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Research Article

Estimation of Reliability for a Two Component Survival Stress-Strength Model

S. B. Munoli[1] and Rohit R. Mutkekar[2]

[1] *Department of Studies in Statistics, Karnatak University, Dharwad, Karnataka 580-003, India*
[2] *KLS-Institute of Management Education and Research, Belgaum, Karnataka 590-011, India*

Correspondence should be addressed to S. B. Munoli, sbmunoli@yahoo.co.in

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The reliability function for a parallel system of two identical components is derived from a stress-strength model, where failure of one component increases the stress on the surviving component of the system. The Maximum Likelihood Estimators of parameters and their asymptotic distribution are obtained. Further the Maximum Likelihood Estimator and Bayes Estimator of reliability function are obtained using the data from a life-testing experiment. Computation of estimators is illustrated through simulation study.

1. Introduction

Several authors have considered estimation of system reliability based on stress-strength models. Here are a few references of contributions towards these models. Church and Harris [1] considered estimation of reliability from stress-strength relationship. Downton [2] considered the case of estimation of reliability for a stress-strength model under normal distribution. Wani and Kabe [3] have considered the problem of estimation of system reliability where life time of each component has gamma distribution. Constantine et al. [4] considered estimation of stress-strength relationship under the assumption of stress-strength random variables following gamma distribution with known shape parameters. Bhattacharya and Johnson [5] have considered estimation of reliability in a multicomponent stress-strength model. Kunchur and Munoli [6] have considered estimation of reliability for a multicomponent survival stress-strength model based on exponential distribution. Dan and Krausz [7] have obtained inference for a multistep stress-strength model of parallel system. Kunchur and Munoli [8] have considered estimation of reliability in Freund's model for a two component system. Hanagal [9] has considered the problem of estimation of system reliability in a two component stress-strength model with cases on distribution of stress as exponential and gamma. Kundu and Gupta [10] have obtained inference of stress-strength relationship for generalized exponential scale family of distribution. Bhattacharya [11] has proposed Bayesian approach to life-testing and reliability estimation. Draper and Guttman [12] have obtained the Bayes estimator of reliability in a multi-component stress-strength model.

In the present study, we are considering a system of two components. The component survives as long as the stress on it is smaller than its strength. The system survives if at least one component functions (parallel system). Here the stress and strength associated with the components of the system are random variables. To carry out the inference, we assume certain probability distribution for these random variables. Let the strength of the two components be X_1 and X_2, where X_1, X_2 are independently and identically distributed gamma random variables with shape parameter γ and scale parameter μ. Let Y_1, Y_2 be the stress on the two components, respectively. Initially Y_1, Y_2 are independently and identically distributed as exponential random variables with parameter θ. Statistically dependent failures are typical for modern systems, which involve complicated interactions among component parts, and it is reasonable to assume that the failure of one component does change the stress on surviving components. The strength on the other hand is dependent

on various inbuilt qualities of the component such as the technology by which it has been manufactured, the raw materials used, and so forth. Failure of one component of the system need not change these inbuilt qualities of the surviving component of the system. Hence, change in the stress of the surviving component is assumed on failure of other components of the system. That is, in a two component parallel system, the distribution of stress of the surviving component changes upon the failure of the other component. Then the exponential distribution with parameter $\alpha\theta(\alpha > 0)$ follows. The system fails whenever both the components of the system fail.

A block diagram for the proposed model is given in Figure 1.

One can quote several examples for two component parallel systems such as pair of kidneys, eyes, hands, and legs, pair of elevators, and pair of engines in an aircraft. To illustrate the function of the proposed model let us consider the example of a pair of kidneys. Here the function of the kidneys is to purify the blood and thus help to maintain the body in healthy condition. Here the pair of kidneys perform the same function as per their natural built-up mechanism (strength). Failure of one kidney increases the purification work load on the surviving kidney (stress). Here the surviving kidney should carry out the purification function the same as it was when both the kidneys were functioning. Taking this scenario under consideration, we consider that failure of one component of the system changes only the stress and not the strength of the surviving component.

The reliability function is derived in Section 2. Life-testing experiment is explained in Section 3. The Maximum Likelihood Estimators (MLEs) are obtained in the same section. Section 4 deals with Bayes estimation of reliability function. Computations of estimators of reliability function (MLE and Bayes) along with the findings of the study are discussed in Section 5. Some results that support the findings of this research paper are proved in the appendix.

2. Reliability Function

In order to find the reliability function of the model discussed in Section 1, let us consider that U is the minimum (Y_1, Y_2) and V is the maximum (Y_1, Y_2) and let $W = V - U$.

Here U follows exponential distribution with parameter (2θ) and W follows exponential distribution with parameter $(\alpha\theta)$.

The reliability function of this system is given by

$$\begin{aligned} R &= P[\mathrm{Max}(X_1, X_2) > \mathrm{Max}(Y_1, Y_2)] \\ &= 1 + \frac{2\cdot\mu^{\gamma}}{\Gamma\gamma}\cdot\left[\frac{2}{(\alpha-2)}\sum_{i=\gamma}^{\infty}\frac{(\mu)^{i}}{i!}\cdot\frac{\Gamma(i+\gamma)}{(\alpha\theta+2\mu)^{(i+\gamma)}}\right. \\ &\qquad \left. -\frac{\alpha}{(\alpha-2)}\cdot\sum_{i=\gamma}^{\infty}\frac{(\mu)^{i}}{i!}\cdot\frac{\Gamma(i+\gamma)}{(2\theta+2\mu)^{(i+\gamma)}}\right]. \end{aligned} \tag{1}$$

The details of these derivations are given in Lemma A.1.

