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Research Article

On the Favard-Type Theorem and the Jackson-Type Theorem (II)

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Let $\mathbf{R} = (-\infty, \infty)$, and let $Q \in \mathbb{C}^1 : \mathbb{R} \to [0, \infty)$ be an even function. We consider the exponential weights $w(x) = e^{-Q(x)}$, $x \in \mathbb{R}$. In this paper we investigate the relations between the Favard-type inequality and the Jackson-type inequality. We also discuss the equivalence of two K-functionals and the modulus of smoothness.

1. Introduction and Preliminaries

Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in \mathbb{C}^1 : \mathbb{R} \to [0, \infty)$ be an even function. We consider the weights $w(x) := \exp(-Q(x))$ satisfying for all $n = 0, 1, 2, \ldots$,

$$\int_0^\infty x^n w(x) dx < \infty. \tag{1.1}$$

First we need the following definition from [1]. We say that $f : \mathbb{R} \to [0, \infty)$ is quasi-increasing if there exists $C > 0$ such that $f(x) \leqslant C f(y)$, $0 < x < y$.

Definition 1.1. One defines, $w = \exp(-Q) \in \mathcal{F}(C^2+)$ *as follows: Let* $Q : \mathbb{R} \to [0, \infty)$ *be continuous and an even function, and satisfy the following properties:*

(a) $Q'(x)$ *is continuous in* $\mathbb{R}$, *with* $Q(0) = 0$;

(b) $Q''(x)$ *exists and is positive in* $\mathbb{R} \setminus \{0\}$;

(c)

$$\lim_{x\to\infty} Q(x) = \infty; \tag{1.2}$$

(d) the function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0 \tag{1.3}$$

is quasi-increasing in $(0,\infty)$ with $T(x) \geqslant \Lambda > 1,\ x \in \mathbb{R}\setminus\{0\}$;

(e) there exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leqslant C_1 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R}\setminus\{0\}. \tag{1.4}$$

Moreover, there also exists a compact subinterval $J(\ni 0)$ of $\mathbb{R}$, and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geqslant C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R}\setminus J. \tag{1.5}$$

Example 1.2. Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$.

(1) If $T(x)$ is bounded, then we call the weight $w = \exp(-Q)$ the Freud-type weight. The following example is the Freud-type weight:

$$w(x) = \exp\left(-|x|^{\alpha}\right), \quad \alpha > 1. \tag{1.6}$$

(2) If $T(x)$ is unbounded, then we call the weight $w = \exp(-Q)$ the Erdös-type weight. The following examples give the Erdős-type weight $w = \exp(-Q)$.

(i) ([1, Example 1.2], [2, Theorem 3.1]). For $\alpha > 1,\ \ell = 1,2,3,\ldots$,

$$Q(x) = Q_{\ell,\alpha}(x) = \exp_{\ell}\left(|x|^{\alpha}\right) - \exp_{l}(0), \tag{1.7}$$

where $\exp_{\ell}(x) = \exp(\exp(\exp\ldots\exp x)\ldots)$, ($\ell$ times). More generally, we define for $\alpha + u > 1,\ \alpha \geqslant 0, u \geqslant 0$, and $l \geqslant 1$,

$$Q_{l,\alpha,u}(x) := |x|^{u}\left(\exp_{l}\left(|x|^{\alpha}\right) - \alpha^{*}\exp_{l}(0)\right), \tag{1.8}$$

where $\alpha^{*} = 0$ if $\alpha = 0$, otherwise $\alpha^{*} = 1$.

(ii) For $\alpha > 1$, $Q(x) = Q_\alpha(x) := (1+|x|)^{|x|^\alpha} - 1$.

We need the Mhaskar-Rakhmanov-Saff numbers a_x defined by

$$x = \frac{2}{\pi}\int_0^1 \frac{a_x u Q'(a_x u)}{(1-u^2)^{1/2}}du, \quad x > 0. \tag{1.9}$$

If $f : \mathbb{R} \to \mathbb{R}$ is measurable, we define

$$\|fw\|_{L_p(\mathbb{R})} := \begin{cases} \left(\int_{-\infty}^{\infty} |f(t)w(t)|^p dt\right)^{1/p}, & \text{if } 0 < p < \infty, \\ \operatorname{ess\,sup}_{t\in\mathbb{R}} |f(t)w(t)|, & \text{if } p = \infty, \end{cases} \tag{1.10}$$

where if $p = \infty$, then we suppose that f is continuous on $\mathbb{R}$ and

$$\lim_{|x|\to\infty} |f(x)w(x)| = 0. \tag{1.11}$$

The class of all functions f for which $\|fw\|_{L_p(\mathbb{R})} < \infty$ will be denoted by $L_{p,w}(\mathbb{R})$, with the usual understanding that two functions are identified if they are equal almost everywhere. For $f \in L_{p,w}(\mathbb{R})$ $(0 < p \leqslant \infty)$, the degree of weighted polynomial approximation is defined by

$$E_{p,n}(w;f) := \inf_{P\in\mathcal{P}_n} \|w(f-P)\|_{L_p(\mathbb{R})}, \tag{1.12}$$

where $\mathcal{P}_n$ denotes the class of all polynomials of degree at most n. There are various estimates for this degree. Among them, in our previous article [3], we discuss the relation between the Favard-type inequality, the Jackson-type inequality, and the estimates by two K-functionals $\mathcal{K}_{r,p}$ and $\widetilde{\mathcal{K}}_{r,p}$. We recall and summarize them in Section 2. In Section 3, we will discuss the equivalence of $\widetilde{\mathcal{K}}_{r,p}$ and modulus of smoothness $\omega_{r,p}$. As the result, we see $\mathcal{K}_{r,p} \sim \widetilde{\mathcal{K}}_{r,p}$ for a weight in a subclass of $\mathcal{F}(C^2+)$. For any nonzero real valued functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if there exists a constant $C > 0$ independent of x such that for all x

$$\left(\frac{1}{C}\right) f(x) \leqslant g(x) \leqslant C f(x). \tag{1.13}$$

Similarly, for any positive numbers $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$ we define $c_n \sim d_n$. Throughout this paper $C, C_1, \ldots, c, c_1, \ldots$ denote positive constants independent of n, x, t, or $P_n(x)$. The same symbols do not necessarily denote the same constants occurrences.

