Research Article

(H, ϕ) - η -Accretive Mappings and a New System of Generalized Variational Inclusions with (H, ϕ) - η -Accretive Mappings in Banach Spaces

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We introduce a new class of generalized accretive mappings, named (H, ϕ) - η -accretive mappings, in Banach spaces. We define a resolvent operator associated with (H, ϕ) - η -accretive mappings and show its Lipschitz continuity. We also introduce and study a new system of generalized variational inclusions with (H, ϕ) - η -accretive mappings in Banach spaces. By using the resolvent operator technique associated with (H, ϕ) - η -accretive mappings, we construct a new iterative algorithm for solving this system of generalized variational inclusions in Banach spaces. We also prove the existence of solutions for the generalized variational inclusions and the convergence of iterative sequences generated by algorithm. Our results improve and generalize many known corresponding results.

1. Introduction

Variational inequalities and variational inclusions are among the most interesting and important mathematical problems and have been studied intensively in the past years since they have wide applications in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences, and so forth (see, e.g., [1–4]).

Recently Ding [5], Huang and Fang [6], Verma [7], Fang and Huang [8, 9], Huang and Fang [10], Fang et al. [11], Kazmi and Khan [12], and Lan et al. [13, 14] introduced the concepts of η -subdifferential operators, maximal η -monotone operators, A -monotone operators, and (H, η) -monotone operators in Hilbert spaces, H -accretive operators, generalized m -accretive mappings, (H, η) -accretive operators, P - η -accretive operators, and (A, η) -accretive mappings in Banach spaces, and their resolvent operators, respectively. In [15], Luo

and Huang introduced a new concept of (H, ϕ) - η -monotone mappings in Banach spaces and defined the proximal mapping associated with (H, ϕ) - η -monotone mappings.

Motivated and inspired by the research work going on this field, in this paper, we introduce a new concept of (H, ϕ) - η -accretive mappings and give the definition of its resolvent operator in Banach spaces. We also introduce and study a new system of generalized variational inclusions with (H, ϕ) - η -accretive mappings in Banach spaces, and we construct a new iterative algorithm for solving this system of generalized variational inclusions in Banach spaces. We also prove the existence of solutions for the generalized variational inclusions and the convergence of iterative sequences generated by algorithm. The results in this paper improve and extend some known results in the literature.

2. Preliminaries

Let E be a real Banach space equipped with norm $\|\cdot\|$; let E^* be the topological dual space of E ; let $\langle \cdot, \cdot \rangle$ be the pair between E and E^* ; let 2^E be the power set of E ; let $\widetilde{H}(\cdot, \cdot)$ be the Hausdorff metric on $CB(E)$ defined by

$$\widetilde{H}(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \right\}, \quad \forall A, B \in CB(E). \quad (2.1)$$

Definition 2.1 (see [16, 17]). For $q > 1$, a mapping $J_q : E \rightarrow 2^{E^*}$ is said to be generalized duality mapping if it is defined by

$$J_q(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^q, \|x\|^{q-1} = \|f\| \right\}, \quad \forall x \in E. \quad (2.2)$$

In particular, J_2 is the usual normalized duality mapping on E .
It is well known that

$$J_q(x) = \|x\|^{q-1} J_2(x), \quad \forall x (\neq 0) \in E. \quad (2.3)$$

Note that if $E = \mathcal{H}$ is a real Hilbert space, then J_2 becomes the identity mapping on $\mathcal{H}$.

Definition 2.2 (see [18]). A Banach space E is called smooth, if for every $x \in E$ with $\|x\| = 1$, there exists a unique $f \in E^*$ such that $\|f\| = f(x) = 1$. The modulus of E is the function $\rho_E : [0, \infty) \rightarrow [0, \infty)$, defined by

$$\rho_E(\tau) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau \right\}. \quad (2.4)$$

Definition 2.3 (see [17]). The Banach space E is said to be

(i) uniformly smooth, if

$$\lim_{\tau \rightarrow 0} \frac{\rho_E(\tau)}{\tau} = 0; \quad (2.5)$$

(ii) q -uniformly smooth, for $q > 1$, if there exists a constant $\tau > 0$ such that

$$\rho_E(\tau) \leq c\tau^q, \quad \tau \in [0, \infty). \quad (2.6)$$

It is well known (see [16]) that

$$L_q(\text{or } l_q) \text{ is } \begin{cases} q\text{-uniformly smooth, if } 1 < q \leq 2, \\ 2\text{-uniformly smooth, if } q \geq 2. \end{cases} \quad (2.7)$$

Note that if E is uniformly smooth, J_q becomes single-valued, and J_q is single-valued if E^* is strictly convex. In the sequel, unless otherwise specified, we always suppose that E is a real Banach space such that J_q is single-valued.

Lemma 2.4 (see [19]). *Let $q > 1$ be a real number and let E be a smooth Banach space. Then E is q -uniformly smooth if and only if there exists a constant $c_q > 0$ such that for every $x, y \in E$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + c_q\|y\|^q. \quad (2.8)$$

Definition 2.5 (see [9, 20]). Let $P : E \rightarrow E$ be a single-valued mapping. P is said to be

(i) accretive if

$$\langle P(x) - P(y), J_q(x - y) \rangle \geq 0, \quad \text{iff } x, y \in E; \quad (2.9)$$

(ii) strictly accretive if P is accretive and

$$\langle P(x) - P(y), J_q(x - y) \rangle = 0 \quad \text{if } x = y; \quad (2.10)$$

(iii) r -strongly accretive if there exists a constant $r > 0$ such that

$$\langle P(x) - P(y), J_q(x - y) \rangle \geq r\|x - y\|^q, \quad \forall x, y \in E; \quad (2.11)$$

(iv) m -relaxed accretive if there exists a constant $m > 0$ such that

$$\langle P(x) - P(y), J_q(x - y) \rangle \geq -m\|x - y\|^q, \quad \forall x, y \in E; \quad (2.12)$$

(v) (α, ξ) -relaxed cocoercive if there exist constants $\alpha, \xi > 0$ such that

$$\langle P(x) - P(y), J_q(x - y) \rangle \geq -\alpha\|P(x) - P(y)\|^q + \xi\|x - y\|^q, \quad \forall x, y \in E; \quad (2.13)$$

(vi) s -Lipschitz continuous if there exists a constant $s > 0$ such that

$$\|P(x) - P(y)\| \leq s\|x - y\|, \quad \forall x, y \in E. \quad (2.14)$$

Definition 2.6 (see [12]). A mapping $\eta : E \times E \rightarrow E$ is said to be τ -Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau \|x - y\|, \quad \forall x, y \in E. \quad (2.15)$$

Definition 2.7 (see [20]). Let $\eta : E \times E \rightarrow E$ and $P : E \rightarrow E$ be single-valued mappings. Then a multi-valued mapping $T : E \rightarrow 2^E$ is said to be

(i) η -accretive if

$$\langle u - v, J_q(\eta(x, y)) \rangle \geq 0, \quad \forall x, y \in E, \quad \forall u \in T(x), \quad v \in T(y); \quad (2.16)$$

(ii) strictly η -accretive if

$$\langle u - v, J_q(\eta(x, y)) \rangle \geq 0, \quad \forall x, y \in E, \quad \forall u \in T(x), \quad v \in T(y), \quad (2.17)$$

and equality holds if and only if $x = y$;

(iii) γ -strongly η -accretive if there exists a constant $\gamma > 0$ such that

$$\langle u - v, J_q(\eta(x, y)) \rangle \geq \gamma \|x - y\|^q, \quad \forall x, y \in E, \quad \forall u \in T(x), \quad v \in T(y); \quad (2.18)$$

(iv) m -relaxed η -accretive if there exists a constant $m > 0$ such that

$$\langle u - v, J_q(\eta(x, y)) \rangle \geq -m \|x - y\|^q, \quad \forall x, y \in E, \quad \forall u \in T(x), \quad v \in T(y); \quad (2.19)$$

(v) η - m -accretive if T is η -accretive and $(I + \rho T)(E) = E$ holds for all $\rho > 0$;

(vi) P - η -accretive if T is η -accretive and $(P + \rho T)(E) = E$ holds for all $\rho > 0$.

Definition 2.8. Let $E_1, E_2, \dots, E_n$ be real Banach spaces, and, for $i = 1, 2, \dots, n$, let $N_i : \prod_{j=1}^n E_j \rightarrow E_i$ be a single-valued mapping. Then N_i is said to be ξ_{ij} -Lipschitz continuous in the j th argument, if there exists a constant $\xi_{ij} > 0$ such that

$$\begin{aligned} & \|N_i(x_1, \dots, x_{j-1}, y_{j1}, x_{j+1}, \dots, x_n) - N_i(x_1, \dots, x_{j-1}, y_{j2}, x_{j+1}, \dots, x_n)\| \\ & \leq \xi_{ij} \|y_{j1} - y_{j2}\|, \quad \forall y_{j1}, y_{j2} \in E_j, \quad x_i \in E_i \quad (i \in \{1, 2, \dots, n\}, i \neq j). \end{aligned} \quad (2.20)$$

Definition 2.9. Let $A : E \rightarrow \text{CB}(E)$ be a set-valued mapping. A is said to be $\widetilde{H}$ -Lipschitz continuous, if there exists a constant $t > 0$ such that

$$\widetilde{H}(A(x), B(y)) \leq t \|x - y\|, \quad \forall x, y \in E, \quad (2.21)$$

where $\widetilde{H}(\cdot, \cdot)$ denotes the Hausdorff metric on $\text{CB}(E)$.