3. Life-Testing Experiment

In order to obtain the estimators of R, suppose that "n" systems whose life distribution is characterized by the reliability function derived in Section 2 are put on life-testing experiment. Here, $X_{1i}, X_{2i}(i = 1, 2\cdots n)$ are observed and X_{1i}, X_{2i} are independently and identically distributed gamma random variables with shape parameter γ and scale parameter μ. Also, the data of stress $U_j = \mathrm{Min}(Y_{1j}, Y_{2j})$ and $V_j = \mathrm{Max}(Y_{1j}, Y_{2j})$, $(j = 1, 2\cdots m)$ are obtained separately from a simulation of conditions of the operating environment. U_j, W_j $(j = 1, 2, 3\cdots m)$ are exponential random variables with parameters 2θ and $\alpha\theta$, respectively.

Now, the joint probability density function of the random variables X_{1i}, $X_{2i}(i = 1, 2\cdots n)$, U_j, W_j $(j = 1, 2\cdots m)$ is given by

$$\begin{aligned} L &= \left(\frac{\mu^{\gamma}}{\Gamma\gamma}\right)^{2n}\cdot\left\{\prod_{i=1}^{n}(x_{1i})^{\gamma-1}\right\}\cdot\left\{\prod_{i=1}^{n}(x_{2i})^{\gamma-1}\right\} \\ &\quad\cdot e^{-\mu(x_1'+x_2')}\cdot(2\theta)^{m}\cdot(\alpha\theta)^{m}\cdot e^{-2\theta u'}\cdot e^{-\alpha\theta w'}, \end{aligned} \tag{2}$$

where

$$x_1' = \sum_{i=1}^{n}x_{1i},\quad x_2' = \sum_{i=1}^{n}x_{2i},\quad u' = \sum_{j=1}^{m}u_j,\quad w' = \sum_{j=1}^{m}w_j. \tag{3}$$

The MLEs (the estimators that maximize the likelihood function) of parameters γ, μ, θ, α are given in the following expressions, respectively,

$$\hat{\gamma} = \frac{n}{2n[\mathcal{A}] - \left[\sum_{i=1}^{n}\log(x_{1i}) + \sum_{i=1}^{n}\log(x_{2i})\right]}, \tag{4}$$

$$\hat{\mu} = \frac{2n^2}{(x_1'+x_2')\left[2n\{\mathcal{A}\} - \left\{\sum_{i=1}^{n}\log(x_{1i}) + \sum_{i=1}^{n}\log(x_{2i})\right\}\right]}, \tag{5}$$

$$\hat{\theta} = \frac{m}{2u'}, \tag{6}$$

$$\hat{\alpha} = \frac{2u'}{w'}, \tag{7}$$

where $\mathcal{A}$ denote $\log(x_1' + x_2') - \log(2n)$

Using the invariance property of MLEs, the MLE of reliability function $\hat{R}$ is obtained by substituting the MLEs of parameters γ, μ, θ, and α in expression (1), that is,

$$\begin{aligned} \hat{R} &= 1 + \frac{2\cdot\hat{\mu}^{\hat{\gamma}}}{\Gamma\hat{\gamma}}\cdot\left[\frac{2}{(\hat{\alpha}-2)}\sum_{i=\hat{\gamma}}^{\infty}\frac{(\hat{\mu})^{i}}{i!}\cdot\frac{\Gamma(i+\hat{\gamma})}{\left(\hat{\alpha}\hat{\theta}+2\hat{\mu}\right)^{(i+\hat{\gamma})}}\right. \\ &\qquad \left. -\frac{\hat{\alpha}}{(\hat{\alpha}-2)}\cdot\sum_{i=\hat{\gamma}}^{\infty}\frac{(\hat{\mu})^{i}}{i!}\cdot\frac{\Gamma(i+\hat{\gamma})}{\left(2\hat{\theta}+2\hat{\mu}\right)^{(i+\hat{\gamma})}}\right]. \end{aligned} \tag{8}$$