2. Known Results and Summarization

Let $r \geqslant 1$ be an integer (in this paper, we suppose that r (or s) is an integer). The rth order K-functional of a function $f \in L_{p,w}(\mathbb{R})$ is defined by the formula

$$\mathcal{K}_{r,p}(w, f, \delta) := \inf\left\{ \|w(f-g)\|_{L_p(\mathbb{R})} + \delta^r \left\| w g^{(r)} \right\|_{L_p(\mathbb{R})} \right\}, \tag{2.1}$$

for $\delta > 0$, where the infimum is taken over all $r-1$ times continuously differentiable g such that $g^{(r-1)}$ is absolutely continuous, and $g^{(r)} \in L_{p,w}(\mathbb{R})$ (this means $wg^{(r)} \in L_p(\mathbb{R})$). Using this rth order K-functional, we can estimate the order of $E_{p,n}(w; f)$.

Theorem 2.1 (see [3]). *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$. Let $r \geqslant 1$, $s \geqslant 0$ be integers, and let $1 \leqslant p \leqslant \infty$. Let f be $s-1$ times continuously differentiable, $f^{(s-1)}$ be absolutely continuous on each compact interval, and $f^{(s)} \in L_{p,w}(\mathbb{R})$ (when $s = 0$, these assumptions state merely that $f \in L_{p,w}(\mathbb{R})$). Then, for every integer $n \geqslant r + s$,*

$$E_{p,n}(w; f) \leqslant C\left(\frac{a_n}{n}\right)^s \mathcal{K}_{r,p}\left(w, f^{(s)}, \frac{a_n}{n}\right). \tag{2.2}$$

Theorem 2.1 was shown by using the following Favard-type inequalities (see [3]).

Theorem 2.2 ([4, Corollary 8]). *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$. Let f be $s-1$ times continuously differentiable, and let $f^{(s-1)}$ for some integer $s \geqslant 1$ be absolutely continuous on each compact interval. Let $1 \leqslant p \leqslant \infty$ and $f^{(s)} \in L_{p,w}(\mathbb{R})$. Then one has*

$$E_{p,n}(w; f) \leqslant C\left(\frac{a_n}{n}\right)^s \left\| w f^{(s)} \right\|_{L_p(\mathbb{R})}, \tag{2.3}$$

equivalently,

$$E_{p,n}(w; f) \leqslant C\left(\frac{a_n}{n}\right)^s E_{p,n-s}\left(w; f^{(s)}\right). \tag{2.4}$$

Remark 2.3. (1) ([4, Remark 11]) Let $w \in \mathcal{F}(C^2+)$ and let $0 < p < 1$. Then there exists a constant C_0 such that for every absolutely continuous function f with $f' \in L_{\infty,w}$ and $w'f \in L_p(\mathbb{R})$, and for every $n \in \mathbb{N}$, we have

$$E_{p,n}(w; f) \leqslant C_0\left(\frac{a_n}{n}\right)\left\{ \|wf'\|_{L_\infty(\mathbb{R})} + \|w'f\|_{L_p(\mathbb{R})} \right\}. \tag{2.5}$$

(2) ([3, Section 4]) As a by-product of the method of the proof for Theorem 2.2, we can obtain the degree of functions which satisfy the Hölder-Lipschitz condition. Let $w \in \mathcal{F}(C^2+), 1 \leqslant p \leqslant \infty$ and $1/p \leqslant \beta \leqslant 1$. Let g be absolutely continuous with

$|g'|^\beta \in L_{p,w}(\mathbb{R})$ (and for $p = \infty$, we require g to be continuous, and gw to vanish at $\pm\infty$). Let $f \in L_{p,w}(\mathbb{R})$ and set

$$f^*_{g,\beta}(x) := \sup_{y\in\mathbb{R}} \frac{|f(x) - f(y)|}{|g(x) - g(y)|^\beta}. \tag{2.6}$$

Then we define

$$\mathbf{F}_{g,\beta}(w,p) := \left\{ f \mid f \in L_{p,w}(\mathbb{R}), \left\| f^*_{g,\beta} \right\|_{L_\infty(\mathbb{R})} < \infty \right\}. \tag{2.7}$$

Now, we have the following theorem:

Theorem 2.4 ([3, Theorem 4.2]). *Let $1 \leqslant p \leqslant \infty$, $1/p \leqslant \beta \leqslant 1$, and let $f \in \mathbf{F}_{g,\beta}(w,p)$. Then one has*

$$E_{p,n}(w;f) \leqslant C\left(\frac{a_n}{n}\right)^\beta \left\| w|g'|^\beta \right\|_{L_p(\mathbb{R})} \left\| f^*_{g,\beta} \right\|_{L_\infty(\mathbb{R})}. \tag{2.8}$$

We define the following class of weights from [5, Definition 1.1].

Definition 2.5. Let $w(x) := \exp(-Q(x))$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, continuous, and Q' is positive in $(0,\infty)$. Then one writes $w = \exp(-Q) \in \mathcal{E}_1$, if the following are satisfied:

(a) *$xQ'(x)$ is strictly increasing in $(0,\infty)$ with $\lim_{x\to 0+} xQ'(x) = 0$;*

(b) *the function*

$$T(x) := \frac{xQ'(x)}{Q(x)} \tag{2.9}$$

is quasi-increasing in (C,∞) for some $C > 0$ and $\lim_{x\to\infty} T(x) = \infty$;

(c) *assume*

$$\frac{yQ'(y)}{xQ'(x)} \leqslant C_1\left(\frac{Q(y)}{Q(x)}\right)^{C_3}, \quad y \geqslant x \geqslant C_2, \tag{2.10}$$

for some positive constants $C_1, C_2,$ and C_3.

Remark 2.6. Let $w(x) := \exp(-Q(x)) \in \mathcal{F}(C^2+)$ and $T(x)$ is unbounded, then we see $w \in \mathcal{E}_1$. In fact, we see this as follows.

From Definition 1.1(e) and (d), we have for $y \geqslant x > 0$,

$$\begin{aligned}
\frac{Q'(y)}{Q'(x)} &= \exp\left(\int_x^y \frac{Q''(t)}{Q'(t)}dt\right), \\
&\leqslant \exp\left(C_1 \int_x^y \frac{Q'(t)}{Q(t)}dt\right) = \left(\frac{Q(y)}{Q(x)}\right)^{C_1}, \\
\frac{y}{x} &= \exp\left(\int_x^y \frac{1}{t}dt\right) \leqslant \exp\left(\frac{1}{\Lambda}\int_x^y \frac{Q'(t)}{Q(t)}dt\right) \\
&= \left(\frac{Q(y)}{Q(x)}\right)^{1/\Lambda}.
\end{aligned} \tag{2.11}$$

Therefore we obtain (c) in Definition 2.5 for $C_3 = C_1 + 1/\Lambda$.