Definition 2.10. Let $\mathcal{H}$ be a Hilbert space, and let $H : \mathcal{H} \rightarrow \mathcal{H}$ be a single-valued mapping. H is said to be

(i) coercive if

$$\lim_{\|x\| \rightarrow \infty} \frac{\langle H(x), x \rangle}{\|x\|} = +\infty; \quad (2.22)$$

(ii) hemicontinuous if for any fixed $x, y, z \in \mathcal{H}$, the function $t \rightarrow \langle H(x + ty), z \rangle$ is continuous at 0^+ .

3. (H, ϕ) - η -Accretive Mappings

In this section, we will introduce a new class of generalized accretive mappings, (H, ϕ) - η -accretive mappings, and discuss some properties of (H, ϕ) - η -accretive mappings.

Definition 3.1. Let E be a Banach space and let $H, \phi : E \rightarrow E$, $\eta : E \times E \rightarrow E$ be single-valued mappings and $M : E \rightarrow 2^E$ a multi-valued mapping. The mapping M is said to be a (H, ϕ) - η -accretive mapping, if $\phi \circ M$ is η -accretive and $(H + \phi \circ M)(E) = E$.

Remark 3.2. (i) If M is η -accretive and $\phi(x) = \lambda x$, for all $x \in E$, $\lambda > 0$, then (H, ϕ) - η -accretive mapping reduces to the P - η -accretive mapping studied by Kazmi and Khan [12].

(ii) If $\eta(x, y) = x - y$, for all $x, y \in E$, M is η -accretive, and $\phi(x) = \lambda x$, for all $x \in E$, $\lambda > 0$, then (H, ϕ) - η -accretive mapping reduces to the H -accretive mapping studied by Fang and Huang [9].

(iii) If M is m -relaxed η -accretive and $\phi(x) = \lambda x$, for all $x \in E$, $\lambda > 0$, then (H, ϕ) - η -accretive mapping reduces to the (A, η) -accretive mapping studied by Lan et al. [14].

Similarly, we give the following definition.

Definition 3.3. Let $H, \phi : E \rightarrow E$ be single-valued mappings and $M : E \rightarrow 2^E$ a multi-valued mapping. The mapping M is said to be a (H, ϕ) -accretive mapping, if $\phi \circ M$ is accretive and $(H + \phi \circ M)(E) = E$.

Example 3.4. Let $\mathcal{H}$ be a Hilbert space and for every $x, y \in \mathcal{H}$, $\eta(x, y) = x - y$, $\phi(x) = \lambda x$, where $\lambda > 0$ is a constant. Let $M : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be a maximal monotone mapping and $H : \mathcal{H} \rightarrow \mathcal{H}$ a bounded, coercive, hemicontinuous, and α -strongly η -accretive mapping. Then it follows from Corollary 32.26 of [21] that M is (H, ϕ) - η -accretive mapping.

Theorem 3.5. Let $\phi : E \rightarrow E$, $\eta : E \times E \rightarrow E$ be single-valued mappings, $H : E \rightarrow E$ a strictly η -accretive mapping, and $M : E \rightarrow 2^E$ a (H, ϕ) - η -accretive mapping. Then $(H + \phi \circ M)^{-1}$ is a single-valued mapping.

Proof. For any given $z \in E$, let $x, y \in (H + \phi \circ M)^{-1}(z)$. It follows that

$$z - H(x) \in \phi \circ M(x), \quad z - H(y) \in \phi \circ M(y). \quad (3.1)$$

Then η -accretivity of $\phi \circ M$ implies that

$$\langle z - H(x) - (z - H(y)), J_q(\eta(x, y)) \rangle = \langle -H(x) + H(y), J_q(\eta(x, y)) \rangle \geq 0. \quad (3.2)$$

This implies that $x = y$ and so $(H + \phi \circ M)^{-1}$ is a single-valued mapping. This completes the proof. $\square$

By Theorem 3.5, we can define the resolvent operator $J_{M,\phi}^{H,\eta}$ associated with an (H, ϕ) - η -accretive mapping M as follows.

Definition 3.6. Let $\phi : E \rightarrow E$, $\eta : E \times E \rightarrow E$ be single-valued mappings, let $H : E \rightarrow E$ be a strictly η -accretive mapping, and let $M : E \rightarrow 2^E$ be a (H, ϕ) - η -accretive mapping. A resolvent operator $J_{M,\phi}^{H,\eta} : E \rightarrow E$ is defined by

$$J_{M,\phi}^{H,\eta}(x) = (H + \phi \circ M)^{-1}(x), \quad \forall x \in E. \quad (3.3)$$

Remark 3.7. (i) If M is η -accretive and $\phi(x) = \lambda x$, for all $x \in E$, $\lambda > 0$, then the resolvent operator $J_{M,\phi}^{H,\eta}$ reduces to the P - η -proximal point mapping $J_{\eta,\rho}^M$ introduced by Kazmi and Khan [12].

(ii) If $\eta(x, y) = x - y$, for all $x, y \in E$, M is η -accretive, and $\phi(x) = \lambda x$, for all $x \in E$, $\lambda > 0$, then the resolvent operator $J_{M,\phi}^{H,\eta}$ reduces to the proximal-point mapping introduced by Fang and Huang [9].

(iii) If M is m -relaxed η -accretive and $\phi(x) = \lambda x$, for all $x \in E$, $\lambda > 0$, then the resolvent operator $J_{M,\phi}^{H,\eta}$ reduces to the resolvent operator $R_{\eta,M}^{\rho,A}$ introduced by Lan et al. [14].

Theorem 3.8. Let $\phi : E \rightarrow E$ be a single-valued mapping, let $\eta : E \times E \rightarrow E$ be a τ -Lipschitz continuous mapping, let $H : E \rightarrow E$ be a γ -strongly η -accretive mapping, and let $M : E \rightarrow 2^E$ be a (H, ϕ) - η -accretive mapping. Then the resolvent operator $J_{M,\phi}^{H,\eta}$ is Lipschitz continuous with constant τ^{q-1}/γ , that is,

$$\|J_{M,\phi}^{H,\eta}(x) - J_{M,\phi}^{H,\eta}(y)\| \leq \frac{\tau^{q-1}}{\gamma} \|x - y\|, \quad \forall x, y \in E. \quad (3.4)$$

Proof. Let x, y be any given points in E , it follows from Definition 3.6 that

$$J_{M,\phi}^{H,\eta}(x) = (H + \phi \circ M)^{-1}(x), \quad J_{M,\phi}^{H,\eta}(y) = (H + \phi \circ M)^{-1}(y). \quad (3.5)$$

This implies that

$$\begin{aligned} x - H(J_{M,\phi}^{H,\eta}(x)) &\in \phi \circ M(J_{M,\phi}^{H,\eta}(x)), \\ y - H(J_{M,\phi}^{H,\eta}(y)) &\in \phi \circ M(J_{M,\phi}^{H,\eta}(y)). \end{aligned} \quad (3.6)$$

Since M is (H, ϕ) - η -accretive, we have

$$\begin{aligned} \langle x - H(J_{M,\phi}^{H,\eta}(x)) - (y - H(J_{M,\phi}^{H,\eta}(y))), J_q(\eta(J_{M,\phi}^{H,\eta}(x), J_{M,\phi}^{H,\eta}(y))) \rangle &= \\ \langle x - y - (H(J_{M,\phi}^{H,\eta}(x)) - H(J_{M,\phi}^{H,\eta}(y))), J_q(\eta(J_{M,\phi}^{H,\eta}(x), J_{M,\phi}^{H,\eta}(y))) \rangle &\geq 0. \end{aligned} \quad (3.7)$$

The inequality above implies that

$$\begin{aligned}
 \|x - y\| \left\| \eta \left(J_{M,\phi}^{H,\eta}(x), J_{M,\phi}^{H,\eta}(y) \right) \right\|^{q-1} &= \|x - y\| \left\| J_q \left(\eta \left(J_{M,\phi}^{H,\eta}(x), J_{M,\phi}^{H,\eta}(y) \right) \right) \right\| \\
 &\geq \left\langle x - y, J_q \left(\eta \left(J_{M,\phi}^{H,\eta}(x), J_{M,\phi}^{H,\eta}(y) \right) \right) \right\rangle \\
 &\geq \left\langle H \left(J_{M,\phi}^{H,\eta}(x) \right) - H \left(J_{M,\phi}^{H,\eta}(y) \right), J_q \left(\eta \left(J_{M,\phi}^{H,\eta}(x), J_{M,\phi}^{H,\eta}(y) \right) \right) \right\rangle \\
 &\geq \gamma \left\| J_{M,\phi}^{H,\eta}(x) - J_{M,\phi}^{H,\eta}(y) \right\|^q.
 \end{aligned} \tag{3.8}$$