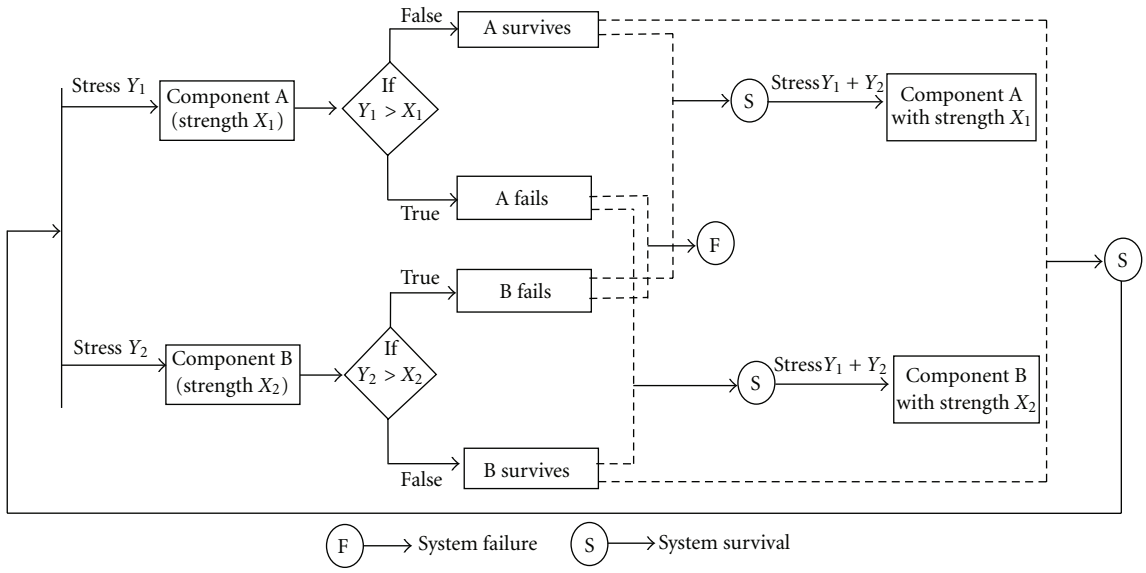

Figure 1

The details of these findings are given in the form of Lemma A.2.

From the asymptotic properties of MLEs under regularity conditions and multivariate central limit theorem, we have $[\sqrt{n}(\hat{\gamma}-\gamma), \sqrt{n}(\hat{\mu}-\mu), \sqrt{m}(\hat{\theta}-\theta), \sqrt{m}(\hat{\alpha}-\alpha)] \rightarrow N_4[0, A(\gamma,\mu,\theta,\alpha)]$, where $A(\gamma,\mu,\theta,\alpha)$ are the corresponding elements of the inverse of Fisher Information Matrix "$I(\gamma,\mu,\theta,\alpha)$" as

$$I(\gamma,\mu,\theta,\alpha) = \begin{bmatrix} \frac{n(2\gamma+1)}{2\gamma^2} & \frac{-2n}{\mu} & 0 & 0 \\ \frac{-2n}{\mu} & \frac{2n}{\mu^2} & 0 & 0 \\ 0 & 0 & \frac{2m}{\theta^2} & \frac{m}{\alpha\theta} \\ 0 & 0 & \frac{m}{\alpha\theta} & \frac{m}{\alpha^2} \end{bmatrix} \tag{9}$$

and its inverse is given by

$$I^{-1} = \begin{bmatrix} \frac{2}{n[(2\gamma+1)/\gamma^2-4]} & \frac{2\mu}{n[(2\gamma+1)/\gamma^2-4]} & 0 & 0 \\ \frac{2\mu}{n[(2\gamma+1)/\gamma^2-4]} & \frac{\mu^2(2\gamma+1)}{n[(2\gamma+1)/\gamma^2-4]} & 0 & 0 \\ 0 & 0 & \frac{\theta^2}{m} & \frac{-\alpha\theta}{m} \\ 0 & 0 & \frac{-\alpha\theta}{m} & \frac{2\alpha^2}{m} \end{bmatrix}. \tag{10}$$

4. Bayes Estimation of Reliability Function

In order to obtain Bayes estimator of reliability function $\widehat{R_B}$, assume that parameters γ and α are known (i. e., $\gamma = \gamma_0$ and $\alpha = \alpha_0$) and consider the prior distribution of parameters μ and θ [13] as

$$\begin{aligned} g(\mu) &= \frac{1}{\Gamma p} \cdot e^{-\mu} \cdot \mu^{p-1}, \quad p \geq 0,\ 0 < \mu < \infty, \\ g(\theta) &= \frac{1}{\Gamma q} \cdot e^{-\theta} \cdot \theta^{q-1}, \quad q \geq 0,\ 0 < \theta < \infty. \end{aligned} \tag{11}$$

The Bayes estimator of reliability function is obtained as the posterior expectation of R and is given by

$$\begin{aligned} \widehat{R_B} &= \iint_0^\infty R \cdot f_2\left(\mu,\theta \mid x_{1i}, x_{2i}, u_j, w_j\right) \mathrm{d}\mu \mathrm{d}\theta \\ &= \iint_0^\infty \left[1 + \frac{2\cdot\mu^{\gamma_0}}{\Gamma\gamma_0} \cdot \left[\frac{2}{(\alpha_0-2)} \sum_{i=\gamma_0}^{\infty} \frac{(\mu)^i}{i!} \cdot \frac{\Gamma(i+\gamma_0)}{(\alpha_0\theta+2\mu)^{(i+\gamma_0)}} - \frac{\alpha_0}{(\alpha_0-2)} \cdot \sum_{i=\gamma_0}^{\infty} \frac{(\mu)^i}{i!} \cdot \frac{\Gamma(i+\gamma_0)}{(2\theta+2\mu)^{(i+\gamma_0)}}\right]\right] \\ &\quad \cdot f_2\left(\mu,\theta \mid x_{1i}, x_{2i}, u_j, w_j\right) \cdot \mathrm{d}\mu \mathrm{d}\theta \\ &= 1 + I_1 - I_2, \end{aligned} \tag{12}$$

