.

Define $H : Y_\infty \rightarrow 2^X$ by

$$H(y) = \overline{T(Y_\infty)} \cap F(y) \quad \text{for each } y \in Y_\infty. \quad (4.2)$$

By condition (ii), $H(y)$ is compact in X , for each $y \in Y_\infty$. We now claim that H is a generalized F - KKM mapping with respect to $T|_{Y_\infty}$. Let $N \in \langle Y_\infty \rangle$ and for each $\{e_{i_0}, \dots, e_{i_k}\} \subset \{e_0, \dots, e_n\}$. By condition (iii), $T|_{Y_\infty}(\varphi_N(\Delta_k)) \subset \overline{T(Y_\infty)} \cap (\bigcup_{j=0}^k F(y_{i_j})) = \bigcup_{j=0}^k H(y_{i_j})$. Thus, we have shown that H is a generalized F - KKM mapping with respect to $T|_{Y_\infty}$. Since $T|_{Y_\infty} \in KKM(Y_\infty, X)$, the family $\{H(y) : y \in Y_\infty\}$ has the finite intersection property. And, since $H(y)$ is compact, $\bigcap_{y \in Y_\infty} H(y) \neq \emptyset$, that is, $\overline{T(Y_\infty)} \cap (\bigcap \{F(y) : y \in Y_\infty\}) \neq \emptyset$. This completes the proof. $\square$

Theorem 4.4. Let (Y, φ_N) be a bounded FC -space and let X be a topological space. If $T, F : Y \rightarrow 2^X$ are two set-valued mappings satisfying the following:

- (i) $T \in KKM(Y, X)$ is a k -set contraction map, $0 < k < 1$, with $\overline{T(Y)} \subset X$,
- (ii) for any $y \in Y$, $F(y)$ is transfer compactly closed in X ,
- (iii) F is a generalized F - KKM mapping with respect to T .

Then

$$\overline{T(Y_\infty)} \cap \left(\bigcap \{F(y) : y \in Y_\infty\} \right) \neq \emptyset, \quad (4.3)$$

where Y_∞ is the precompact-inducing FC -subspace of Y .

Proof. Define a mapping $\text{ccl } F : Y_\infty \rightarrow 2^X$ by $(\text{ccl } F)(y) = \text{ccl } F(y)$ for each $y \in Y_\infty$, it is easy to see that $\text{ccl } F$ is also a generalized F - KKM mapping with respect to $T|_{Y_\infty}$ with compactly closed values. By the same process of the proof Theorem 3.10, we get a compact subset $\overline{T(Y_\infty)}$ of Y . By Theorem 4.3, $\overline{T(Y_\infty)} \cap (\bigcap \{\text{ccl } F(y) : y \in Y_\infty\}) \neq \emptyset$. And since for any $y \in Y$, $F(y)$ is transfer compactly closed in X , by Lemma 2.2 [9], we have $\overline{T(Y_\infty)} \cap (\bigcap \{F(y) : y \in Y_\infty\}) = \overline{T(Y_\infty)} \cap (\bigcap \{\text{ccl } F(y) : y \in Y_\infty\}) \neq \emptyset$. This completes the proof. $\square$

Remark 4.5. Theorem 4.3 generalizes [5, Theorem 3] from a nonempty bounded convex subset of a Hausdorff topological vector space to a bounded FC -space and [9, Theorem 3.1]. Theorem 4.4 generalizes [5, Theorem 3] in several aspects and [9, Theorem 3.2].

The following results are the generalization of the Ky Fan matching theorem and coincidence theorems.

Theorem 4.6. Let (X, φ_N) be a bounded FC -space. If $T, H : X \rightarrow 2^X$ are two set-valued mappings satisfying the following:

- (i) $T \in KKM(X, X)$ is a k -set contraction map, $0 < k < 1$, with $\overline{T(X)} \subset X$,
- (ii) for any $x \in X$, $H(x)$ is compactly open in X ,
- (iii) for the precompact-inducing FC -subspace X_∞ of X , $\overline{T(X_\infty)} \subset H(X_\infty)$.