If $f : \mathbb{R} \to \mathbb{R}$, and $h > 0$, then we define the differences of f inductively by the formula

$$\begin{aligned}
\Delta_h^0 f(x) &= f(x), \\
\Delta_h^1 f(x) &= \Delta_h f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right), \\
\Delta_h^k f(x) = \Delta_h^{k-1}(\Delta_h f)(x) &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} f\left(x + \frac{kh}{2} - jh\right), \quad k = 2, 3, .., x \in \mathbb{R}.
\end{aligned} \tag{2.12}$$

We set

$$\begin{aligned}
\sigma(t) &:= \inf\left\{a_u : \frac{a_u}{u} \leqslant t\right\}, \quad t > 0, \\
\Phi_t(x) &:= \sqrt{\left|1 - \frac{|x|}{\sigma(t)}\right|} + T(\sigma(t))^{-1/2}, \quad x \in \mathbb{R}.
\end{aligned} \tag{2.13}$$

By [6], when $w = \exp(-Q)$ is the Erdős-type weight we define for $f \in L_{p,w}(\mathbb{R}), 0 < p \leqslant \infty$,

$$\begin{aligned}
\omega_{r,p}(w, f, t) := &\sup_{0<h\leqslant t} \left\| w \Delta_{h\Phi_t(x)}^r (f) \right\|_{L_p(|x|\leqslant\sigma(2t))} \\
&+ \inf_{R\in\mathcal{P}_{r-1}} \left\| w(x)(f - R)(x) \right\|_{L_p(|x|\geqslant\sigma(4t))}.
\end{aligned} \tag{2.14}$$

If $w = \exp(-Q)$ is the Freud-type weights, then we define

$$\begin{aligned}
\omega_{r,p}(w, f, t) := &\sup_{0<h\leqslant t} \left\| w \Delta_h^r (f) \right\|_{L_p(|x|\leqslant\sigma(h))} \\
&+ \inf_{R\in\mathcal{P}_{r-1}} \left\| w(x)(f - R)(x) \right\|_{L_p(|x|\geqslant\sigma(t))}.
\end{aligned} \tag{2.15}$$

The following Jackson-type inequality is known.

Theorem 2.7 (see [5, Theorem 1.2], [6, Corollary 1.4]). *Let $w \in \mathcal{E}_1$. Let $0 < p \leqslant \infty$. Then for $f : \mathbb{R} \to \mathbb{R}$ for which $f \in L_{p,w}(\mathbb{R})$ (and for $p = \infty$, we require f to be continuous, and fw to vanish at $\pm\infty$), one has for $n \geqslant C_3$,*

$$E_{p,n}(f;w) \leqslant C_1 \omega_{r,p}\left(w, f, C_2 \frac{a_n}{n}\right), \tag{2.16}$$

where C_j, $j = 1, 2, 3$, do not depend on f and n.

We also consider the following class of weights which are called the Freud weights.

Definition 2.8 ([7, Definition 3.3]). *Let $w = \exp(-Q)$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, and $Q\prime$ exists and is positive on $(0, \infty)$. Moreover, assume that $xQ'(x)$ is strictly increasing, with right limit 0 at 0, and for some $\lambda, A, B > 1$, $C > 0$,*

$$A \leqslant \frac{Q'(\lambda x)}{Q'(x)} \leqslant B, \qquad x \geqslant C. \tag{2.17}$$

Then we call w Freud weight, and write $w \in \mathcal{F}^*$.

Remark 2.9. Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$. Then we see the following.

(1) If $T(x) = (xQ'(x))/Q(x)$ is bounded, then we say that $w = \exp(-Q)$ is the Freud-type weight, and we write $w \in \mathcal{F}$. Then we see $\mathcal{F}^* \cap \mathcal{F}(C^2+) = \mathcal{F}$. In fact, when $w \in \mathcal{F}$, $1 < \nu$ and $x > 0$ large enough, by Definition 1.1(e),

$$\frac{Q'(\nu x)}{Q'(x)} = \exp\left(\int_x^{\nu x} \frac{Q''(u)}{Q'(u)} du\right) \leqslant \exp\left(C_1 \int_x^{\nu x} \frac{Q'(u)}{Q(u)} du\right), \tag{2.18}$$

and then since $T(x)$ is bounded, there exists $C > 0$ such that

$$\frac{Q'(\nu x)}{Q'(x)} \leqslant \exp\left(C_1 C \int_x^{\nu x} \frac{1}{u} du\right) = \nu^{C_1 C}. \tag{2.19}$$

Similarly,

$$\begin{aligned} \frac{Q'(\nu x)}{Q'(x)} &= \exp\left(\int_x^{\nu x} \frac{Q''(u)}{Q'(u)} du\right) \geqslant \exp\left(C_2 \int_x^{\nu x} \frac{Q'(u)}{Q(u)} du\right) \\ &\geqslant \exp\left(C_2 \Lambda \int_x^{\nu x} \frac{1}{u} du\right) = \nu^{C_2 \Lambda}. \end{aligned} \tag{2.20}$$

Therefore, if we take $\lambda := \nu > 1$ large enough, then we have (2.17).

Conversely, to show $\mathcal{F}^* \cap \mathcal{F}(C^2+) \subset \mathcal{F}$ we suppose that there exists $w = \exp(-Q) \in \mathcal{F}(C^2+)$ such that $T(x) = (xQ'(x))/Q(x)$ is unbounded. Then since $T(x)$ is quasi-increasing

in $(0,\infty)$, we see that $T(x) \to \infty$ as $x \to \infty$. So, for any $M > 0$ there exists $L > 0$ (large enough) such that $T(x) > M$ for $x > L$. Therefore, we have for $x > L$ and any $\lambda > 1$,

$$\begin{aligned}\frac{Q'(\lambda x)}{Q'(x)} &= \exp\left(\int_x^{\lambda x} \frac{Q''(u)}{Q'(u)} du\right) \geqslant \exp\left(C_3 \int_x^{\lambda x} \frac{Q'(u)}{Q(u)} du\right) \\ &= \exp\left(C_3 \int_x^{\lambda x} \frac{T(u)}{u} du\right) = \lambda^{C_3 M},\end{aligned} \tag{2.21}$$

where $C_3 > 0$ is a constant, that is, (1) does not hold. Hence we have $\mathcal{F}^* \cap \mathcal{F}(C^2+) \subset \mathcal{F}$. Consequently, we have $\mathcal{F} = \mathcal{F}^* \cap \mathcal{F}(C^2+)$.