where

$$I_1 = \iint_0^\infty \frac{4\cdot\mu^{\gamma_0}}{(\alpha_0-2)\cdot\Gamma\gamma_0} \cdot \sum_{i=\gamma_0}^{\infty} \frac{(\mu)^i}{i!} \cdot \frac{\Gamma(i+\gamma_0)}{(\alpha_0\theta+2\mu)^{(i+\gamma_0)}} \times f_2\left(\mu,\theta \mid x_{1i}, x_{2i}, u_j, w_j\right) \cdot \mathrm{d}\mu \mathrm{d}\theta$$

$$= \frac{4.\mu^{\gamma_0}}{(\alpha_0 - 2)\cdot\Gamma\gamma_0\cdot\Gamma(2n\gamma_0 + p)\cdot\Gamma(2m+q)} \cdot \sum_{i=\gamma_0}^{\infty}\frac{\Gamma(i+\gamma_0)}{i!}\cdot\iint_0^{\infty} A_1 \mathrm{d}\mu\,\mathrm{d}\theta \tag{13}$$

with

$$A_1 = \left(\frac{e^{-\mu(x_1{}'+x_2{}'+1)}\cdot\mu^{2n\gamma_0+p+i-1}\cdot e^{-\theta(2u'+\alpha_0 w'+1)}}{(\alpha_0\theta+2\mu)^{(i+\gamma_0)}}\right) \cdot\left(\theta^{(2m+q-1)}\cdot(x_1{}'+x_2{}'+1)^{(2n\gamma_0+p)} \cdot(2u'+\alpha_0 w'+1)^{(2m+q)}\right),$$

$$I_2 = \iint_0^{\infty}\frac{2\cdot\alpha_0\cdot\mu^{\gamma_0}}{(\alpha_0-2)\cdot\Gamma\gamma_0}\cdot\sum_{i=\gamma_0}^{\infty}\frac{(\mu)^i}{i!}\cdot\frac{\Gamma(i+\gamma_0)}{(2\theta+2\mu)^{(i+\gamma_0)}} \times f_2\left(\mu,\theta \mid x_{1i}, x_{2i}, u_j, w_j\right)\cdot \mathrm{d}\mu\,\mathrm{d}\theta$$

$$= \frac{2\cdot\alpha_0\cdot\mu^{\gamma_0}}{(\alpha_0-2)\cdot\Gamma\gamma_0\cdot\Gamma(2n\gamma_0+p)\cdot\Gamma(2m+q)} \cdot\sum_{i=\gamma_0}^{\infty}\frac{\Gamma(i+\gamma_0)}{i!}\cdot\iint_0^{\infty} A_2 \mathrm{d}\mu\mathrm{d}\theta \tag{14}$$

with

$$A_2 = \left(\frac{e^{-\mu(x_1{}'+x_2{}'+1)}\cdot\mu^{2n\gamma_0+p+i-1}\cdot e^{-\theta(2u'+\alpha_0 w'+1)}}{(2\theta+2\mu)^{(i+\gamma_0)}}\right) \cdot\left(\theta^{(2m+q-1)}\cdot(x_1{}'+x_2{}'+1)^{(2n\gamma_0+p)} \cdot(2u'+\alpha_0 w'+1)^{(2m+q)}\right). \tag{15}$$

Substituting values of expressions (13), and (14) in expression (12) we obtain the Bayes estimator of reliability function $\widehat{R_B}$.

The details of these findings are given in the form of Lemma A.3.

5. Computation of Estimators

For the ith system, the random variables x_{1i}, x_{2i} (with respect to strength) and random variables u_i, w_i (with respect to stress) are generated independently as follows.

Step 1. Initialize $j_1 = 1$, $x_{1i} = 0.0$, $x_{2i} = 0.0$, $n = n_0$, $\gamma = \gamma_0$ for the 1st and 2nd components of the system. Uniform random numbers $U_1[j_1], V_1[j_1]$ are generated from $U(0,1)$. Further for a given value of $= \mu_0$, exponential random variables $U_2[j_1] = (-1/\mu_0)\cdot\ln(1 - U_1[j_1])$, $V_2[j_1] = (-1/\mu_0)\cdot\ln(1 - V_1[j_1])$ are obtained for the 1st and the 2nd components of the ith system, respectively. Now, j_1 is incremented by 1 and the process of generating the exponential random variables is repeated for both the components of the ith system. This repetition process is continued until $j_1 \le \gamma_0$ and the subsequent exponential random variable values generated are noted for both the components of the ith system. Here $x_{1i} = x_{1i} + U_2[j_1]$ and $x_{2i} = x_{2i} + V_2[j_1]$ for $j_1 = 1, 2\cdots\gamma_0$. Here x_{1i}, x_{2i} denote gamma random variables with shape parameter $\gamma = \gamma_0$ and scale parameter $\mu = \mu_0$. We also compute the values $\ln(x_{1i})$ and $\ln(x_{2i})$.

Step 2. The whole procedure in Step 1 is repeated for $n = n_0$ number of systems, and the statistics $x_1{}' = \sum_{i=1}^{n} x_{1i}, x_2{}' = \sum_{i=1}^{n} x_{2i}, \sum_{i=1}^{n}\log(x_{1i}), \sum_{i=1}^{n}\log(x_{2i})$ are computed.