Then for the precompact-inducing FC-subspace X_∞ of X satisfying the following condition:

$$T(X_\infty) \cap \left(\bigcap \{H(x) : x \in M\} \right) \neq \emptyset \quad \text{for some } M \in \langle X_\infty \rangle. \quad (4.4)$$

Proof. Let $\{V_i\}_{i \in I}$ be a basis of symmetric open entourages for the uniformity $\mathcal{U}$. By the same process of the proof Theorem 3.10, we get a compact subset $\overline{T(X_\infty)}$ of X , and $T|_{X_\infty} \in KKM(X_\infty, X)$, since $T \in KKM(X, X)$.

We claim that there exists $M \in \langle X_\infty \rangle$ such that $T(X_\infty) \cap (\bigcap \{H(x) : x \in M\}) \neq \emptyset$. On the contrary, assume that $T(X_\infty) \cap (\bigcap \{H(x) : x \in M\}) = \emptyset$ for any $M \in \langle X_\infty \rangle$, then $T(X_\infty) \subset \bigcap_{x \in M} H^c(x)$. Since X_∞ is FC-subspace of X , $M \in \langle X_\infty \rangle$ and for any $\{x_{i_0}, \dots, x_{i_k}\} \subset M$, we have $T(\varphi_N(\Delta_k)) \subset T(X_\infty) \subset \bigcap_{x \in M} H^c(x)$, where $\Delta_k = \text{co}(\{e_{i_j} : j = 0, \dots, k\})$. This implies that H^c is a generalized F - KKM mapping with respect to T . By condition (ii), for any $x \in X$, $H^c(x)$ is compactly closed in X . Follows Theorem 4.3, we have $\overline{T(X_\infty)} \cap (\bigcap \{H^c(x) : x \in X_\infty\}) \neq \emptyset$, which implies $\overline{T(X_\infty)} \not\subseteq \bigcup_{x \in X_\infty} H(x)$, a contradiction to condition (iii). This completes the proof. $\square$

Theorem 4.7. Let $(X, \mathcal{U}, \varphi_N)$ be a locally FC-space. Assume that

- (i) $T \in KKM(X, X)$ is a k -set contraction map, $0 < k < 1$, with $\overline{T(X)} \subset X$,
- (ii) $F : X \rightarrow 2^X$ is a Φ -mapping.

Then there exists $(\bar{x}, \bar{y}) \in X \times X$ such that $\bar{y} \in T(\bar{x})$ and $\bar{x} \in F(\bar{y})$.

Proof. By the same process of the proof Theorem 3.10, we get a compact subset $\overline{T(X_\infty)}$ of X , and $T|_{X_\infty} \in KKM(X_\infty, X)$, since $T \in KKM(X, X)$.

Let $K = \overline{T(X_\infty)}$ is compact. Then $F|_K$ is a Φ -mapping, and by Lemma 3.3, $F|_K$ has a continuous selection $f : K \rightarrow X$. So, by Lemma 3.5, we have $fT|_{X_\infty} \in KKM(X_\infty, X)$ and so, it follows from Theorem 3.10 that there exists $x \in \overline{T(X_\infty)}$ such that $\bar{x} \in fT(\bar{x}) \subset FT(\bar{x})$; that is, there exists $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in F(\bar{y})$. This completes the proof. $\square$

Theorem 4.8. Let $(X, \mathcal{U}, \varphi_N)$ be a locally FC-space, and let Y be a topological space. Assume that

- (i) $T \in KKM(X, Y)$ is compact and closed,
- (ii) $F : Y \rightarrow 2^X$ is a Φ -mapping.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in T(\bar{x})$ and $\bar{x} \in F(\bar{y})$.