(2) If $T(x) = (xQ'(x))/Q(x)$ is bounded, then for $x \geqslant 1$, there exists $c, C > 0$ such that

$$x^{\Lambda} \leqslant Q(x) \leqslant Cx^{c}. \tag{2.22}$$

Theorem 2.10 ([7, Theorem 3.5]). *Let $w \in \mathcal{F}^*$. Let $0 < p \leqslant \infty$. Then for $f : \mathbb{R} \to \mathbb{R}$ for which $f \in L_{p,w}(\mathbb{R})$ (and for $p = \infty$, one requires f to be continuous, and fw to vanish at $\pm\infty$), one has for $n \geqslant C_3$,*

$$E_{p,n}(f;w) \leqslant C_1 \omega_{r,p}\left(w, f, C_2 \frac{a_n}{n}\right), \tag{2.23}$$

where C_j, $j = 1,2,3$, do not depend on f and n.

Damelin [6] introduces the following K-functional: let $f \in L_{p,w}(\mathbb{R})$, $0 < p \leqslant \infty$ and $r \geqslant 1$ be an integer, then we define

$$\widetilde{\mathcal{K}}_{r,p}(w, f, \delta) := \inf_{P \in \mathcal{P}_n} \left\{ \left\| w(f - P) \right\|_{L_p(\mathbb{R})} + \delta^r \left\| wP^{(r)} \right\|_{L_p(\mathbb{R})} \right\}, \tag{2.24}$$

where $\delta > 0$ are chosen in advance and

$$n = n(\delta) := \inf\left\{ k : \frac{a_k}{k} \leqslant \delta \right\}. \tag{2.25}$$

Then Damelin gives the following.

Theorem 2.11 ([6, Theorem 1.3 (b)]). *Let $w \in \mathcal{E}_1$, $r \geqslant 1$, $0 < p \leqslant \infty$, and let $fw \in L_p(\mathbb{R})$ (and for $p = \infty$, one requires f to be continuous, and fw to vanish at $\pm\infty$). Then one has*

$$\omega_{r,p}(w, f, t) \sim \widetilde{\mathcal{K}}_{r,p}(w, f, \delta). \tag{2.26}$$

For the Freud-type weights we have also the followings.

Theorem 2.12 ([7, Theorem 3.9, 3.10]). *Let $w \in \mathcal{F}^*$, $r \geqslant 1$, $0 < p \leqslant \infty$, and let $fw \in L_p(\mathbb{R})$ (and for $p = \infty$, we require f to be continuous, and fw to vanish at $\pm\infty$). Then we have*

$$\mathcal{K}_{r,p}(w, f, \delta) \sim \omega_{r,p}(w, f, t) \sim \widetilde{\mathcal{K}}_{r,p}(w, f, \delta). \tag{2.27}$$

For $w \in \mathcal{F}(C^2+)$, we see easily that

$$\mathcal{K}_{r,p}(w, f, \delta) \leqslant \widetilde{\mathcal{K}}_{r,p}(w, f, \delta). \tag{2.28}$$

So, from Theorem 2.1 we obtain the following corollary.

Corollary 2.13. *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$. Let $r \geqslant 1, s \geqslant 0$ be integers, and let $1 \leqslant p \leqslant \infty$. Let f be $s-1$ times continuously differentiable, $f^{(s-1)}$ be absolutely continuous, and $f^{(s)} \in L_{p,w}(\mathbb{R})$ (when $s = 0$, these assumptions state merely that $f \in L_{p,w}(\mathbb{R})$). Then, for every integer $n \geqslant r + s$,*

$$E_{p,n}(w; f) \leqslant C\left(\frac{a_n}{n}\right)^s \widetilde{\mathcal{K}}_{r,p}\left(w, f^{(s)}, \frac{a_n}{n}\right). \tag{2.29}$$

The main theme in [3] is to summarize the above theorems. Let $w \in \mathcal{F}(C^2+), r \geqslant 1$ be an integer, and let $1 \leqslant p \leqslant \infty$. We have the following succession of the theorems. We use the constant $C_i > 0,\ i = 1, 2, 3, \ldots$ which do not depend on f and n.

(a) [Theorem 2.7 with $r = 1$ (the Erdős-type case)], [Theorem 2.10 with $r = 1$ (the Freud case)]:

let $f : \mathbb{R} \to \mathbb{R}$ and if $1 \leqslant p < \infty$, assume that $fw \in L_p(\mathbb{R})$. If $p = \infty$, assume in addition that f is continuous and that fw has limit 0 at $\pm\infty$. Then we have

$$E_{p,n}(w; f) \leqslant C_1 \omega_{1,p}\left(w, f, C_2 \frac{a_n}{n}\right). \tag{2.30}$$

(b) [Theorem 2.2]:

let f be $s-1$ times continuously differentiable, and let for some integer $s \geqslant 1$, $f^{(s-1)}(x)$ be absolutely continuous on each compact interval. Let $1 \leqslant p \leqslant \infty$ and $f^{(s)} \in L_{p,w}(\mathbb{R})$. Then we have

$$E_{p,n}(w; f) \leqslant C_3\left(\frac{a_n}{n}\right)^s \left\| w f^{(s)} \right\|_{L_p(\mathbb{R})}, \tag{2.31}$$

equivalently,

$$E_{p,n}(w; f) \leqslant C_4\left(\frac{a_n}{n}\right)^s E_{p,n}\left(w; f^{(s)}\right). \tag{2.32}$$

(c) [Theorem 2.1]:

let $s \geqslant 0$ be an integer, and let f be $s-1$ times continuously differentiable, $f^{(s-1)}$ be absolutely continuous, and $f^{(s)} \in L_{p,w}(\mathbb{R})$ (when $s=0$, these assumptions state merely that $f \in L_{p,w}(\mathbb{R})$). Then, for every integer $r \geqslant 1$ and $n \geqslant r+s$,

$$E_{p,n}(w;f) \leqslant C_5\left(\frac{a_n}{n}\right)^s \mathcal{K}_{r,p}\left(w, f^{(s)}, \frac{a_n}{n}\right). \tag{2.33}$$

(d) [Theorem 2.7, 2.11 (the Erdős-type case)], [Theorem 2.10, 2.12 (the Freud case)]:

for $f : \mathbb{R} \to \mathbb{R}$ with $f \in L_{p,w}(\mathbb{R})$ (and for $p=\infty$ we require f to be continuous, and fw to vanish at $\pm\infty$), for every n large enough and for every integer $r \geqslant 1$ we have

$$E_{p,n}(w;f) \leqslant C_6\widetilde{\mathcal{K}}_{r,p}\left(w, f, \frac{a_n}{n}\right). \tag{2.34}$$

Consequently we find an interesting fact as follows:

(b) is shown by the use of (a) (see [4]). Using (b), we proved (c). If we use [6, Theorem 1.3 (b)] and [7, Theorems 3.9 and 3.10], we see easily that (c) means (d) with Theorem 2.11. Trivially, we have (a) from (d).

3. Equivalence of K-Functional and Modulus of Smoothness

In this section we consider a subclass of $\mathcal{F}(C^2+)$.