Step 3. Initialize $i = 1$, $u_i = 0.0$, $w_i = 0.0$, $m = m_0$. Uniform random numbers $U_3[i]$, $U_4[i]$ are generated from $U(0,1)$. Further for a given value of $\theta = \theta_0$, $\alpha = \alpha_0$ exponential random variables $u_i = (-1/\theta_0)\cdot\ln(1 - U_3[i])$, $w_i = (-1/\alpha_0\cdot\theta_0)\cdot\ln(1 - U_4[i])$ are obtained. The value of i is incremented by 1, and the above process of generated exponential random variables is repeated. This repetition process is continued until $i \le m$ and the subsequent exponential random variable values generated u_i, w_i for $i = 1, 2\cdots m$ are computed. The statistics $u' = \sum_{i=1}^{m} u_i$, $w' = \sum_{i=1}^{m} w_i$ are computed.

Step 4. With the help of the statistics $x_1{}'$, $x_2{}'$, $\sum_{i=1}^{n}\log(x_{1i})$, $\sum_{i=1}^{n}\log(x_{2i})$, u' and w', the MLEs of parameters $\gamma, \mu, \theta, \alpha$ of the model are obtained. Using these MLEs in the expression of reliability function, the MLE of reliability function is obtained.

For the parameter values $\gamma = \gamma_0$, $\mu = \mu_0$, $\theta = \theta_0$, and $\alpha = \alpha_0$ the value of reliability function is also obtained.

The Bayes estimator of reliability function is obtained using the simulated values of $x_1{}'$, $x_2{}'$, $\sum_{i=1}^{n}\log(x_{1i})$, $\sum_{i=1}^{n}\log(x_{2i})$, u' and w' for given values of p and q.

Tables 1, 2, and 3 give the results of the above simulation experiment for different values of n and m.

6. Conclusion

The reliability function for the proposed model is evaluated in terms of stress-strength relationship rather than considering the time factor, as it is realistic to observe reliability of a system functioning under the influence of external factors (stress) as compared to longevity of working time associated with the system.

Though the expression for reliability function involves sum of infinite series, it is observed that for given values of the parameters the value of the reliability function stabilizes at the 15th value ($i = 15$) of the running variable involved in the sums.

Stress and strength associated with the components of the system possess different physical properties; thereby data on stress and strength in the life-testing experiment is observed separately based on their corresponding operative environments.

The MLEs are sufficient, efficient and also maximizes the likelihood of the joint distribution function. Further, using

Table 1: $\gamma = 3$, $\mu = 1.0$, $\theta = 0.5$, $n = 5$, $m = 5$, $p = 10$, and $q = 10$.

α	R	x_1'	x_2'	$\sum_{i=1}^{n}\log(x_{2i})$	$\sum_{i=1}^{n}\log(x_{2i})$	u'	w'	$\hat{R}$	$\widehat{R_B}$	Bias (MLE)	Bias (Bayes)
0.5	0.476	13.01744	13.69945	3.94646	4.04823	4.88118	8.93451	0.619	0.562	2.05×10^{-2}	7.40×10^{-3}
1.0	0.676	15.00853	21.81265	4.64973	7.07659	3.29156	7.66388	0.798	0.744	1.49×10^{-2}	4.62×10^{-3}
2.0	0.774	16.13645	15.41186	5.42399	4.94180	5.01072	3.71295	0.866	0.792	8.46×10^{-3}	3.24×10^{-4}

Table 2: $\gamma = 3$, $\mu = 1.0$, $\theta = 0.5$, $n = 6$, $m = 6$, $p = 11$, and $q = 11$.

α	R	x_1'	x_2'	$\sum_{i=1}^{n}\log(x_{2i})$	$\sum_{i=1}^{n}\log(x_{2i})$	u'	w'	$\hat{R}$	$\widehat{R_B}$	Bias (MLE)	Bias (Bayes)
0.5	0.476	12.88996	17.43603	4.119318	5.50226	2.30250	20.4892	0.574	0.556	9.61×10^{-3}	6.40×10^{-3}
1.0	0.676	14.82036	16.22916	4.578078	5.22255	3.76145	10.4876	0.759	0.731	6.89×10^{-3}	3.03×10^{-3}
2.0	0.774	10.17496	16.28889	2.750896	5.10211	4.32817	4.35386	0.822	0.787	2.31×10^{-3}	1.69×10^{-4}

Table 3: $\gamma = 3$, $\mu = 1.0$, $\theta = 0.5$, $n = 7$, $m = 7$, $p = 12$, and $q = 12$.

α	R	x_1'	x_2'	$\sum_{i=1}^{n}\log(x_{2i})$	$\sum_{i=1}^{n}\log(x_{2i})$	u'	w'	$\hat{R}$	$\widehat{R_B}$	Bias (MLE)	Bias (Bayes)
0.5	0.476	23.09795	17.29346	6.941847	5.754879	4.19987	11.8371	0.512	0.494	1.29×10^{-3}	3.24×10^{-4}
1.0	0.676	12.50140	34.21716	2.203496	10.363429	1.61698	14.5710	0.747	0.724	5.04×10^{-3}	2.30×10^{-3}
2.0	0.774	20.37072	25.06768	7.069134	7.447618	4.01297	6.64931	0.807	0.780	1.09×10^{-3}	3.60×10^{-5}

the invariance property of MLE, it is easy to obtain the MLE of reliability function.

Bayes estimator is based on the prior information obtained through certain pilot study that helps in synthesizing the information to be generated for the system under function. Hence, Bayes estimator of reliability function is obtained by considering certain prior information for the parameters. But the process of obtaining Bayes estimator of reliability function is quite tedious as it involves lengthy numerical calculations.