Proof. Since T is compact, we have $K = \overline{T(X)}$ is compact in Y . By condition (ii), we have $F|_K$ is also a Φ -mapping. By Lemma 3.3, $F|_K$ has a continuous selection $f : K \rightarrow X$. So, by Lemma 3.5, we have $fT \in KKM(X, X)$, and so, it follows from Corollary 3.7 that there exists $\bar{x} \in X$ such that $\bar{x} \in fT(\bar{x}) \subset FT(\bar{x})$; that is, there exists $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in F(\bar{y})$. This completes the proof. $\square$

As a consequence of the above Theorem 4.6, we have the following generalized variational inequality.

Corollary 4.9. Let (X, φ_N) be a bounded FC-space, and let $T \in KKM(X, X)$ be a k -set contraction map, $0 < k < 1$, with $\overline{T(X)} \subset X$. If $\varphi, \psi : X \times X \rightarrow R \cup \{-\infty, +\infty\}$ are two real-valued mappings satisfying the following:

- (i) $\varphi(x, y) \leq 0$ for each $(x, y) \in \mathcal{G}_T$,
- (ii) for fixed $x \in X$, the mapping $y \mapsto \varphi(x, y)$ is lower semicontinuous on K for each compact subset K of X ,
- (iii) for fixed $y \in X$, for each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$ and $N_1 = \{x_{i_0}, \dots, x_{i_k}\} \subset N$, $\bar{x} \in \varphi_N(\Delta_k)$ such that $\varphi(\bar{x}, y) \leq 0$ implies that there exists $x \in N_1$ such that $\varphi(x, y) \leq 0$.

Then for the precompact-inducing FC-subspace X_∞ of X , there exists $\bar{y} \in X_\infty$ such that $\varphi(x, \bar{y}) \leq 0$ for each $x \in X_\infty$.

Proof. Define $F, S : X \rightarrow 2^X$ by

$$\begin{aligned} S(x) &= \{y \in X : \varphi(x, y) \leq 0\} \quad \text{for each } x \in X, \\ F(x) &= \{y \in X : \varphi(x, y) \leq 0\} \quad \text{for each } x \in X. \end{aligned} \tag{4.5}$$

By condition (i), we have $\mathcal{G}_T \subset \mathcal{G}_S$, and by condition (ii), $F(x)$ is compactly closed for each $x \in X$. The condition (iii) implies that for each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$ and $N_1 = \{x_{i_0}, \dots, x_{i_k}\} \subset N$, $S(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^k F(x_{i_j})$, and then $T(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^k F(x_{i_j})$; that is, F is a generalized F -KKM mapping with respect to T .

Thus, all conditions of Theorem 4.3 are satisfied. By Theorem 4.3, and for the precompact-inducing FC-subspace X_∞ of X , we have that $\overline{T(X_\infty)} \cap (\bigcap \{F(x) : x \in X_\infty\}) \neq \emptyset$. Let $\bar{y} \in \overline{T(X_\infty)} \cap (\bigcap \{F(x) : x \in X_\infty\})$, and hence we have $\varphi(x, \bar{y}) \leq 0$ for each $x \in X_\infty$. This completes the proof. $\square$

Applying Theorem 4.8 and Lemma 3.4, we have the following result.

Corollary 4.10. *Let $(X, \mathcal{U}, \varphi_N)$ be a locally FC-space, and let Y be a topological space. If $T : X \rightarrow 2^Y$, $F : Y \rightarrow 2^X$ are two Φ -mappings, and T is compact and closed, then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in T(\bar{x})$ and $\bar{x} \in F(\bar{y})$.*

From Corollary 4.10, we have the following result.