Assumption 3.1. Let $r \geqslant 2$ be an integer. Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$. Then we suppose that $Q \in C^{r-1}(\mathbb{R})$, $Q^{(r)}(x)$ exists in $\mathbb{R} \setminus \{0\}$, furthermore, the following inequalities hold

$$\left|Q^{(j)}(x)\right| \leqslant C_j\left|Q^{(j-1)}(x)\right|\left|\frac{Q'(x)}{Q(x)}\right|, \quad j = 1, 2, \ldots, r, \tag{3.1}$$

where for $j=r$ we suppose that (3.1) hold almost everywhere on $x \in \mathbb{R} \setminus \{0\}$. Then we write $w \in \mathcal{F}(C^r+)$.

Remark 3.2. Example 1.2, (i), (ii) satisfy Assumption 3.1, moreover $Q(x) = x^{2m}$, m: a positive integer, satisfies Assumption 3.1.

In Section 2, we know

$$\mathcal{K}_{r,p}(w,f,t) \leqslant \widetilde{\mathcal{K}}_{r,p}(w,f,t) \sim \omega_{r,p}(w,f,t). \tag{3.2}$$

The equivalences mentioned in the last of Section 2 give a certain suggestion, that is,

$$\mathcal{K}_{r,p}(w,f,t) \sim \omega_{r,p}(w,f,t). \tag{3.3}$$

In fact, this is true.

Theorem 3.3. *Let $r \geqslant 2$ be a positive integer, and let $w \in \mathcal{F}(C^r+)$. Let $f \in L_{p,w}(\mathbb{R})$. Then one has* (3.3). *Similarly, when $w \in \mathcal{F}(C^2+)$ and $r = 1$, one has* (3.3).

To prove Theorem 3.3, we need some lemmas.

Lemma 3.4. *Let $r \geqslant 1$ be an integer. If $f^{(r-1)}(x)$ is absolutely continuous on $\mathbb{R}$, then for $k = 1, 2, \ldots, r$, one has the following representation:*

$$\Delta^k_{\Phi_t(x)h} f(x) = (\Phi_t(x))^k \left\{ \int_{-h/2}^{h/2} \right\}^k f^{(k)}(x + \Phi_t(x)(u_1 + u_2 + \cdots + u_k)) du_1 du_2 \cdots du_k. \tag{3.4}$$

Proof.

$$\begin{aligned} \int_{h/2}^{h/2} f^{(1)}(x + \Phi_t(x)u_1) du_1 &= (\Phi_t(x))^{-1} \int_{x-\Phi_t(x)h/2}^{x+\Phi_t(x)h/2} f^{(1)}(v_1) dv_1 \\ &= (\Phi_t(x))^{-1} \triangle_{\Phi t(x)h} f(x). \end{aligned} \tag{3.5}$$

Therefore, for $r = 1$ we have (3.4). For some $k \geqslant 1$ we suppose (3.4). Then we have

$$\begin{aligned} &\left\{ \int_{-h/2}^{h/2} \right\}^{k+1} f^{(k+1)}(u) du_1 du_2 \cdots du_{k+1} \\ &\quad = \int_{-h/2}^{h/2} \left[\left\{ \int_{-h/2}^{h/2} \right\}^k (f')^{(k)}(\tilde{u}) du_1 du_2 \cdots du_k \right] du_{k+1} \\ &\quad = (\Phi_t(x))^{-k} \int_{-h/2}^{h/2} \Delta^k_{\Phi_t(x)h} f'(x + \Phi_t(x)u_{k+1}) du_{k+1} \\ &\quad = (\Phi_t(x))^{-(k+1)} \Delta^k_{\Phi_t(x)h} \left(\Delta_{\Phi_t(x)h} f(x) \right) \\ &\quad = (\Phi_t(x))^{-(k+1)} \Delta^{k+1}_{\Phi_t(x)h} f(x), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} u &= u(x; u_1, \ldots, u_{k+1}) = x + \Phi_t(x)(u_1 + u_2 + \cdots + u_{k+1}), \\ \tilde{u} &= \tilde{u}(x; u_1, \ldots, u_{k+1}) = x + \Phi_t(x)u_{k+1} + \Phi_t(x)(u_1 + u_2 + \cdots + u_k). \end{aligned} \tag{3.7}$$

Hence we have (3.4) for $r = 1, 2, 3, \ldots$. □

Lemma 3.5 (see [1, Lemma 3.4 (3.17)]). *Uniformly for $t > 0$, one has*

$$Q'(a_t) \sim \frac{t\sqrt{T(a_t)}}{a_t}. \tag{3.8}$$

Lemma 3.6. *Let $w \in \mathcal{F}(C^r+)$ and $r \geqslant 2$ be an integer. Then for any integer k, $1 \leqslant k \leqslant r-1$, there exist $c_1, c_2 > 0$ and $A > 0$ such that for $|x| \geqslant A$,*

$$c_1 \leqslant (-1)^k \frac{w^{(k)}(x)}{\left(1+Q'(x)^2\right)^{k/2} w(x)} \leqslant c_2. \tag{3.9}$$

Also, there exist $c_3, c_4 > 0$ and $B > 0$ such that for $|x| \geqslant B$,

$$c_3 \leqslant (-1)^k \frac{\left(w^{-1}\right)^{(k)}(x)}{\left(1+Q'(x)^2\right)^{k/2} w^{-1}(x)} \leqslant c_4. \tag{3.10}$$

Furthermore, for $k = r$, (3.9) and (3.10) hold almost everywhere on $|x| \geqslant A$ and $|x| \geqslant B$, respectively. When $w \in \mathcal{F}(C^2+)$ and $r = 1$, one also has (3.9) and (3.10).

Proof. First, we will see that for $\mu = \pm 1$ the following equations hold:

$$\lim_{|x| \to} \frac{\left(w^{\mu}\right)^{(k)}(x)}{Q'(x)^k w^{\mu}(x)} = (-\mu)^k, \quad k = 1, \ldots, r-1. \tag{3.11}$$

For $k = r$ we take $\tilde{Q}(x)$ as follows;

$$\begin{gathered} \tilde{Q}^{(j)}(x) = Q^{(j)}(x), \quad x \in \mathbb{R},\ j = 1, 2, \ldots, r-1, \\ \tilde{Q}^{(r)}(x) = Q^{(r)}(x), \quad \text{a.e. } x \in \mathbb{R} \setminus \{0\}, \end{gathered} \tag{3.12}$$

and for all $j = 1, 2, \ldots, r$, $\tilde{Q}(x)$ satisfies (3.1) for all $x \neq 0$. Then we obtain that exchanging Q with $\tilde{Q}$, (3.11) also holds.