From Tables 1, 2, and 3, it is clear that for greater values of "m" and "n" (large sample size) both MLE and Bayes estimators perform better. In the majority of the cases both the estimators overestimate the true value of reliability function "R." Here we observe that Bayes estimator is a better estimator in terms of bias for the given data set.

Appendix

Lemma A.1. *The reliability function given in expression* (1) *is derived as follows. The reliability function for the system under study is given by*

$$\begin{aligned}
R &= P[\mathrm{Max}(X_1,\ X_2) > \mathrm{Max}(Y_1, Y_2)] \\
&= P[\mathrm{Max}(X_1,\ X_2) > V] \\
&= P[U + W < \mathrm{Max}(X_1,\ X_2)] \\
&= P[U < \mathrm{Max}(X_1,\ X_2) - \ W] \\
&= P[U < Z - W],\ \text{where } Z = \mathrm{Max}(X_1,\ X_2) \\
&= \int_0^{\infty}\int_0^{z}\left[1 - e^{-2\theta(z-w)}\right] \cdot f_W(w) \cdot f_Z(z)dwdz \\
&= \int_0^{\infty}\int_0^{z}\left[1 - e^{-2\theta(z-w)}\right] \cdot \alpha\theta \cdot e^{-\alpha\theta w} \cdot \ f_Z(z)dwdz \\
&= \int_0^{\infty}\int_0^{z}\alpha\theta \cdot e^{-\alpha\theta w} \cdot f_Z(z)dw\,dz - \int_0^{\infty}\int_0^{z} e^{-2\theta(z-w)} \cdot \alpha\theta \\
&\quad \cdot e^{-\alpha\theta w} \cdot f_Z(z)\ dwdz \\
&= \int_0^{\infty}\left[1 - e^{-\alpha\theta z}\right] \cdot f_Z(z)dz - \ \alpha\theta\int_0^{\infty}\int_0^{z} e^{-2\theta(z-w)-\alpha\theta w} \\
&\quad \cdot f_Z(z)dwdz \\
&= 1 - \int_0^{\infty} e^{-\alpha\theta z} \cdot f_Z(z)\ dz - \alpha\theta\int_0^{\infty}\int_0^{z} e^{-2\theta(z-w)-\alpha\theta w} \\
&\quad \cdot f_Z(z)dwdz.
\end{aligned} \tag{A.1}$$

As $Z = \mathrm{Max}(X_1,\ X_2)$, *its distribution is given by*

$$f_Z(z) = 2 \cdot [F(z)] \cdot f(z). \tag{A.2}$$

Here $F(z)$ *represents the distribution function and* $f(z)$ *represents probability density function of random variable* Z, *and as* Z *follows gamma distribution with shape parameter* γ *and scale parameter* μ, *one has*

$$f_Z(z) = 2 \cdot \left[\int_0^{z}\frac{\mu^{\gamma}}{\Gamma\gamma} \cdot \ e^{-\mu x} \cdot \ x^{\gamma-1}dx\right] \cdot \ \frac{\mu^{\gamma}}{\Gamma\gamma} \cdot e^{-\mu z} \cdot z^{\gamma-1}. \tag{A.3}$$

Using the relationship between incomplete gamma distribution and Poisson sum, one has

$$\int_0^{z}\frac{\mu^{\gamma}}{\Gamma\gamma} \cdot e^{-\mu x} \cdot x^{\gamma-1}dx = \sum_{i=\gamma}^{\infty}\frac{e^{-\mu z} \cdot (\mu z)^{i}}{i!}. \tag{A.4}$$

Substituting the above integral value in expression (A.3), *one gets*

$$f_Z(z) = 2 \cdot \left[\sum_{i=\gamma}^{\infty} \frac{e^{-\mu z} \cdot (\mu z)^i}{i!}\right] \cdot \frac{\mu^\gamma}{\Gamma\gamma} \cdot e^{-\mu z} \cdot z^{\gamma-1}$$
$$= \frac{2 \cdot \mu^\gamma \cdot e^{-2\mu z}}{\Gamma\gamma} \cdot \sum_{i=\gamma}^{\infty} \frac{(\mu)^i}{i!} \cdot z^{(i+\gamma-1)}. \tag{A.5}$$

Now, substituting the value of $f_Z(z)$ *from expression* (A.5) *in the expression for R (expression* (A.1)*), one will solve the integrals associated with expression* (A.1) *as follows:*

$$\int_0^\infty e^{-\alpha\theta z} \cdot f_Z(z)dz$$
$$= \frac{2 \cdot \mu^\gamma}{\Gamma\gamma} \cdot \sum_{i=\gamma}^{\infty} \frac{(\mu)^i}{i!} \cdot \int_0^\infty e^{-(\alpha\theta+2\mu)z} \cdot z^{(i+\gamma-1)} dz$$
$$= \frac{2 \cdot \mu^\gamma}{\Gamma\gamma} \cdot \sum_{i=\gamma}^{\infty} \frac{(\mu)^i}{i!} \cdot \frac{\Gamma(i+\gamma)}{(\alpha\theta+2\mu)^{(i+\gamma)}}, \tag{A.6}$$