Since η is Lipschitz continuous with a constant τ , we have

$$\left\| \eta \left(J_{M,\phi}^{H,\eta}(x), J_{M,\phi}^{H,\eta}(y) \right) \right\| \leq \tau \left\| J_{M,\phi}^{H,\eta}(x) - J_{M,\phi}^{H,\eta}(y) \right\|. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\|x - y\| \tau^{q-1} \left\| J_{M,\phi}^{H,\eta}(x) - J_{M,\phi}^{H,\eta}(y) \right\|^{q-1} \geq \gamma \left\| J_{M,\phi}^{H,\eta}(x) - J_{M,\phi}^{H,\eta}(y) \right\|^q. \tag{3.10}$$

Hence, we get

$$\left\| J_{M,\phi}^{H,\eta}(x) - J_{M,\phi}^{H,\eta}(y) \right\| \leq \frac{\tau^{q-1}}{\gamma} \|x - y\|. \tag{3.11}$$

This completes the proof. $\square$

4. A New System of Generalized Variational Inclusions

In this section, we will introduce a new system of generalized variational inclusions with (H, ϕ) - η -accretive mappings and construct a new iterative algorithm for solving this system of generalized variational inclusions. In what follows, for each $i = 1, 2, 3$, suppose that E_i is a Banach space, $H_i, f_i, \phi_i, p_i : E_i \rightarrow E_i$, $\eta_i : E_i \times E_i \rightarrow E_i$, $F_i, N_i : \prod_{k=1}^3 E_k \rightarrow E_i$ are single-valued mappings, $T_{ki} : E_k \rightarrow \text{CB}(E_i)$ is a set-valued mapping, and $M_i : E_i \times E_i \rightarrow 2^{E_i}$ is a (H_i, ϕ_i) - η_i -accretive mapping in the second argument. Assume that $f_i(E_i) \cap \text{dom } M(w_i, \cdot) \neq \emptyset$, for each $w_i \in E_i$. We consider the following system of generalized variational inclusions. Find $(x_1, x_2, x_3, u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33})$ such that for each $i = 1, 2, 3$, $x_i \in E_i$, $u_{1i} \in T_{1i}(x_1)$, $u_{2i} \in T_{2i}(x_2)$, $u_{3i} \in T_{3i}(x_3)$, and

$$0 \in F_i(x_1 - p_1(x_1), x_2 - p_2(x_2), x_3 - p_3(x_3)) + N_i(u_{i1}, u_{i2}, u_{i3}) + M_i(x_i, f_i(x_i)). \tag{4.1}$$

The following are some special cases of problem (4.1).

(i) If $E_1 = \mathcal{H}_1$, $E_2 = \mathcal{H}_2$, and $E_3 = \mathcal{H}_3$ are three Hilbert spaces, and, for each $i = 1, 2, 3$, $p_i = 0$, $N_i = 0$, $M_1(x, f_1(x)) = \partial_{\eta_1} \varphi_1(x)$, for all $x \in E_1$, $M_2(y, f_2(y)) = \partial_{\eta_2} \varphi_2(y)$, for all $y \in E_2$, and $M_3(z, f_3(z)) = \partial_{\eta_3} \varphi_3(z)$, for all $z \in E_3$, where $\varphi_i : E_i \rightarrow R \cup \{\infty\}$ is a proper lower

semicontinuous and η_i -subdifferential function, $\partial_{\eta_1}\varphi_1(x)$ is the η_1 -subdifferential of φ_1 at x , $\partial_{\eta_2}\varphi_2(y)$ is the η_2 -subdifferential of φ_2 at y , and $\partial_{\eta_3}\varphi_3(z)$ is the η_3 -subdifferential of φ_3 at z , then SGVIP (4.1) reduces to the following system of variational inequalities, which is to find $(x_1, x_2, x_3) \in E_1 \times E_2 \times E_3$ such that

$$\begin{aligned} \langle F_1(x_1, x_2, x_3), \eta_1(a, x_1) \rangle + \varphi_1(a) - \varphi_1(x_1) &\geq 0, \quad \forall a \in E_1, \\ \langle F_2(x_1, x_2, x_3), \eta_2(b, x_2) \rangle + \varphi_2(b) - \varphi_2(x_2) &\geq 0, \quad \forall b \in E_2, \\ \langle F_3(x_1, x_2, x_3), \eta_3(c, x_3) \rangle + \varphi_3(c) - \varphi_3(x_3) &\geq 0, \quad \forall c \in E_3. \end{aligned} \quad (4.2)$$

If $\eta_1(a, x) = a - x$, for all $a, x \in E_1$, $\eta_2(b, y) = b - y$, for all $b, y \in E_2$, $\eta_3(c, z) = c - z$, for all $c, z \in E_3$, and $M_1(x, f_1(x)) = \partial\varphi_1(x)$ is the subdifferential of φ_1 at x , $M_2(y, f_2(y)) = \partial\varphi_2(y)$ is the subdifferential of φ_2 at y , and $M_3(z, f_3(z)) = \partial\varphi_3(z)$ is the subdifferential of φ_3 at z , then problem (4.2) reduces to the following system of variational inequalities, which is to find $(x_1, x_2, x_3) \in E_1 \times E_2 \times E_3$ such that

$$\begin{aligned} \langle F_1(x_1, x_2, x_3), a - x_1 \rangle + \varphi_1(a) - \varphi_1(x_1) &\geq 0, \quad \forall a \in E_1, \\ \langle F_2(x_1, x_2, x_3), b - x_2 \rangle + \varphi_2(b) - \varphi_2(x_2) &\geq 0, \quad \forall b \in E_2, \\ \langle F_3(x_1, x_2, x_3), c - x_3 \rangle + \varphi_3(c) - \varphi_3(x_3) &\geq 0, \quad \forall c \in E_3. \end{aligned} \quad (4.3)$$

If $M_1(x, f_1(x)) = \partial\delta_{K_1}(x)$, $M_2(y, f_2(y)) = \partial\delta_{K_2}(y)$, and $M_3(z, f_3(z)) = \partial\delta_{K_3}(z)$, for all $x \in K_1$, $y \in K_2$, and $z \in K_3$, where $K_1 \subset E_1$, $K_2 \subset E_2$, and $K_3 \subset E_3$ are three nonempty, closed, and convex subsets, δ_{K_1} , δ_{K_2} , and δ_{K_3} denote the indicator functions of K_1 , K_2 , and K_3 , respectively, then problem (4.3) reduces to the following system of variational inequalities, which is to find $(x_1, x_2, x_3) \in E_1 \times E_2 \times E_3$ such that

$$\begin{aligned} \langle F_1(x_1, x_2, x_3), a - x_1 \rangle &\geq 0, \quad \forall a \in K_1, \\ \langle F_2(x_1, x_2, x_3), b - x_2 \rangle &\geq 0, \quad \forall b \in K_2, \\ \langle F_3(x_1, x_2, x_3), c - x_3 \rangle &\geq 0, \quad \forall c \in K_3. \end{aligned} \quad (4.4)$$

If $E_1 = E_2 = E_3 = \mathcal{H}$ is a Hilbert space, $K_1 = K_2 = K_3 = K$ is a nonempty, closed, and convex subset, $F_1(x, y, z) = sT_1(y, z, x) + x - y$, $F_2(x, y, z) = tT_2(z, x, y) + y - z$, and $F_3(x, y, z) = rT_3(x, y, z) + z - x$, for all $x, y, z \in K$, where $T_1, T_2, T_3 : K \times K \times K \rightarrow \mathcal{H}$ are mappings on $K \times K \times K$, $r, s, t > 0$ are three numbers, then problem (4.4) reduces to the following system of variational inequalities, which is to find $x_1, x_2, x_3 \in K$ such that

$$\begin{aligned} \langle sT_1(x_2, x_3, x_1) + x_1 - x_2, a - x_1 \rangle &\geq 0, \quad \forall a \in K, \\ \langle tT_2(x_3, x_1, x_2) + x_2 - x_3, a - x_2 \rangle &\geq 0, \quad \forall a \in K, \\ \langle rT_3(x_1, x_2, x_3) + x_3 - x_1, a - x_3 \rangle &\geq 0, \quad \forall a \in K. \end{aligned} \quad (4.5)$$

Problem (4.5) was introduced and studied by Cho and Qin [22].