Let $\mu = 1$, and let $1 \leqslant k \leqslant r-1$.

$$\begin{gathered} w'(x) = -Q'(x) w(x), \\ w''(x) = \left(-Q''(x) + Q'(x)^2\right) w(x), \\ w'''(x) = \left(-Q'''(x) + 3Q'(x)Q''(x) - Q'(x)^3\right) w(x), \end{gathered} \tag{3.13}$$

and we continue this manner, so

$$w^{(k)}(x) = \left(\sum_{\substack{i_1 + 2i_2 + \cdots + k i_k = k, \\ i_1 < k}} c_{i_1, i_2, \ldots, i_k} Q'(x)^{i_1} Q''(x)^{i_2} \cdots Q^{(k)}(x)^{i_k} + (-1)^k Q'(x)^k \right) w(x), \tag{3.14}$$

where $c_{i_1,i_2,\ldots,i_k}$ are coefficients. Here, from (3.1), for $|x| \geqslant A > 0$ large enough and $i_j \neq 0$, $2 \leqslant j \leqslant k$,

$$\left|Q^{(j)}(x)\right|^{i_j} \leqslant C_j\left(\left|\frac{Q'(x)}{Q(x)}\right|^{j-1}\left|Q'(x)\right|\right)^{i_j} = C_j\left(\frac{|Q'(x)|^j}{Q(x)^{j-1}}\right)^{i_j}. \tag{3.15}$$

Hence, from (3.14) and (3.15) we have

$$\left|\lim_{|x|\to\infty}\frac{w^{(k)}(x)}{Q'(x)^k w(x)} - (-1)^k\right| \leqslant C\lim_{|x|\to\infty}\frac{1}{Q(x)} = 0, \quad k = 1,\ldots,r-1, \tag{3.16}$$

where C is a positive constant. Therefore, we have (3.11) with $\mu = 1$. Similarly, for $\mu = -1$,

$$\lim_{|x|\to\infty}\frac{\left(w^{-1}\right)^{(k)}(x)}{Q'(x)^k w^{-1}(x)} = 1, \quad k = 1,\ldots,r-1. \tag{3.17}$$

Therefore, we have (3.11) for $k = 1,2,\ldots,r-1$, and hence we also have (3.9) and (3.10) for $|x|$ large enough. If in (3.11), we replace Q with $\widetilde{Q}$, then repeating the above proof we also obtain (3.11), so for $k = r$ we conclude (3.9) and (3.10) with a.e. x large enough, that is,

$$\begin{aligned} c_1 &\leqslant (-1)^r\frac{w^{(r)}(x)}{\left(1+Q'(x)^2\right)^{r/2}w(x)} \leqslant c_2, \quad \text{a.e. } |x| \geqslant A, \\ c_3 &\leqslant (-1)^r\frac{\left(w^{-1}\right)^{(r)}(x)}{\left(1+Q'(x)^2\right)^{r/2}w^{-1}(x)} \leqslant c_4, \quad \text{a.e. } |x| \geqslant B. \end{aligned} \tag{3.18}$$

For $r = 1$, the lemma is trivial. □

Lemma 3.7. *Let $r \geqslant 1$ be an integer. There exists $C > 0$ such that for every integer $k = 1,2,\ldots,r$,*

$$\left|Q'(x)\right|^k w(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty, \tag{3.19}$$

$$\frac{w^{-1}(x)}{\left(1+Q'(x)^2\right)^{(k-1)/2}}\int_{|x|}^{\infty}\left(1+Q'(t)^2\right)^{k/2} w(t)dt \leqslant C, \quad x \in \mathbb{R}, \tag{3.20}$$

and for $x \in \mathbb{R}$,

$$\left(1+Q'(x)^2\right)^{k/2}w(x)\int_0^{|x|}\left(1+Q'(t)^2\right)^{-(k-1)/2}w^{-1}(t)dt \leqslant C, \quad x \in \mathbb{R}. \tag{3.21}$$

Proof. From Definition 1.1(e) we see that there exist $C > 0$ and $\lambda > 0$ such that

$$\left|Q'(x)\right| \leqslant CQ(x)^{\lambda}, \quad x \in \mathbb{R}, \tag{3.22}$$

so we have

$$\left|Q'(x)\right|^{k} w(x) \leqslant C^{k} Q(x)^{\lambda k} w(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty. \tag{3.23}$$

Hence, we have (3.19). From (3.9) with $k-1$, we see that for $|x| \geqslant A > 0$ and constants $c_1, c_2, c_3 > 0$,

$$\begin{aligned} &\frac{w^{-1}(x)}{\left(1+Q'(x)^2\right)^{(k-1)/2}} \int_{|x|}^{\infty} \left(1+Q'(t)^2\right)^{k/2} w(t)dt \\ &\quad \leqslant c_1 \frac{w^{-1}(x)}{\left(1+Q'(x)^2\right)^{(k-1)/2}} \int_{|x|}^{\infty} (-1)^k w^{(k)}(t)dt \\ &\quad \leqslant c_2 \frac{w^{-1}(x)}{\left(1+Q'(x)^2\right)^{(k-1)/2}} (-1)^{k-1} w^{(k-1)}(x) \leqslant c_3. \end{aligned} \tag{3.24}$$

Especially, if we use (3.24) with $|x| = A$, then we have

$$\frac{w^{-1}(A)}{\left(1+Q'(A)^2\right)^{(k-1)/2}} \int_{A}^{\infty} \left(1+Q'(t)^2\right)^{k/2} w(t)dt \leqslant c_3. \tag{3.25}$$

Then, for $|x| \leqslant A$ there exists a constant $C = C(A)$ such that

$$\frac{w^{-1}(x)}{\left(1+Q'(x)^2\right)^{(k-1)/2}} \leqslant C \frac{w^{-1}(A)}{\left(1+Q'(A)^2\right)^{(k-1)/2}}. \tag{3.26}$$

In fact, for $|x| \leqslant A$ we have

$$1+Q'(A)^2 \leqslant \left(1+Q'(A)^2\right)\left(1+Q'(x)^2\right), \tag{3.27}$$

that is,

$$\left(1+Q'(A)^2\right)^{(k-1)/2} \leqslant C(A)\left(1+Q'(x)^2\right)^{(k-1)/2}, \tag{3.28}$$

where $C(A) = (1+Q'(A)^2)^{(k-1)/2}$. Hence we have (3.26).