$$\alpha\theta \int_0^\infty \int_0^z e^{-2\theta(z-w)-\alpha\theta w} \cdot f_Z(z)dwdz$$
$$= \frac{\alpha}{\alpha-2} \int_0^\infty e^{-2\theta z} \cdot f_Z(z)dz - \frac{\alpha}{\alpha-2} \int_0^\infty e^{-\alpha\theta z} \cdot f_Z(z)dz, \tag{A.7}$$

where,

$$\int_0^\infty e^{-2\theta z} \cdot f_Z(z)dz = \frac{2.\mu^\gamma}{\Gamma\gamma} \cdot \sum_{i=\gamma}^{\infty} \frac{(\mu)^i}{i!} \cdot \int_0^\infty e^{-(2\theta+2\mu)z} \cdot z^{(i+\gamma-1)} dz$$
$$= \frac{2 \cdot \mu^\gamma}{\Gamma\gamma} \cdot \sum_{i=\gamma}^{\infty} \frac{(\mu)^i}{i!} \cdot \frac{\Gamma(i+\gamma)}{(2\theta+2\mu)^{(i+\gamma)}}. \tag{A.8}$$

Using the results of expressions (A.8) *and* (A.6) *in expression* (A.7), *one has*

$$\alpha\theta \int_0^\infty \int_0^z e^{-2\theta(z-w)-\alpha\theta w} \cdot f_Z(z)dwdz$$
$$= \frac{\alpha}{\alpha-2} \cdot \left\{\frac{2 \cdot \mu^\gamma}{\Gamma\gamma} \cdot \sum_{i=\gamma}^{\infty} \frac{(\mu)^i}{i!} \cdot \frac{\Gamma(i+\gamma)}{(2\theta+2\mu)^{(i+\gamma)}}\right\} - \frac{\alpha}{\alpha-2} \cdot \left\{\frac{2 \cdot \mu^\gamma}{\Gamma\gamma} \cdot \sum_{i=\gamma}^{\infty} \frac{(\mu)^i}{i!} \cdot \frac{\Gamma(i+\gamma)}{(\alpha\theta+2\mu)^{(i+\gamma)}}\right\}$$
$$= \frac{2 \cdot \alpha \cdot \mu^\gamma}{(\alpha-2) \cdot \Gamma\gamma} \times \left\{\sum_{i=\gamma}^{\infty} \frac{(\mu)^i}{i!} \cdot \frac{\Gamma(i+\gamma)}{(2\theta+2\mu)^{(i+\gamma)}} - \sum_{i=\gamma}^{\infty} \frac{(\mu)^i}{i!} \cdot \frac{\Gamma(i+\gamma)}{(\alpha\theta+2\mu)^{(i+\gamma)}}\right\}. \tag{A.9}$$

Substituting the results of expressions (A.6) *and* (A.9) *in the expression* (A.1), *one gets the reliability function given in expression* (1).

Lemma A.2. *The MLEs of parameters* $\gamma, \mu, \theta,$ *and* α *given in expressions* (4), (5), (6), *and* (7) *and further used to determine the MLE of reliability function in expression* (8) *are derived as follows.*

The log-likelihood function of expression (2) *in is*

$$\log L = 2n \log\left(\frac{\mu^\gamma}{\Gamma\gamma}\right) + \log\left\{\prod_{i=1}^{n} (x_{1i})^{\gamma-1}\right\} + \log\left\{\prod_{i=1}^{n} (x_{2i})^{\gamma-1}\right\} - \mu(x_1{}' + x_2{}') + m\log(2\theta) + m\log(\alpha\theta) - 2\theta u' - \alpha\theta w'$$
$$= 2n\gamma\log\mu - 2n\log(\Gamma\gamma) + (\gamma-1)\sum_{i=1}^{n}\log(x_{1i}) + (\gamma-1)\sum_{i=1}^{n}\log(x_{2i}) - \mu(x_1{}' + x_2{}') + m\log(2\theta) + m\log(\alpha\theta) - 2\theta u' - \alpha\theta w'. \tag{A.10}$$

Now,

$$\frac{\partial \log L}{\partial \mu} = 0$$
$$\Longrightarrow \frac{2n\gamma}{\mu} - (x_1' + x_2') = 0$$
$$\Longrightarrow \frac{2n\gamma}{\mu} = (x_1' + x_2') \tag{A.11}$$
$$\Longrightarrow \log(2n) + \log\left(\frac{\gamma}{\mu}\right) = \log(x_1' + x_2').$$

Similarly,

$$\frac{\partial \log L}{\partial \gamma} = 0$$
$$\Longrightarrow 2n\log\mu - 2n\frac{\partial}{\partial\gamma}(\log(\Gamma\gamma)) + \sum_{i=1}^{n}\log(x_{1i}) + \sum_{i=1}^{n}\log(x_{2i}) = 0. \tag{A.12}$$

Now

$$\begin{aligned}
&\frac{\partial}{\partial\gamma}(\log(\Gamma\gamma)) \simeq \log\gamma - \frac{1}{2\gamma}\\
&\Longrightarrow 2n\log\mu - 2n\left\{\log\gamma - \frac{1}{2\gamma}\right\}\\
&+\sum_{i=1}^{n}\log(x_{1i}) + \sum_{i=1}^{n}\log(x_{2i}) = 0\\
&\Longrightarrow 2n\log\mu - 2n\log\gamma + \frac{n}{\gamma} + \sum_{i=1}^{n}\log(x_{1i}) + \sum_{i=1}^{n}\log(x_{2i}) = 0\\
&\Longrightarrow 2n[\log\gamma - \log\mu] = \frac{n}{\gamma} + \sum_{i=1}^{n}\log(x_{1i}) + \sum_{i=1}^{n}\log(x_{2i})\\
&\Longrightarrow 2n\log\left(\frac{\gamma}{\mu}\right) = \frac{n}{\gamma} + \sum_{i=1}^{n}\log(x_{1i}) + \sum_{i=1}^{n}\log(x_{2i}).
\end{aligned} \tag{A.13}$$