(ii) If $E_2 = E_3$, $F_1(\cdot, \cdot, \cdot) = F_1(\cdot, \cdot)$, $F_2(\cdot, \cdot, \cdot) = F_2(\cdot, \cdot)$, and $F_2 = F_3$ and for each $i = 1, 2, 3$, $p_i = 0$, $N_i = 0$, $M_2(y, f_2(y)) = M_3(z, f_3(z))$, for all $(y, z) \in E_2 \times E_3$, then SGVIP (4.1) reduces to the following system of variational inclusions, which is to find $(x_1, x_2) \in E_1 \times E_2$ such that

$$\begin{aligned} 0 &\in F_1(x_1, x_2) + M_1(x_1, f_1(x_1)), \\ 0 &\in F_2(x_1, x_2) + M_2(x_2, f_2(x_2)). \end{aligned} \quad (4.6)$$

If $E_1 = \mathcal{H}_1$ and $E_2 = \mathcal{H}_2$ are two Hilbert spaces and $M_1(x, f_1(x)) = \partial_{\eta_1} \varphi_1(x)$, for all $x \in E_1$, $M_2(y, f_2(y)) = \partial_{\eta_2} \varphi_2(y)$, for all $y \in E_2$, then the problem (4.6) reduces to the following system of variational inequalities, which is to find $(x_1, x_2) \in E_1 \times E_2$ such that

$$\begin{aligned} \langle F_1(x_1, x_2), \eta_1(a, x_1) \rangle + \varphi_1(a) - \varphi_1(x_1) &\geq 0, \quad \forall a \in E_1, \\ \langle F_2(x_1, x_2), \eta_2(b, x_2) \rangle + \varphi_2(b) - \varphi_2(x_2) &\geq 0, \quad \forall b \in E_2. \end{aligned} \quad (4.7)$$

If $\eta_1(a, x) = a - x$, for all $a, x \in E_1$, $\eta_2(b, y) = b - y$, for all $b, y \in E_2$, $M_1(x, f_1(x)) = \partial \varphi_1(x)$ is the subdifferential of φ_1 at x , and $M_2(y, f_2(y)) = \partial \varphi_2(y)$ is the subdifferential of φ_2 at y , then the problem (4.7) reduces to the following system of variational inequalities, which is to find $(x_1, x_2) \in E_1 \times E_2$ such that

$$\begin{aligned} \langle F_1(x_1, x_2), a - x_1 \rangle + \varphi_1(a) - \varphi_1(x_1) &\geq 0, \quad \forall a \in E_1, \\ \langle F_2(x_1, x_2), b - x_2 \rangle + \varphi_2(b) - \varphi_2(x_2) &\geq 0, \quad \forall b \in E_2. \end{aligned} \quad (4.8)$$

Problem (4.8) was introduced and studied by Cho et al. [23].

If $M_1(x, f_1(x)) = \partial \delta_{K_1}(x)$, $M_2(y, f_2(y)) = \partial \delta_{K_2}(y)$, for all $x \in K_1$, $y \in K_2$, where $K_1 \subset E_1$, $K_2 \subset E_2$ are two nonempty, closed, and convex subsets and $\partial \delta_{K_1}$ and $\partial \delta_{K_2}$ denote the indicator functions of K_1 , K_2 , respectively, then problem (4.8) reduces to the following system of variational inequalities, which is to find $(x_1, x_2) \in E_1 \times E_2$ such that

$$\begin{aligned} \langle F_1(x_1, x_2), a - x_1 \rangle &\geq 0, \quad \forall a \in K_1, \\ \langle F_2(x_1, x_2), b - x_2 \rangle &\geq 0, \quad \forall b \in K_2. \end{aligned} \quad (4.9)$$

Problem (4.9) is just the problem [24] with F_1 and F_2 being single-valued.

If $E_1 = E_2 = \mathcal{H}$ is a Hilbert space and $K_1 = K_2 = K$ is a nonempty, closed, and convex subset, $F_1(x, y) = \rho T(y, x) + x - y$ and $F_2(x, y) = \lambda T(x, y) + y - x$, for all $x, y \in K$, where $T : K \times K \rightarrow \mathcal{H}$ is a mapping on $K \times K$, $\rho, \lambda > 0$ are two numbers, then problem (4.9) reduces to the following problem, which is to find $x_1, x_2 \in K$ such that

$$\begin{aligned} \langle \rho T(x_2, x_1) + x_1 - x_2, a - x_1 \rangle &\geq 0, \quad \forall a \in K, \\ \langle \lambda T(x_1, x_2) + x_2 - x_1, a - x_2 \rangle &\geq 0, \quad \forall a \in K. \end{aligned} \quad (4.10)$$

Problem (4.10) was introduced and studied by Verma [25].

Lemma 4.1. Let, for $i = 1, 2, 3$, $\phi_i : E_i \rightarrow E_i$ be a single-valued mapping satisfying $\phi_i(x + y) = \phi_i(x) + \phi_i(y)$ and $\ker \phi_i = \{0\}$, $\eta_i : E_i \times E_i \rightarrow E_i$ a single-valued mapping, $H_i : E_i \rightarrow E_i$ a strictly η_i -accretive mapping and $M_i : E_i \times E_i \rightarrow 2^{E_i}$ a (H_i, ϕ_i) - η_i -accretive mapping in the second argument. Then $(x_1, x_2, x_3, u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33})$ in which $x_i \in E_i$, $u_{1i} \in T_{1i}(x_1)$, $u_{2i} \in T_{2i}(x_2)$, $u_{3i} \in T_{3i}(x_3)$ ($i = 1, 2, 3$) is a solution of the problem (4.1) if and only if

$$f_i(x_i) = J_{M_i(x_i, \cdot), \phi_i}^{H_i, \eta_i} [H_i \circ f_i(x_i) - \phi_i \circ F_i(x_1 - p_1(x_1), x_2 - p_2(x_2), x_3 - p_3(x_3)) - \phi_i \circ N_i(u_{i1}, u_{i2}, u_{i3})], \quad (4.11)$$

where $J_{M_i(x_i, \cdot), \phi_i}^{H_i, \eta_i} = (H_i + \phi_i \circ M_i(x_i, \cdot))^{-1}$.

Proof. The fact directly follows from Definition 3.6. $\square$

Algorithm 4.2. For any $x_i^0 \in E_i$, take $u_{1i}^0 \in T_{1i}(x_1^0)$, $u_{2i}^0 \in T_{2i}(x_2^0)$, and $u_{3i}^0 \in T_{3i}(x_3^0)$ ($i = 1, 2, 3$). For $i = 1, 2, 3$, let

$$x_i^1 = x_i^0 - f_i(x_i^0) + J_{M_i(x_i^0, \cdot), \phi_i}^{H_i, \eta_i} [H_i \circ f_i(x_i^0) - \phi_i \circ F_i(x_1^0 - p_1(x_1^0), x_2^0 - p_2(x_2^0), x_3^0 - p_3(x_3^0)) - \phi_i \circ N_i(u_{i1}^0, u_{i2}^0, u_{i3}^0)]. \quad (4.12)$$

Since $u_{1i}^0 \in T_{1i}(x_1^0)$, $u_{2i}^0 \in T_{2i}(x_2^0)$, $u_{3i}^0 \in T_{3i}(x_3^0)$ ($i = 1, 2, 3$), by Nodler's theorem [26], there exist $u_{1i}^1 \in T_{1i}(x_1^1)$, $u_{2i}^1 \in T_{2i}(x_2^1)$, $u_{3i}^1 \in T_{3i}(x_3^1)$ ($i = 1, 2, 3$), such that, for each $i = 1, 2, 3$,

$$\begin{aligned} \|u_{1i}^1 - u_{1i}^0\| &\leq (1 + 1) \widetilde{H}(T_{1i}(x_1^1), T_{1i}(x_1^0)), \\ \|u_{2i}^1 - u_{2i}^0\| &\leq (1 + 1) \widetilde{H}(T_{2i}(x_2^1), T_{2i}(x_2^0)), \\ \|u_{3i}^1 - u_{3i}^0\| &\leq (1 + 1) \widetilde{H}(T_{3i}(x_3^1), T_{3i}(x_3^0)). \end{aligned} \quad (4.13)$$

For $i = 1, 2, 3$, let

$$x_i^2 = x_i^1 - f_i(x_i^1) + J_{M_i(x_i^1, \cdot), \phi_i}^{H_i, \eta_i} [H_i \circ f_i(x_i^1) - \phi_i \circ F_i(x_1^1 - p_1(x_1^1), x_2^1 - p_2(x_2^1), x_3^1 - p_3(x_3^1)) - \phi_i \circ N_i(u_{i1}^1, u_{i2}^1, u_{i3}^1)]. \quad (4.14)$$

Again by Nodler's theorem [26], there exist $u_{1i}^2 \in T_{1i}(x_1^2)$, $u_{2i}^2 \in T_{2i}(x_2^2)$, $u_{3i}^2 \in T_{3i}(x_3^2)$ ($i = 1, 2, 3$), such that, for each $i = 1, 2, 3$,

$$\begin{aligned} \|u_{1i}^2 - u_{1i}^1\| &\leq \left(1 + \frac{1}{2}\right) \widetilde{H}(T_{1i}(x_1^2), T_{1i}(x_1^1)), \\ \|u_{2i}^2 - u_{2i}^1\| &\leq \left(1 + \frac{1}{2}\right) \widetilde{H}(T_{2i}(x_2^2), T_{2i}(x_2^1)), \\ \|u_{3i}^2 - u_{3i}^1\| &\leq \left(1 + \frac{1}{2}\right) \widetilde{H}(T_{3i}(x_3^2), T_{3i}(x_3^1)). \end{aligned} \quad (4.15)$$