Now, by (3.26), if $|x| \leqslant A$, then we have

$$\frac{w^{-1}(x)}{\left(1+Q'(x)^2\right)^{(k-1)/2}} \int_A^{\infty} \left(1+Q'(t)^2\right)^{k/2} w(t)dt \leqslant c_4(A). \tag{3.29}$$

When $|x| < A$, we see easily that

$$\frac{w^{-1}(x)}{\left(1+Q'(x)^2\right)^{(k-1)/2}} \int_{|x|}^{A} \left(1+Q'(t)^2\right)^{k/2} w(t)dt \leqslant c_5 = c_5(A). \tag{3.30}$$

Hence, with (3.29) we have for $|x| \leqslant A$,

$$\frac{w^{-1}(x)}{\left(1+Q'(x)^2\right)^{(k-1)/2}} \int_{|x|}^{\infty} \left(1+Q'(t)^2\right)^{k/2} w(t)dt \leqslant c_6(A). \tag{3.31}$$

Consequently, from (3.24) and (3.31) we have (3.20). We need to show (3.21). For $|x|$ large enough, we see

$$Q''(x) \sim \frac{Q'(x)^2}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R}, \tag{3.32}$$

(see Definition 1.1(e)) so we can select $|x| \geqslant A > 0$ large enough such that

$$Q''(x) < \frac{1}{2k} Q'(x)^2, \quad \text{a.e. } x \in \mathbb{R}. \tag{3.33}$$

We show (3.21) for $\int_A^{|x|}$, $|x| \geqslant A > 0$. For $|x| \geqslant A$, we have by (3.33),

$$\begin{aligned} I &:= \int_A^{|x|} \frac{w^{-1}(t)}{Q'(t)^{k-1}} dt = \int_A^{|x|} \frac{Q'(t)w^{-1}(t)}{Q'(t)^k} dt \\ &\leqslant \frac{w^{-1}(x)}{Q'(|x|)^k} + k \int_A^{|x|} w^{-1}(t) \frac{Q''(t)}{Q'(t)^{k+1}} dt \\ &\leqslant \frac{w^{-1}(x)}{Q'(|x|)^k} + \frac{1}{2} I. \end{aligned} \tag{3.34}$$

Hence, we have for a certain constant $c_7 > 0$,

$$\left(1+Q'(x)^2\right)^{k/2} w(x) \int_A^{|x|} \frac{w^{-1}(t)}{\left(1+Q'(t)^2\right)^{(k-1)/2}} dt \leqslant c_7. \tag{3.35}$$

For $\int_0^A$ we have (3.21) by the use of (3.19). In fact, there exists $c_8 = c_8(A)$ such that

$$\left(1+Q'(x)^2\right)^{k/2} w(x) \int_0^A \frac{w^{-1}(t)}{\left(1+Q'(t)^2\right)^{(k-1)/2}} dt \leqslant c_8. \tag{3.36}$$

So, with (3.35) we conclude (3.21). □

To prove Theorem 3.3 we further need the following lemma.

Lemma 3.8. *For $r = 1$, one lets $w \in \mathcal{F}(C^2+)$, and for integer $r \geqslant 2$ one lets $w \in \mathcal{F}(C^r+)$. Let $1 \leqslant p \leqslant \infty$, and $1 \leqslant k \leqslant r$ be an integer. If $g : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, $g(0) = 0$, and $|Q'(x)|^{k-1} g' \in L_{p,w}(\mathbb{R})$, then*

$$\left\| \left(1+Q'(x)^2\right)^{k/2} wg \right\|_{L_p(\mathbb{R})} \leqslant C \left\| \left(1+Q'(x)^2\right)^{(k-1)/2} wg' \right\|_{L_p(\mathbb{R})}. \tag{3.37}$$

Proof. We will prove (3.37) for $p = 1$ and $p = \infty$, and then we use the Riesz-Thorin interpolation theorem. Let

$$\varphi(t) := \left(1+Q'(t)^2\right)^{(k-1)/2} w(t) g'(t), \quad t \in \mathbb{R}. \tag{3.38}$$

Then for almost all $x \geqslant 0$,

$$|g(x)| + |g(-x)| \leqslant \int_0^x \frac{w^{-1}(t)}{\left(1+Q'(t)^2\right)^{(k-1)/2}} \{|\varphi(t)| + |\varphi(-t)|\} dt. \tag{3.39}$$

Denoting

$$\Psi(t) := |\varphi(t)| + |\varphi(-t)|, \tag{3.40}$$

we get

$$\begin{aligned}
\left\| \left(1+Q'^2\right)^{k/2} wg \right\|_{L_1(\mathbb{R})} &= \int_0^\infty \left(1+Q'(x)^2\right)^{k/2} w(x) \{|g(x)| + |g(-x)|\} dx \\
&\leqslant \int_0^\infty \left(1+Q'(x)^2\right)^{k/2} w(x) \int_0^{|x|} \frac{w^{-1}(t)}{\left(1+Q'(t)^2\right)^{(k-1)/2}} \Psi(t) dt\, dx \\
&= \int_0^\infty \left\{ \frac{w^{-1}(x)}{\left(1+Q'(x)^2\right)^{(k-1)/2}} \int_{|x|}^\infty \left(1+Q'(t)^2\right)^{k/2} w(t) dt \right\} \Psi(x) dx,
\end{aligned} \tag{3.41}$$

by changing of the integral order. Hence, from (3.20) we have

$$\left\|(1+Q'^2)^{k/2}wg\right\|_{L_1(\mathbb{R})} \leqslant C\int_0^\infty \Psi(x)dx = \|\varphi\|_{L_1(\mathbb{R})}. \tag{3.42}$$

By the definition of (3.38), we have (3.37) for $p = 1$.

Next, we show (3.37) for $p = \infty$. From (3.39) we see that

$$\begin{aligned}\left|\left(1+Q'(x)^2\right)^{k/2}w(x)g(x)\right| &\leqslant \left(1+Q'(x)^2\right)^{k/2}w(x)\{|g(x)|+|g(-x)|\}\\ &\leqslant \left(1+Q'(x)^2\right)^{k/2}w(x)\int_0^{|x|}\frac{w^{-1}(t)}{\left(1+Q'(t)^2\right)^{(k-1)/2}}\{|\varphi(t)|+|\varphi(-t)|\}dt\\ &\leqslant 2\|\varphi\|_{L_\infty(\mathbb{R})}\left(1+Q'(x)^2\right)^{k/2}w(x)\int_0^{|x|}\frac{w^{-1}(t)}{\left(1+Q'(t)^2\right)^{(k-1)/2}}dt\\ &\leqslant C\|\varphi\|_{L_\infty(\mathbb{R})},\end{aligned} \tag{3.43}$$

by (3.21). So we have (3.37) for $p = \infty$. Let $\phi \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}) \subset L_p(\mathbb{R}),\ 1 \leqslant p \leqslant \infty$, then we set

$$g(x) := \int_0^x \frac{\phi(t)}{\left(1+Q'(t)^2\right)^{(k-1)/2}w(t)}dt. \tag{3.44}$$