Solving (A.11) *and* (A.13) *simultaneously one obtains the MLEs of parameters of γ and μ given in expression* (4) *and* (5), *respectively.*

Now again,

$$\begin{aligned}
&\frac{\partial\log L}{\partial\theta} = 0\\
&\Longrightarrow \frac{m}{\theta} + \frac{m}{\theta} - 2u' - \alpha w' = 0\\
&\Longrightarrow \frac{2m}{\theta} - 2u' - \alpha w' = 0.
\end{aligned} \tag{A.14}$$

Similarly,

$$\begin{aligned}
&\frac{\partial\log L}{\partial\alpha} = 0\\
&\Longrightarrow \frac{m}{\alpha} - \theta w' = 0.
\end{aligned} \tag{A.15}$$

Solving (A.14) *and* (A.15) *simultaneously one obtains the MLEs of parameters of θ and α given in expression* (6) *and* (7), *respectively.*

Lemma A.3. *The posterior distribution of μ and θ, that is, $f_2(\mu,\theta \mid x_{1i}, x_{2i}, u_j, w_j)$ used in expression* (12), *is derived as follows.*

The joint probability density function of random variables, X_{1i}, X_{2i} $(i = 1, 2\cdots n)$, U_j, W_j $(j = 1, 2\cdots m)$, μ and θ is given by X_{1i}, X_{2i} $(i = 1, 2\cdots n)$, U_j, W_j $(j = 1, 2\cdots m)$, μ and θ is given by

$$\begin{aligned}
&f\left(x_{1i}, x_{2i}, u_j, w_j, \mu, \theta\right)\\
&= \left\{\left(\frac{\mu^{\gamma_0}}{\Gamma\gamma_0}\right)^{2n}\cdot\left\{\prod_{i=1}^{n}(x_{1i})^{\gamma_0-1}\right\}\cdot\left\{\prod_{i=1}^{n}(x_{2i})^{\gamma_0-1}\right\}\cdot e^{-\mu(x_1'+x_2')}\right\}\\
&\quad\cdot\left\{(2\theta)^m\cdot(\alpha_0\theta)^m\cdot e^{-2\theta u'}\cdot e^{-\alpha_0\theta w'}\cdot\frac{1}{\Gamma p}\right.\\
&\quad\left.\cdot e^{-\mu}\cdot\mu^{p-1}\cdot\frac{1}{\Gamma q}\cdot e^{-\theta}\cdot\theta^{q-1}\right\}.
\end{aligned} \tag{A.16}$$

Integrating $f(x_{1i}, x_{2i}, u_j, w_j, \mu, \theta)$ with respect to μ and θ over their respective range one gets, the joint probability distribution of X_{1i}, $X_{2i}(i = 1, 2\cdots n)$, U_j, W_j $(j = 1, 2\cdots m)$ as

$$\begin{aligned}
&f_1\left(x_{1i}, x_{2i}, u_j, w_j\right)\\
&= \left[\frac{1}{(\Gamma\gamma_0)^{2n}}\cdot\left\{\prod_{i=1}^{n}(x_{1i})^{\gamma_0-1}\right\}\cdot\left\{\prod_{i=1}^{n}(x_{2i})^{\gamma_0-1}\right\}\right]\\
&\quad\cdot\left[\frac{2^m\cdot\alpha_0{}^m}{\Gamma p\cdot\Gamma q}\cdot\frac{\Gamma(2n\gamma_0+p)}{(x_1'+x_1'+1)^{(2n\gamma_0+p)}}\right.\\
&\quad\left.\cdot\frac{\Gamma(2m+q)}{(2u'+\alpha_0 w'+1)^{(2m+q)}}\right].
\end{aligned} \tag{A.17}$$

Dividing $f(x_{1i}, x_{2i}, u_j, w_j, \mu, \theta)$ in expression (A.16) *by $f_1(x_{1i}, x_{2i}, u_j, w_j)$ in expression* (A.17) *one gets the posterior distribution of μ and θ $f_2(\mu,\theta \mid x_{1i}, x_{2i}, u_j, w_j)$ as*

$$\begin{aligned}
&f_2\left(\mu,\theta \mid x_{1i}, x_{2i}, u_j, w_j\right)\\
&= \left\{\frac{e^{-\mu(x_1'+x_2'+1)}\cdot\mu^{2n\gamma_0+p-1}\cdot e^{-\theta(2u'+\alpha_0 w'+1)}}{\Gamma(2n\gamma_0+p)\cdot\Gamma(2m+q)}\right\}\\
&\quad\cdot\left\{\theta^{(2m+q-1)}\cdot(x_1'+x_2'+1)^{(2n\gamma_0+p)}\right.\\
&\quad\left.\cdot(2u'+\alpha_0 w'+1)^{(2m+q)}\right\}.
\end{aligned} \tag{A.18}$$

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