By induction, we can compute the sequences $x_i^n, u_{1i}^n, u_{2i}^n, u_{3i}^n$ ($i = 1, 2, 3$) by the following iterative schemes such that, for each $i = 1, 2, 3$,

$$\begin{aligned}
 x_i^{n+1} &= x_i^n - f_i(x_i^n) + J_{M_i(x_i^n, \cdot), \phi_i}^{H_i, \eta_i} [H_i \circ f_i(x_i^n) - \phi_i \circ F_i(x_1^n - p_1(x_1^n), x_2^n - p_2(x_2^n), x_3^n - p_3(x_3^n)) \\
 &\quad - \phi_i \circ N_i(u_{1i}^n, u_{2i}^n, u_{3i}^n)], \\
 u_{1i}^n &\in T_{1i}(x_1^n), \quad \|u_{1i}^n - u_{1i}^{n-1}\| \leq \left(1 + \frac{1}{n}\right) \widetilde{H}(T_{1i}(x_1^n), T_{1i}(x_1^{n-1})), \\
 u_{2i}^n &\in T_{2i}(x_2^n), \quad \|u_{2i}^n - u_{2i}^{n-1}\| \leq \left(1 + \frac{1}{n}\right) \widetilde{H}(T_{2i}(x_2^n), T_{2i}(x_2^{n-1})), \\
 u_{3i}^n &\in T_{3i}(x_3^n), \quad \|u_{3i}^n - u_{3i}^{n-1}\| \leq \left(1 + \frac{1}{n}\right) \widetilde{H}(T_{3i}(x_3^n), T_{3i}(x_3^{n-1})),
 \end{aligned} \tag{4.16}$$

for all $n = 0, 1, 2, \dots$

Now we prove the existence of solution of the SGVIP (4.1) and the convergence of Algorithm 4.2.

Theorem 4.3. Let, for $i = 1, 2, 3$, E_i be a q -uniformly smooth Banach space and let $\phi_i : E_i \rightarrow E_i$ be a θ_i -Lipschitz continuous mapping satisfying $\phi_i(x+y) = \phi_i(x) + \phi_i(y)$ and $\ker \phi_i = \{0\}$. Let $\eta_i : E_i \times E_i \rightarrow E_i$ be τ_i -Lipschitz continuous, let $H_i : E_i \rightarrow E_i$ be a γ_i -strongly η_i -accretive and s_i -Lipschitz continuous mapping, $f_i : E_i \rightarrow E_i$ be (α_i, μ_i) -relaxed cocoercive and ξ_i -Lipschitz continuous, and let $p_i : E_i \rightarrow E_i$ be a strongly accretive mapping with constant δ_{p_i} and Lipschitz continuous with constant λ_{p_i} . Suppose that $F_i : \prod_{k=1}^3 E_k \rightarrow E_i$ is β_{ik} -Lipschitz continuous in the k th argument and $N_i : \prod_{k=1}^3 E_k \rightarrow E_i$ be ζ_{ik} -Lipschitz continuous in the k th argument for $k = 1, 2, 3$, $M_i : E_i \times E_i \rightarrow 2^{E_i}$ is a (H_i, ϕ_i) - η_i -accretive mapping in the second argument, and set-valued mappings $T_{1i} : E_1 \rightarrow CB(E_i)$, $T_{2i} : E_2 \rightarrow CB(E_i)$, $T_{3i} : E_3 \rightarrow CB(E_i)$ are $\widetilde{H}$ -Lipschitz continuous with constants $t_{1i} > 0$, $t_{2i} > 0$, $t_{3i} > 0$, respectively. In addition if for all $i = 1, 2, 3$, one has

$$\begin{aligned}
 &\|J_{M_i(x_{i1}, \cdot), \phi_i}^{H_i, \eta_i}(z_i) - J_{M_i(x_{i2}, \cdot), \phi_i}^{H_i, \eta_i}(z_i)\| \leq \delta_i \|x_{i1} - x_{i2}\|, \quad \forall x_{i1}, x_{i2}, z_i \in E_i, \tag{4.17} \\
 &0 < \left(1 + q\alpha_1 \xi_1^q - q\mu_1 + c_q \xi_1^q\right)^{1/q} + \delta_1 + \frac{\tau_1^{q-1}}{\gamma_1} \\
 &\quad \times \left(\left(s_1^q \xi_1^q + q\theta_1 \beta_{11} s_1^{q-1} \xi_1^{q-1} (1 - q\delta_{p_1} + c_q \lambda_{p_1}^q)^{1/q} + c_q \theta_1^q \beta_{11}^q (1 - q\delta_{p_1} + c_q \lambda_{p_1}^q)^{1/q} \right)^{1/q} \right. \\
 &\quad \left. + \sum_{j=1}^3 \theta_1 \xi_{1j} t_{1j} \right) + \left(\theta_2 \beta_{21} \frac{\tau_2^{q-1}}{\gamma_2} + \theta_3 \beta_{31} \frac{\tau_3^{q-1}}{\gamma_3} \right) (1 - q\delta_{p_1} + c_q \lambda_{p_1}^q)^{1/q} < 1, \\
 &0 < \left(1 + q\alpha_2 \xi_2^q - q\mu_2 + c_q \xi_2^q\right)^{1/q} + \delta_2 + \frac{\tau_2^{q-1}}{\gamma_2}
 \end{aligned}$$

$$\begin{aligned}
& \times \left(\left(s_2^q \xi_2^q + q\theta_2 \beta_{22} s_2^{q-1} \xi_2^{q-1} (1 - q\delta_{p_2} + c_q \lambda_{p_2}^q)^{1/q} + c_q \theta_2^q \beta_{22}^q (1 - q\delta_{p_2} + c_q \lambda_{p_2}^q) \right)^{1/q} \right. \\
& \quad \left. + \sum_{j=1}^3 \theta_2 \xi_{2j} t_{2j} \right) + \left(\theta_1 \beta_{12} \frac{\tau_1^{q-1}}{\gamma_1} + \theta_3 \beta_{32} \frac{\tau_3^{q-1}}{\gamma_3} \right) (1 - q\delta_{p_2} + c_q \lambda_{p_2}^q)^{1/q} < 1, \\
0 & < \left(1 + q\alpha_3 \xi_3^q - q\mu_3 + c_q \xi_3^q \right)^{1/q} + \delta_3 + \frac{\tau_3^{q-1}}{\gamma_3} \\
& \times \left(\left(s_3^q \xi_3^q + q\theta_3 \beta_{33} s_3^{q-1} \xi_3^{q-1} (1 - q\delta_{p_3} + c_q \lambda_{p_3}^q)^{1/q} + c_q \theta_3^q \beta_{33}^q (1 - q\delta_{p_3} + c_q \lambda_{p_3}^q) \right)^{1/q} \right. \\
& \quad \left. + \sum_{j=1}^3 \theta_3 \xi_{3j} t_{3j} \right) + \left(\theta_1 \beta_{13} \frac{\tau_1^{q-1}}{\gamma_1} + \theta_2 \beta_{23} \frac{\tau_2^{q-1}}{\gamma_2} \right) (1 - q\delta_{p_3} + c_q \lambda_{p_3}^q)^{1/q} < 1. \quad (4.18)
\end{aligned}$$

Then the problem (4.1) admits a solution $(x_1, x_2, x_3, u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33})$ and sequences $x_1^n, x_2^n, x_3^n, u_{11}^n, u_{12}^n, u_{13}^n, u_{21}^n, u_{22}^n, u_{23}^n, u_{31}^n, u_{32}^n, u_{33}^n$ converge to $x_1, x_2, x_3, u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33}$, respectively, where $x_i^n, u_{1i}^n, u_{2i}^n, u_{3i}^n$ ($i = 1, 2, 3$) are sequences generated by Algorithm 4.2.

Proof. For $i = 1, 2, 3$, let

$$D_i^n = H_i \circ f_i(x_i^n) - \phi_i \circ F_i(x_1^n - p_1(x_1^n), x_2^n - p_2(x_2^n), x_3^n - p_3(x_3^n)) - \phi_i \circ N_i(u_{i1}^n, u_{i2}^n, u_{i3}^n). \quad (4.19)$$