So we have

$$\left(1+Q'(x)^2\right)^{(k-1)/2}w(x)g'(x) = \phi(x) \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}) \subset L_p(\mathbb{R}), \quad 1 \leqslant p \leqslant \infty. \tag{3.45}$$

Since $L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ is dense in $L_p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, using the Riesz-Thorin interpolation theorem for the linear operator,

$$\phi \longmapsto \left(1+Q'(x)^2\right)^{k/2}w(x)\int_0^x \frac{w^{-1}(t)}{\left(1+Q'(t)^2\right)^{(k-1)/2}}\phi(t)dt, \quad \phi \in L_p(\mathbb{R}), \tag{3.46}$$

we have the result. □

Corollary 3.9. *For $r = 1$, one lets $w \in \mathcal{F}(C^2+)$, and for integer $r \geqslant 2$ one lets $w \in \mathcal{F}(C^r+)$. Let $1 \leqslant p \leqslant \infty$, and $1 \leqslant k$ be an integer. If $g : \mathbb{R} \to \mathbb{R}$ is absolutely continuous on $\mathbb{R}$, $g^{(j)}(0) = 0,\ j = 0, \ldots, k-1$, and $g^{(k)} \in L_{p,w}(\mathbb{R})$, then*

$$\left\|\left(1+Q'(x)^2\right)^{k/2}wg\right\|_{L_p(\mathbb{R})} \leqslant C\left\|wg^{(k)}\right\|_{L_p(\mathbb{R})}. \tag{3.47}$$

Lemma 3.10 ([4, Lemma 7]). *Let $w \in \mathcal{F}(C^2+)$. For a certain constant $C_1 > 0$, let u satisfy*

$$0 < C_1 \leqslant u, \quad t = \frac{a_u}{u}. \tag{3.48}$$

Then there exists a constant $C_2 > 1$ such that for every x and y which satisfy $|x| \leqslant \sigma(2t)$ and $|x-y| \leqslant t\Phi_t(x)/2$, one has

$$\frac{1}{C_2} w(y) \leqslant w(x) \leqslant C_2 w(y). \tag{3.49}$$

Proof of Theorem 3.3. For a given $\varepsilon > 0$, we can select g, where $g^{(r-1)}(x)$ is absolutely continuous on $\mathbb{R}$, and $g^{(r)} \in L_{p,w}(\mathbb{R})$ such that

$$\|w(f-g)\|_{L_p(\mathbb{R})} + t^r \left\| w g^{(r)} \right\|_{L_p(\mathbb{R})} - \varepsilon \leqslant \mathcal{K}_{r,p}(w, f, t). \tag{3.50}$$

Let $f = h + g$, where $h \in L_{p,w}(\mathbb{R})$. Then we have

$$\omega_{r,p}(h,t) \leqslant C \|wh\|_{L_p(\mathbb{R})}. \tag{3.51}$$

Let $0 < s \leqslant t/r$. From Lemma 3.4, 3.10 and the Hőlder-Minkowski inequality, we have

$$\begin{aligned}
&\left\| w \Delta^r_{s\Phi_t(x)} g(x) \right\|_{L_p(|x| \leqslant \sigma(2t))}, \\
&C \left\| w(x) \left\{ \int_{-s/2}^{s/2} \right\}^r \left| g^{(r)}(u) \right| du_1 du_2 \ldots du_r \right\|_{L_p(|x| \leqslant \sigma(2t))} \qquad (\text{note } \Phi_t(x) \leqslant C) \\
&\quad \leqslant C \left\| \left\{ \int_{-s/2}^{s/2} \right\}^r \left| \left(w g^{(r)} \right)(u) \right| du_1 du_2 \cdots du_r \right\|_{L_p(|x| \leqslant \sigma(2t))} \\
&\quad \leqslant C \left[\left\{ \int_{-s/2}^{s/2} \right\}^r \left\| (w g^{(r)})(u) \right\|_{L_p(|x| \leqslant \sigma(2t))} du_1 du_2 \cdots du_r \right] \\
&\quad \leqslant C s^r \left\| w g^{(r)} \right\|_{L_p(\mathbb{R})} \leqslant C t^r \left\| w g^{(r)} \right\|_{L_p(\mathbb{R})},
\end{aligned} \tag{3.52}$$

where $u = x + \Phi_t(x)(u_1 + u_2 + \cdots + u_r)$. We estimate

$$\inf_{P \in \mathcal{P}_{r-1}} \left\| (g(x) - P(x)) w(x) \right\|_{L_p(|x| \geqslant \sigma(4t))}. \tag{3.53}$$

Then we may suppose

$$g^{(j)}(0) - P^{(j)}(0) = 0, \quad j = 0, 1, \ldots, r-1. \tag{3.54}$$

Using Lemma 3.5, we see

$$\left|Q'(\sigma(4t))\right|^{-1} \leqslant Ct, \tag{3.55}$$

(see [7, page 12]). By Corollary 3.9 with $k = r$ and (3.55) we have

$$\begin{aligned}
&\left\|w(g-P)\right\|_{L_p(|x|\geqslant\sigma(4t))} \\
&\quad\leqslant \left|Q'(\sigma(4t))\right|^{-r}\left\|\left(1+Q'(x)^2\right)^{r/2} w(g-P)\right\|_{L_p(|x|\geqslant\sigma(4t))} \\
&\quad\leqslant Ct^r\left\|wg^{(r)}\right\|_{L_p(\mathbb{R})}.
\end{aligned} \tag{3.56}$$

Hence, from (3.51), (3.52), and (3.56) we have

$$\begin{aligned}
\omega_{r,p}(w,f,t) - C\varepsilon &\leqslant C\left(\|wh\|_{L_p(\mathbb{R})} + t^r\left\|wg^{(r)}\right\|_{L_p(\mathbb{R})} - \varepsilon\right) \\
&\leqslant C\mathcal{K}_{r,p}(w,f,t).
\end{aligned} \tag{3.57}$$

Consequently, we have

$$\omega_{r,p}(w,f,t) \leqslant C\mathcal{K}_{r,p}(w,f,t). \tag{3.58}$$

Therefore, from (3.2) we have (3.3). □

Corollary 3.11. *For $r = 1$, one lets $w \in \mathcal{F}(C^2+)$, and for integer $r \geqslant 2$ one lets $w \in \mathcal{F}(C^r+)$. Let $f \in L_{p,w}(\mathbb{R})$. Then one has*

$$\omega_{r,p}(w,f,t) \sim \mathcal{K}_{r,p}(w,f,t) \sim \widetilde{\mathcal{K}}_{r,p}(w,f,t). \tag{3.59}$$

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