Corollary 4.11. *Let $(X, \mathcal{U}, \varphi_N)$ be a locally FC-space, and let (Y, φ_M) be a compact FC-space. If $f, g, p, q : X \times Y \rightarrow \mathbb{R}$ are four real-valued functions, and a, b be two real numbers. Suppose the following conditions hold:*

- (i) $g(x, y) \leq f(x, y)$ and $p(x, y) \leq q(x, y)$ for all $x \in X, y \in Y$,
- (ii) for each $x \in X$, f be FC-quasiconcave on Y and for each $y \in Y$, p be FC-quasiconvex on X ,
- (iii) for each $x \in X$, g be transfer compactly lower semicontinuous in x and for each $y \in Y$, q be transfer compactly upper semicontinuous in y ,
- (iv) f is upper semicontinuous on $X \times Y$.

Then one of the following statements holds:

- (1) there exists $\mu \in X$ such that $g(\mu, y) \leq a$ for each $y \in Y$,
- (2) there exists $\nu \in Y$ such that $q(x, \nu) \geq b$ for each $x \in X$,
- (3) there exists $(\mu, \nu) \in X \times Y$ such that $f(\mu, \nu) \geq a$ and $p(\mu, \nu) \leq b$.

Proof. Let $S, T : X \rightarrow 2^Y$ and $H, F : Y \rightarrow 2^X$ be defined by

$$\begin{aligned} S(x) &= \{y \in Y : g(x, y) - a > 0\} \quad \text{for each } x \in X, \\ T(x) &= \{y \in Y : f(x, y) - a \geq 0\} \quad \text{for each } x \in X, \\ H(y) &= \{x \in X : q(x, y) - b < 0\} \quad \text{for each } y \in Y, \\ F(y) &= \{x \in X : p(x, y) - b \leq 0\} \quad \text{for each } y \in Y. \end{aligned} \tag{4.6}$$

By the condition (iii), S^{-1} is transfer compactly open valued on Y and H^{-1} is transfer compactly open valued on X . By the condition (i), we have $S(x) \subset T(x)$ for each $x \in X$ and $H(y) \subset F(y)$ for each $y \in Y$. By the condition (ii), $T(x)$ is FC-subspace of Y for each $x \in X$, and so $F(x)$ is FC-subspace of X for each $y \in Y$.

Suppose that the Statements (1) and (2) are false. Then $S(x) \neq \emptyset$ for each $x \in X$ and $H(y) \neq \emptyset$ for each $y \in Y$. So, we conclude that T is a Φ -mapping with a companion mapping S and F is a Φ -mapping with a companion mapping H . By the Condition (iv), T is closed. Thus, all conditions of Corollary 4.9 are satisfied. By Corollary 4.9, there exists $(\mu, \nu) \in X \times Y$ such that $\nu \in T(\mu)$ and $\mu \in F(\nu)$; that is, $f(\mu, \nu) \geq a$ and $p(\mu, \nu) \leq b$. This completes the proof. $\square$

From Corollary 4.11, we have the following minimax theorem.

Corollary 4.12. *Let $(X, \mathcal{M}, \varphi_N)$ be a locally FC-space, and let (Y, φ_M) be a compact FC-space. If $f, g, p, q : X \times Y \rightarrow \mathbb{R}$ are four real-valued functions, and let a, b be two real numbers. Suppose the following conditions hold*

- (i) $g(x, y) \leq f(x, y) \leq p(x, y) \leq q(x, y)$ for all $x \in X, y \in Y$,
- (ii) for each $x \in X$, let f be FC-quasiconcave on Y and for each $y \in Y$, let p be FC-quasiconvex on X ,
- (iii) for each $x \in X$, let g be transfer compactly lower semicontinuous in x and for each $y \in Y$, let q be transfer compactly upper semicontinuous in y ,
- (iv) f is upper semicontinuous on $X \times Y$.