By Algorithm 4.2 and (4.17), we have

$$\begin{aligned}
& \|x_i^{n+1} - x_i^n\| \\
& = \left\| x_i^n - f_i(x_i^n) + J_{M_i(x_i^n, \cdot), \phi_i}^{H_i, \eta_i} [H_i \circ f_i(x_i^n) - \phi_i \circ F_i(x_1^n - p_1(x_1^n), x_2^n - p_2(x_2^n), x_3^n - p_3(x_3^n)) \right. \\
& \quad \left. - \phi_i \circ N_i(u_{i1}^n, u_{i2}^n, u_{i3}^n)] \right. \\
& \quad \left. - (x_i^{n-1} - f_i(x_i^{n-1}) \right. \\
& \quad \left. + J_{M_i(x_i^{n-1}, \cdot), \phi_i}^{H_i, \eta_i} [H_i \circ f_i(x_i^{n-1}) \right. \\
& \quad \left. - \phi_i \circ F_i(x_1^{n-1} - p_1(x_1^{n-1}), x_2^{n-1} - p_2(x_2^{n-1}), x_3^{n-1} - p_3(x_3^{n-1})) \right. \\
& \quad \left. - \phi_i \circ N_i(u_{i1}^{n-1}, u_{i2}^{n-1}, u_{i3}^{n-1})] \right\| \\
& \leq \|x_i^n - x_i^{n-1} - [f_i(x_i^n) - f_i(x_i^{n-1})]\| + \|J_{M_i(x_i^n, \cdot), \phi_i}^{H_i, \eta_i}(D_i^n) - J_{M_i(x_i^{n-1}, \cdot), \phi_i}^{H_i, \eta_i}(D_i^{n-1})\|
\end{aligned}$$

$$\begin{aligned}
& + \left\| J_{M_i(x_i^n, \cdot), \phi_i}^{H_i, \eta_i}(D_i^{n-1}) - J_{M_i(x_i^{n-1}, \cdot), \phi_i}^{H_i, \eta_i}(D_i^{n-1}) \right\| \\
& \leq \left\| x_i^n - x_i^{n-1} - [f_i(x_i^n) - f_i(x_i^{n-1})] \right\| + \frac{\tau_i^{q-1}}{\gamma_i} \|D_i^n - D_i^{n-1}\| + \delta_i \|x_i^n - x_i^{n-1}\|.
\end{aligned} \tag{4.20}$$

Since $f_i : E_i \rightarrow E_i$ is (α_i, μ_i) -relaxed cocoercive and ξ_i -Lipschitz continuous and $p_i : E_i \rightarrow E_i$ is δ_{p_i} -strongly accretive and λ_{p_i} -Lipschitz continuous, we have

$$\begin{aligned}
& \left\| x_i^n - x_i^{n-1} - [f_i(x_i^n) - f_i(x_i^{n-1})] \right\|^q \\
& \leq \left\| x_i^n - x_i^{n-1} \right\|^q - q \langle f_i(x_i^n) - f_i(x_i^{n-1}), J_q(x_i^n - x_i^{n-1}) \rangle + c_q \|f_i(x_i^n) - f_i(x_i^{n-1})\|^q \\
& \leq (1 + q\alpha_i\xi_i^q - q\mu_i + c_q\xi_i^q) \|x_i^n - x_i^{n-1}\|^q, \\
& \left\| x_i^n - x_i^{n-1} - [p_i(x_i^n) - p_i(x_i^{n-1})] \right\|^q \\
& \leq \left\| x_i^n - x_i^{n-1} \right\|^q - q \langle p_i(x_i^n) - p_i(x_i^{n-1}), J_q(x_i^n - x_i^{n-1}) \rangle + c_q \|p_i(x_i^n) - p_i(x_i^{n-1})\|^q \\
& \leq (1 - q\delta_{p_i} + c_q\lambda_{p_i}^q) \|x_i^n - x_i^{n-1}\|^q.
\end{aligned} \tag{4.21}$$

From (4.19), we have

$$\begin{aligned}
& \|D_i^n - D_i^{n-1}\| \\
& = \left\| H_i \circ f_i(x_i^n) - \phi_i \circ F_i(x_1^n - p_1(x_1^n), x_2^n - p_2(x_2^n), x_3^n - p_3(x_3^n)) - \phi_i \circ N_i(u_{i1}^n, u_{i2}^n, u_{i3}^n) \right. \\
& \quad \left. - (H_i \circ f_i(x_i^{n-1}) - \phi_i \circ F_i(x_1^{n-1} - p_1(x_1^{n-1}), x_2^{n-1} - p_2(x_2^{n-1}), x_3^{n-1} - p_3(x_3^{n-1})) \right. \\
& \quad \left. - \phi_i \circ N_i(u_{i1}^{n-1}, u_{i2}^{n-1}, u_{i3}^{n-1})) \right\| \\
& \leq \left\| H_i \circ f_i(x_i^n) - H_i \circ f_i(x_i^{n-1}) - (\phi_i \circ F_i(x_1^n - p_1(x_1^n), x_2^n - p_2(x_2^n), x_3^n - p_3(x_3^n)) \right. \\
& \quad \left. - \phi_i \circ F_i(x_1^{n-1} - p_1(x_1^{n-1}), x_2^{n-1} - p_2(x_2^{n-1}), x_3^{n-1} - p_3(x_3^{n-1}))) \right\| \\
& \quad + \left\| \phi_i \circ N_i(u_{i1}^n, u_{i2}^n, u_{i3}^n) - \phi_i \circ N_i(u_{i1}^{n-1}, u_{i2}^{n-1}, u_{i3}^{n-1}) \right\|.
\end{aligned} \tag{4.22}$$

Since $F_i : \prod_{k=1}^3 E_k \rightarrow E_i$ is β_{ik} -Lipschitz continuous in the k th argument and by continuity of $H_i, f_i, \phi_i (i = 1, 2, 3)$, we have

$$\begin{aligned}
& \left\| H_i \circ f_i(x_i^n) - H_i \circ f_i(x_i^{n-1}) - (\phi_i \circ F_i(x_1^n - p_1(x_1^n), x_2^n - p_2(x_2^n), x_3^n - p_3(x_3^n)) \right. \\
& \quad \left. - \phi_i \circ F_i(x_1^{n-1} - p_1(x_1^{n-1}), x_2^{n-1} - p_2(x_2^{n-1}), x_3^{n-1} - p_3(x_3^{n-1}))) \right\| \\
& \leq \left(s_i^q \xi_i^q + q\theta_i \beta_{ii} s_i^{q-1} \xi_i^{q-1} (1 - q\delta_{p_i} + c_q \lambda_{p_i}^q)^{1/q} + c_q \theta_i^q \beta_{ii}^q (1 - q\delta_{p_i} + c_q \lambda_{p_i}^q)^{1/q} \right) \|x_i^n - x_i^{n-1}\| \\
& \quad + \sum_{j=1}^{i-1} \theta_i \beta_{ij} (1 - q\delta_{p_j} + c_q \lambda_{p_j}^q)^{1/q} \|x_j^n - x_j^{n-1}\| + \sum_{j=i+1}^3 \theta_i \beta_{ij} (1 - q\delta_{p_j} + c_q \lambda_{p_j}^q)^{1/q} \|x_j^n - x_j^{n-1}\|.
\end{aligned} \tag{4.23}$$

It follows from the Lipschitz continuity of N_i and the $\tilde{H}$ -Lipschitz continuity of T_{ij} that

$$\begin{aligned}
& \left\| N_i(u_{i1}^n, u_{i2}^n, u_{i3}^n) - N_i(u_{i1}^{n-1}, u_{i2}^{n-1}, u_{i3}^{n-1}) \right\| \\
& \leq \left\| N_i(u_{i1}^n, u_{i2}^n, u_{i3}^n) - N_i(u_{i1}^{n-1}, u_{i2}^n, u_{i3}^n) \right\| \\
& \quad + \left\| N_i(u_{i1}^{n-1}, u_{i2}^n, u_{i3}^n) - N_i(u_{i1}^{n-1}, u_{i2}^{n-1}, u_{i3}^n) \right\| \\
& \quad + \left\| N_i(u_{i1}^{n-1}, u_{i2}^{n-1}, u_{i3}^n) - N_i(u_{i1}^{n-1}, u_{i2}^{n-1}, u_{i3}^{n-1}) \right\| \\
& \leq \xi_{i1} \|u_{i1}^n - u_{i1}^{n-1}\| + \xi_{i2} \|u_{i2}^n - u_{i2}^{n-1}\| + \xi_{i3} \|u_{i3}^n - u_{i3}^{n-1}\| \\
& = \sum_{j=1}^3 \xi_{ij} \|u_{ij}^n - u_{ij}^{n-1}\| \leq \sum_{j=1}^3 \xi_{ij} \left(1 + \frac{1}{n}\right) \tilde{H}(T_{ij}(x_i^n), T_{ij}(x_i^{n-1})) \\
& \leq \sum_{j=1}^3 \xi_{ij} t_{ij} \left(1 + \frac{1}{n}\right) \|x_i^n - x_i^{n-1}\|, \quad i = 1, 2, 3.
\end{aligned} \tag{4.24}$$

It follows from (4.20)–(4.24) that, for each $i = 1, 2, 3$,

$$\begin{aligned}
& \|x_i^{n+1} - x_i^n\| \\
& \leq \left[\left(1 + q\alpha_i \xi_i^q - q\mu_i + c_q \xi_i^q\right)^{1/q} + \delta_i \right. \\
& \quad \left. + \frac{\tau_i^{q-1}}{\gamma_i} \left(\left(s_i^q \xi_i^q + q\theta_i \beta_{ii} s_i^{q-1} \xi_i^{q-1} (1 - q\delta_{p_i} + c_q \lambda_{p_i}^q)^{1/q} \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + c_q \theta_i^q \beta_{ii}^q \left(1 - q\delta_{p_i} + c_q \lambda_{P_i}^q \right)^{1/q} + \sum_{j=1}^3 \theta_i \zeta_{ij} t_{ij} \left(1 + \frac{1}{n} \right) \Bigg] \\
& \times \left\| x_i^n - x_i^{n-1} \right\| + \sum_{j=1}^{i-1} \theta_i \beta_{ij} \frac{\tau_i^{q-1}}{\gamma_i} \left(1 - q\delta_{p_j} + c_q \lambda_{P_j}^q \right)^{1/q} \left\| x_j^n - x_j^{n-1} \right\| \\
& + \sum_{j=i+1}^3 \theta_i \beta_{ij} \frac{\tau_i^{q-1}}{\gamma_i} \left(1 - q\delta_{p_j} + c_q \lambda_{P_j}^q \right)^{1/q} \left\| x_j^n - x_j^{n-1} \right\|.
\end{aligned} \tag{4.25}$$

Therefore,

$$\begin{aligned}
& \sum_{i=1}^3 \left\| x_i^{n+1} - x_i^n \right\| \\
& \leq \sum_{i=1}^3 \left\{ \left[\left(1 + q\alpha_i \xi_i^q - q\mu_i + c_q \xi_i^q \right)^{1/q} + \delta_i \right. \right. \\
& \quad + \frac{\tau_i^{q-1}}{\gamma_i} \left(\left(s_i^q \xi_i^q + q\theta_i \beta_{ii} s_i^{q-1} \xi_i^{q-1} \left(1 - q\delta_{p_i} + c_q \lambda_{P_i}^q \right)^{1/q} + c_q \theta_i^q \beta_{ii}^q \right. \right. \\
& \quad \times \left. \left. \left(1 - q\delta_{p_i} + c_q \lambda_{P_i}^q \right)^{1/q} + \sum_{j=1}^3 \theta_i \zeta_{ij} t_{ij} \left(1 + \frac{1}{n} \right) \right) \right] \\
& \quad \times \left\| x_i^n - x_i^{n-1} \right\| + \sum_{j=1}^{i-1} \theta_i \beta_{ij} \frac{\tau_i^{q-1}}{\gamma_i} \left(1 - q\delta_{p_j} + c_q \lambda_{P_j}^q \right)^{1/q} \left\| x_j^n - x_j^{n-1} \right\| \\
& \quad \left. + \sum_{j=i+1}^3 \theta_i \beta_{ij} \frac{\tau_i^{q-1}}{\gamma_i} \left(1 - q\delta_{p_j} + c_q \lambda_{P_j}^q \right)^{1/q} \left\| x_j^n - x_j^{n-1} \right\| \right\} \\
& = \left[\left(1 + q\alpha_1 \xi_1^q - q\mu_1 + c_q \xi_1^q \right)^{1/q} + \delta_1 + \frac{\tau_1^{q-1}}{\gamma_1} \right. \\
& \quad \times \left(\left(s_1^q \xi_1^q + q\theta_1 \beta_{11} s_1^{q-1} \xi_1^{q-1} \left(1 - q\delta_{p_1} + c_q \lambda_{P_1}^q \right)^{1/q} + c_q \theta_1^q \beta_{11}^q \left(1 - q\delta_{p_1} + c_q \lambda_{P_1}^q \right)^{1/q} \right. \right. \\
& \quad \left. \left. + \sum_{j=1}^3 \theta_1 \zeta_{1j} t_{1j} \left(1 + \frac{1}{n} \right) \right) + \left(\theta_2 \beta_{21} \frac{\tau_2^{q-1}}{\gamma_2} + \theta_3 \beta_{31} \frac{\tau_3^{q-1}}{\gamma_3} \right) \left(1 - q\delta_{p_1} + c_q \lambda_{P_1}^q \right)^{1/q} \right] \\
& \quad \times \left\| x_1^n - x_1^{n-1} \right\| + \left[\left(1 + q\alpha_2 \xi_2^q - q\mu_2 + c_q \xi_2^q \right)^{1/q} + \delta_2 + \frac{\tau_2^{q-1}}{\gamma_2} \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(\left(s_2^q \xi_2^q + q\theta_2 \beta_{22} s_2^{q-1} \xi_2^{q-1} (1 - q\delta_{p_2} + c_q \lambda_{p_2}^q) \right)^{1/q} \right. \\
& \quad \left. + c_q \theta_2^q \beta_{22}^q (1 - q\delta_{p_2} + c_q \lambda_{p_2}^q) \right)^{1/q} + \sum_{j=1}^3 \theta_2 \xi_{2j} t_{2j} \left(1 + \frac{1}{n} \right) \\
& \quad + \left(\theta_1 \beta_{12} \frac{\tau_1^{q-1}}{\gamma_1} + \theta_3 \beta_{32} \frac{\tau_3^{q-1}}{\gamma_3} \right) (1 - q\delta_{p_2} + c_q \lambda_{p_2}^q)^{1/q} \Big] \\
& \times \left\| x_2^n - x_2^{n-1} \right\| + \left[\left(1 + q\alpha_3 \xi_3^q - q\mu_3 + c_q \xi_3^q \right)^{1/q} + \delta_3 + \frac{\tau_3^{q-1}}{\gamma_3} \right. \\
& \quad \times \left(\left(s_3^q \xi_3^q + q\theta_3 \beta_{33} s_3^{q-1} \xi_3^{q-1} (1 - q\delta_{p_3} + c_q \lambda_{p_3}^q) \right)^{1/q} \right. \\
& \quad \left. + c_q \theta_3^q \beta_{33}^q (1 - q\delta_{p_3} + c_q \lambda_{p_3}^q) \right)^{1/q} + \sum_{j=1}^3 \theta_3 \xi_{3j} t_{3j} \left(1 + \frac{1}{n} \right) \\
& \quad \left. + \left(\theta_1 \beta_{13} \frac{\tau_1^{q-1}}{\gamma_1} + \theta_2 \beta_{23} \frac{\tau_2^{q-1}}{\gamma_2} \right) (1 - q\delta_{p_3} + c_q \lambda_{p_3}^q)^{1/q} \right] \\
& \quad \times \left\| x_3^n - x_3^{n-1} \right\| \\
& \leq \theta_n \left(\sum_{i=1}^n \left\| x_i^n - x_i^{n-1} \right\| \right),
\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}
\theta_n = \max \Bigg\{ & \left(1 + q\alpha_1 \xi_1^q - q\mu_1 + c_q \xi_1^q \right)^{1/q} + \delta_1 + \frac{\tau_1^{q-1}}{\gamma_1} \left(\left(s_1^q \xi_1^q + q\theta_1 \beta_{11} s_1^{q-1} \xi_1^{q-1} (1 - q\delta_{p_1} + c_q \lambda_{p_1}^q) \right)^{1/q} \right. \\
& \quad \left. + c_q \theta_1^q \beta_{11}^q (1 - q\delta_{p_1} + c_q \lambda_{p_1}^q) \right)^{1/q} \\
& \quad \left. + \sum_{j=1}^3 \theta_1 \xi_{1j} t_{1j} \left(1 + \frac{1}{n} \right) \right) \\
& + \left(\theta_2 \beta_{21} \frac{\tau_2^{q-1}}{\gamma_2} + \theta_3 \beta_{31} \frac{\tau_3^{q-1}}{\gamma_3} \right) (1 - q\delta_{p_1} + c_q \lambda_{p_1}^q)^{1/q},
\end{aligned}$$

$$\begin{aligned}
& \left(1 + q\alpha_2 \xi_2^q - q\mu_2 + c_q \xi_2^q\right)^{1/q} + \delta_2 + \frac{\tau_2^{q-1}}{\gamma_2} \left(\left(s_2^q \xi_2^q + q\theta_2 \beta_{22} s_2^{q-1} \xi_2^{q-1} \left(1 - q\delta_{p_2} + c_q \lambda_{p_2}^q\right)^{1/q} \right. \right. \\
& \quad \left. \left. + c_q \theta_2^q \beta_{22}^q \left(1 - q\delta_{p_2} + c_q \lambda_{p_2}^q\right) \right)^{1/q} \right. \\
& \quad \left. + \sum_{j=1}^3 \theta_2 \xi_{2j} t_{2j} \left(1 + \frac{1}{n}\right) \right) \\
& + \left(\theta_1 \beta_{12} \frac{\tau_1^{q-1}}{\gamma_1} + \theta_3 \beta_{32} \frac{\tau_3^{q-1}}{\gamma_3} \right) \left(1 - q\delta_{p_2} + c_q \lambda_{p_2}^q\right)^{1/q}, \\
& \left(1 + q\alpha_3 \xi_3^q - q\mu_3 + c_q \xi_3^q\right)^{1/q} + \delta_3 + \frac{\tau_3^{q-1}}{\gamma_3} \left(\left(s_3^q \xi_3^q + q\theta_3 \beta_{33} s_3^{q-1} \xi_3^{q-1} \left(1 - q\delta_{p_3} + c_q \lambda_{p_3}^q\right)^{1/q} \right. \right. \\
& \quad \left. \left. + c_q \theta_3^q \beta_{33}^q \left(1 - q\delta_{p_3} + c_q \lambda_{p_3}^q\right) \right)^{1/q} \right. \\
& \quad \left. + \sum_{j=1}^3 \theta_3 \xi_{3j} t_{3j} \left(1 + \frac{1}{n}\right) \right) \\
& + \left(\theta_1 \beta_{13} \frac{\tau_1^{q-1}}{\gamma_1} + \theta_2 \beta_{23} \frac{\tau_2^{q-1}}{\gamma_2} \right) \left(1 - q\delta_{p_3} + c_q \lambda_{p_3}^q\right)^{1/q} \Bigg\}.
\end{aligned} \tag{4.27}$$