Then

$$\inf_{x \in X} \sup_{y \in Y} g(x, y) \leq \sup_{y \in Y} \inf_{x \in X} q(x, y). \tag{4.7}$$

Proof. Let $\varepsilon > 0$ and let

$$a = \inf_{x \in X} \sup_{y \in Y} g(x, y) - \varepsilon, \quad b = \sup_{y \in Y} \inf_{x \in X} q(x, y) + \varepsilon. \tag{4.8}$$

Then for each $x \in X$, there exists $y \in Y$ such that $g(x, y) > a$, and for each $y \in Y$, there exists $x \in X$ such that $q(x, y) < b$. Therefore, Conclusions (1) and (2) of Corollary 4.11 are false. So there exist $(\mu, \nu) \in X \times Y$ such that $f(\mu, \nu) \geq a$ and $p(\mu, \nu) \leq b$; that is

$$f(\mu, \nu) \geq \inf_{x \in X} \sup_{y \in Y} g(x, y) - \varepsilon, \quad p(\mu, \nu) \leq \sup_{y \in Y} \inf_{x \in X} q(x, y) + \varepsilon. \quad (4.9)$$

So, by Condition (i), we have

$$\inf_{x \in X} \sup_{y \in Y} g(x, y) - \varepsilon < \sup_{y \in Y} \inf_{x \in X} q(x, y) + \varepsilon. \quad (4.10)$$

Since ε is arbitrary positive number, by letting $\varepsilon \downarrow 0$, we get

$$\inf_{x \in X} \sup_{y \in Y} g(x, y) \leq \sup_{y \in Y} \inf_{x \in X} q(x, y). \quad (4.11)$$

This completes the proof. □

Remark 4.13. Theorem 4.6 (resp., Theorems 4.7 and 4.8, Corollaries 4.9, 4.10, 4.11, and 4.12) generalize Theorem 4 (resp., Theorems 7, 6, 5, 8, 9, and 10) in [5] in several aspects.

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References

- [1] H. Knaster, C. Kuratowski, and S. Mazurkiewicz, "Ein Beweis des Fixpunktsatzes für n-dimensional simplexe," *Fundamenta Mathematicae*, vol. 14, pp. 132–137, 1929.
- [2] K. Fan, "A generalization of Tychonoff's fixed point theorem," *Mathematische Annalen*, vol. 142, pp. 305–310, 1961.
- [3] T. H. Chang and C. L. Yen, "KKM property and fixed point theorems," *Journal of Mathematical Analysis and Applications*, vol. 203, no. 1, pp. 224–235, 1996.
- [4] L. J. Lin and H. I. Chen, "Coincidence theorems for families of multimaps and their applications to equilibrium problems," *Abstract and Applied Analysis*, vol. 2003, no. 5, pp. 295–309, 2003.
- [5] C. M. Chen and T. H. Chang, "Some results for the family $KKM(X, Y)$ and the Φ -mapping," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 92–101, 2007.
- [6] A. Amini, M. Fakhar, and J. Zafarani, "KKM mappings in metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 60, no. 6, pp. 1045–1052, 2005.
- [7] X. P. Ding, "Abstract convexity and generalizations of Himmelberg type fixed-point theorems," *Computers & Mathematics with Applications*, vol. 41, no. 3-4, pp. 497–504, 2001.
- [8] X. P. Ding, "Generalizations of Himmelberg type fixed point theorems in locally FC-spaces," *Journal of Sichuan Normal University*, vol. 29, no. 1, pp. 1–6, 2006.
- [9] X. P. Ding and T. M. Ding, "KKM type theorems and generalized vector equilibrium problems in noncompact FC-spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 1230–1245, 2007.
- [10] J. C. Jeng, H. C. Hsu, and Y. Y. Huang, "Fixed point theorems for multifunctions having KKM property on almost convex sets," *Journal of Mathematical Analysis and Applications*, vol. 319, no. 1, pp. 187–198, 2006.

- [11] X. P. Ding, "System of generalized vector quasi-equilibrium problems in locally FC -space," *Acta Mathematica Sinica*. In press.
- [12] X. P. Ding, "Maximal element theorems in product FC -spaces and generalized games," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 29–42, 2005.
- [13] X. P. Ding, "Collective fixed points and a system of coincidence theorems in product FC -spaces and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 11, pp. 2604–2617, 2007.