Let

$$\begin{aligned}
\theta = \max & \left\{ \left(1 + q\alpha_1 \xi_1^q - q\mu_1 + c_q \xi_1^q\right)^{1/q} + \delta_1 + \frac{\tau_1^{q-1}}{\gamma_1} \right. \\
& \times \left(\left(s_1^q \xi_1^q + q\theta_1 \beta_{11} s_1^{q-1} \xi_1^{q-1} \left(1 - q\delta_{p_1} + c_q \lambda_{p_1}^q\right)^{1/q} + c_q \theta_1^q \beta_{11}^q \left(1 - q\delta_{p_1} + c_q \lambda_{p_1}^q\right) \right)^{1/q} \right. \\
& \quad \left. + \sum_{j=1}^3 \theta_1 \xi_{1j} t_{1j} \right) + \left(\theta_2 \beta_{21} \frac{\tau_2^{q-1}}{\gamma_2} + \theta_3 \beta_{31} \frac{\tau_3^{q-1}}{\gamma_3} \right) \left(1 - q\delta_{p_1} + c_q \lambda_{p_1}^q\right)^{1/q}, \\
& \left(1 + q\alpha_2 \xi_2^q - q\mu_2 + c_q \xi_2^q\right)^{1/q} + \delta_2 + \frac{\tau_2^{q-1}}{\gamma_2} \\
& \times \left(\left(s_2^q \xi_2^q + q\theta_2 \beta_{22} s_2^{q-1} \xi_2^{q-1} \left(1 - q\delta_{p_2} + c_q \lambda_{p_2}^q\right)^{1/q} + c_q \theta_2^q \beta_{22}^q \left(1 - q\delta_{p_2} + c_q \lambda_{p_2}^q\right) \right)^{1/q} \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^3 \theta_2 \xi_{2j} t_{2j} \Bigg) + \left(\theta_1 \beta_{12} \frac{\tau_1^{q-1}}{\gamma_1} + \theta_3 \beta_{32} \frac{\tau_3^{q-1}}{\gamma_3} \right) (1 - q\delta_{p_2} + c_q \lambda_{p_2}^q)^{1/q}, \\
& \left(1 + q\alpha_3 \xi_3^q - q\mu_3 + c_q \xi_3^q \right)^{1/q} + \delta_3 + \frac{\tau_3^{q-1}}{\gamma_3} \\
& \times \left(\left(s_3^q \xi_3^q + q\theta_3 \beta_{33} s_3^{q-1} \xi_3^{q-1} (1 - q\delta_{p_3} + c_q \lambda_{p_3}^q)^{1/q} + c_q \theta_3^q \beta_{33}^q (1 - q\delta_{p_3} + c_q \lambda_{p_3}^q) \right)^{1/q} \right. \\
& \left. + \sum_{j=1}^3 \theta_3 \xi_{3j} t_{3j} \right) + \left(\theta_1 \beta_{13} \frac{\tau_1^{q-1}}{\gamma_1} + \theta_2 \beta_{23} \frac{\tau_2^{q-1}}{\gamma_2} \right) (1 - q\delta_{p_3} + c_q \lambda_{p_3}^q)^{1/q} \Bigg\}. \tag{4.28}
\end{aligned}$$

We know that $\theta_n \rightarrow \theta$ as $n \rightarrow \infty$.

Define $\|\cdot\|_*$ on $E = \prod_{k=1}^3 E_k$ by

$$\|(z_1, z_2, z_3)\|_* = \|z_1\| + \|z_2\| + \|z_3\|, \quad \forall (z_1, z_2, z_3) \in E. \tag{4.29}$$

It is easy to see that $(E, \|\cdot\|_*)$ is a Banach space. Define $z^{n+1} = (x_1^{n+1}, x_2^{n+1}, x_3^{n+1})$. Then we have

$$\|z^{n+1} - z^n\|_* = \|x_1^{n+1} - x_1^n\| + \|x_2^{n+1} - x_2^n\| + \|x_3^{n+1} - x_3^n\|. \tag{4.30}$$

It follows from (4.18) that $0 < \theta < 1$, and hence there exists an $n_0 > 0$ and $\theta_0 \in (0, 1)$ such that $\theta_n \leq \theta_0$, for all $n \geq n_0$. Therefore, by (4.26) and (4.30), we have

$$\|z^{n+1} - z^n\|_* \leq \theta_0^{n-n_0} \|z^{n_0+1} - z^{n_0}\|_*. \tag{4.31}$$

Hence, for any $m \geq n \geq n_0$, it follows that

$$\begin{aligned}
\|x_1^m - x_1^n\| & \leq \|z^m - z^n\|_* \\
& \leq \sum_{i=n}^{m-1} \|z^{i+1} - z^i\|_* \\
& \leq \sum_{i=n}^{m-1} \theta_0^{i-n_0} \|z^{n_0+1} - z^{n_0}\|_*. \tag{4.32}
\end{aligned}$$

Since $0 < \theta_0 < 1$, it follows from (4.32) that $\|x_1^m - x_1^n\| \rightarrow 0$ as $n \rightarrow \infty$, and hence $\{x_1^n\}$ is a Cauchy sequence in E_1 . By the same argument, we also have $\{x_i^n\}$ is a Cauchy sequence in E_i ($i = 2, 3$). Thus, there exist $x_1 \in E_1$, $x_2 \in E_2$, $x_3 \in E_3$ such that $x_1^n \rightarrow x_1$, $x_2^n \rightarrow x_2$, $x_3^n \rightarrow x_3$ as $n \rightarrow \infty$.

Now we prove that $u_{1i}^n \rightarrow u_{1i} \in T_{1i}(x_1)$, $u_{2i}^n \rightarrow u_{2i} \in T_{2i}(x_2)$, $u_{3i}^n \rightarrow u_{3i} \in T_{3i}(x_3)$ ($i = 1, 2, 3$). In fact, it follows from the Lipschitz continuity of T_{1i} , T_{2i} , T_{3i} that for, $i = 1, 2, 3$,

$$\begin{aligned}\|u_{1i}^n - u_{1i}^{n-1}\| &\leq \left(1 + \frac{1}{n}\right)t_{1i}\|x_1^n - x_1^{n-1}\|, \\ \|u_{2i}^n - u_{2i}^{n-1}\| &\leq \left(1 + \frac{1}{n}\right)t_{2i}\|x_2^n - x_2^{n-1}\|, \\ \|u_{3i}^n - u_{3i}^{n-1}\| &\leq \left(1 + \frac{1}{n}\right)t_{3i}\|x_3^n - x_3^{n-1}\|.\end{aligned}\tag{4.33}$$

From (4.33), we have that $u_{1i}^n, u_{2i}^n, u_{3i}^n$ ($i = 1, 2, 3$) are also Cauchy sequences. Therefore, there exist $u_{1i} \in E_i$, $u_{2i} \in E_i$, $u_{3i} \in E_i$ such that $u_{1i}^n \rightarrow u_{1i}$, $u_{2i}^n \rightarrow u_{2i}$, $u_{3i}^n \rightarrow u_{3i}$ as $n \rightarrow \infty$. Further, for $i = 1, 2, 3$,

$$\begin{aligned}d(u_{1i}, T_{1i}(x_1)) &\leq \|u_{1i} - u_{1i}^n\| + d(u_{1i}^n, T_{1i}(x_1)) \\ &\leq \|u_{1i} - u_{1i}^n\| + \widetilde{H}(T_{1i}(x_1^n), T_{1i}(x_1)) \\ &\leq \|u_{1i} - u_{1i}^n\| + t_{1i}\|x_1^n - x_1\| \rightarrow 0.\end{aligned}\tag{4.34}$$

Since $T_{1i}(x_1)$ is closed, we have $u_{1i} \in T_{1i}(x_1)$ ($i = 1, 2, 3$). Similarly $u_{2i} \in T_{2i}(x_2)$, $u_{3i} \in T_{3i}(x_3)$ ($i = 1, 2, 3$). By continuity of f_i , H_i , ϕ_i , F_i , N_i , T_{1i} , T_{2i} , T_{3i} , $J_{M_i(x_i), \phi_i}^{H_i, \eta_i}$ and Algorithm 4.2, we know that $x_1, x_2, x_3, u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33}$ satisfy the following relation:

$$\begin{aligned}f_i(x_i) &= J_{M_i(x_i), \phi_i}^{H_i, \eta_i} [H_i \circ f_i(x_i) - \phi_i \circ F_i(x_1 - p_1(x_1), x_2 - p_2(x_2), x_3 - p_3(x_3)) \\ &\quad - \phi_i \circ N_i(u_{i1}, u_{i2}, u_{i3})], \quad i = 1, 2, 3.\end{aligned}\tag{4.35}$$

By Lemma 4.1, $(x_1, x_2, x_3, u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33})$ is a solution of the SGVIP (4.1). This completes the proof. $\square$

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