Research Article

On the Favard-Type Theorem and the Jackson-Type Theorem (II)

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Received 30 September 2011; Accepted 20 October 2011

Academic Editors: C. Lu and E. Yee

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Let \( R = (-\infty, \infty) \), and let \( Q \in C^1 : R \to [0, \infty) \) be an even function. We consider the exponential weights

\[
w(x) = e^{-Q(x)}, \quad x \in R.
\]

In this paper we investigate the relations between the Favard-type inequality and the Jackson-type inequality. We also discuss the equivalence of two \( K \)-functionals and the modulus of smoothness.

1. Introduction and Preliminaries

Let \( R = (-\infty, \infty) \), and let \( Q \in C^1 : R \to [0, \infty) \) be an even function. We consider the weights

\[
w(x) = \exp(-Q(x)), \quad x \in R.
\]

In this paper we investigate the relations between the Favard-type inequality and the Jackson-type inequality. We also discuss the equivalence of two \( K \)-functionals and the modulus of smoothness.

First we need the following definition from [1]. We say that \( f : R \to [0, \infty) \) is quasi-increasing if there exists \( C > 0 \) such that \( f(x) \leq C f(y) \), \( 0 < x < y \).

Definition 1.1. One defines, \( w = \exp(-Q) \in \mathcal{F}(C^2) \) as follows: Let \( Q : R \to [0, \infty) \) be continuous and an even function, and satisfy the following properties:

(a) \( Q'(x) \) is continuous in \( R \), with \( Q(0) = 0 \);

(b) \( Q''(x) \) exists and is positive in \( R \setminus \{0\} \);
(c) 

\[ \lim_{x \to \infty} Q(x) = \infty; \quad (1.2) \]

(d) the function

\[ T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0 \quad (1.3) \]

is quasi-increasing in \((0, \infty)\) with \(T(x) \geq \Lambda > 1, \ x \in \mathbb{R} \setminus \{0\};\)

(e) there exists \(C_1 > 0\) such that

\[ \frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus \{0\}. \quad (1.4) \]

Moreover, there also exists a compact subinterval \(J(\ni 0)\) of \(\mathbb{R}\), and \(C_2 > 0\) such that

\[ \frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J. \quad (1.5) \]

**Example 1.2.** Let \(w = \exp(-Q) \in \mathcal{F}(C^2+)\).

(1) If \(T(x)\) is bounded, then we call the weight \(w = \exp(-Q)\) the Freud-type weight. The following example is the Freud-type weight:

\[ w(x) = \exp(-|x|^\alpha), \quad \alpha > 1. \quad (1.6) \]

(2) If \(T(x)\) is unbounded, then we call the weight \(w = \exp(-Q)\) the Erdős-type weight. The following examples give the Erdős-type weight \(w = \exp(-Q)\).

(i) ([1, Example 1.2], [2, Theorem 3.1]). For \(\alpha > 1, \ \ell = 1, 2, 3, \ldots,\)

\[ Q(x) = Q_{\ell, x}(x) = \exp_\ell(|x|^\alpha) - \exp_{\ell - 1}(0), \quad (1.7) \]

where \(\exp_\ell(x) = \exp(\exp(\exp \ldots \exp x) \ldots), (\ell \text{ times}).\) More generally, we define for \(\alpha + u > 1, \ \alpha \geq 0, u \geq 0, \text{ and } l \geq 1,\)

\[ Q_{l, x, u}(x) := |x|^u (\exp_{\ell}(|x|^\alpha) - \alpha^* \exp_{\ell}(0)), \quad (1.8) \]

where \(\alpha^* = 0\) if \(\alpha = 0,\) otherwise \(\alpha^* = 1.\)
(ii) For $\alpha > 1$, $Q(x) = Q_\alpha(x) := (1 + |x|^{\alpha}) - 1$.

We need the Mhaskar-Rakhmanov-Saff numbers $a_x$ defined by

$$x = \frac{2}{\pi} \int_0^1 a_x u Q(a_x u) \frac{du}{(1 - u^2)^{1/2}}, \quad x > 0.$$  \hspace{1cm} (1.9)

If $f : \mathbb{R} \to \mathbb{R}$ is measurable, we define

$$\|fw\|_{L^p(\mathbb{R})} := \begin{cases} \left( \int_{-\infty}^\infty |f(t)w(t)|^p dt \right)^{1/p}, & \text{if } 0 < p < \infty, \\ \text{ess sup}_{t \in \mathbb{R}} |f(t)w(t)|, & \text{if } p = \infty, \end{cases}$$  \hspace{1cm} (1.10)

where if $p = \infty$, then we suppose that $f$ is continuous on $\mathbb{R}$ and

$$\lim_{|x| \to \infty} |f(x)w(x)| = 0.$$  \hspace{1cm} (1.11)

The class of all functions $f$ for which $\|fw\|_{L^p(\mathbb{R})} < \infty$ will be denoted by $L_{p,w}(\mathbb{R})$, with the usual understanding that two functions are identified if they are equal almost everywhere. For $f \in L_{p,w}(\mathbb{R})$ ($0 < p \leq \infty$), the degree of weighted polynomial approximation is defined by

$$E_{p,n}(w; f) := \inf_{P \in \mathcal{P}_n} \|w(f - P)\|_{L^p(\mathbb{R})},$$  \hspace{1cm} (1.12)

where $\mathcal{P}_n$ denotes the class of all polynomials of degree at most $n$. There are various estimates for this degree. Among them, in our previous article [3], we discuss the relation between the Favard-type inequality, the Jackson-type inequality, and the estimates by two $K$-functionals $K_{r,p}$ and $\tilde{K}_{r,p}$. We recall and summarize them in Section 2. In Section 3, we will discuss the equivalence of $K_{r,p}$ and modulus of smoothness $\omega_{r,p}$. As the result, we see $K_{r,p} \sim \tilde{K}_{r,p}$ for a weight in a subclass of $\mathcal{F}(C^2+)$. For any nonzero real valued functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if there exists a constant $C > 0$ independent of $x$ such that for all $x$

$$\left( \frac{1}{C} \right) f(x) \leq g(x) \leq Cf(x).$$  \hspace{1cm} (1.13)

Similarly, for any positive numbers $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ we define $c_n \sim d_n$. Throughout this paper $C, C_1, \ldots, c, c_1, \ldots$ denote positive constants independent of $n, x, t$, or $P_n(x)$. The same symbols do not necessarily denote the same constants occurrences.
2. Known Results and Summarization

Let $r \geq 1$ be an integer (in this paper, we suppose that $r$ (or $s$) is an integer). The $r$th order $K$-functional of a function $f \in L_{p,w}(\mathbb{R})$ is defined by the formula

$$
\mathcal{K}_{r,p}(w,f,\delta) := \inf \left\{ \|w(f-g)\|_{L_p(\mathbb{R})} + \delta \|w g^{(r)}\|_{L_p(\mathbb{R})} \right\},
$$

for $\delta > 0$, where the infimum is taken over all $r - 1$ times continuously differentiable $g$ such that $g^{(r-1)}$ is absolutely continuous, and $g^{(r)} \in L_{p,w}(\mathbb{R})$ (this means $w g^{(r)} \in L_p(\mathbb{R})$). Using this $r$th order $K$-functional, we can estimate the order of $E_{p,n}(w,f)$.

**Theorem 2.1** (see [3]). Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$. Let $r \geq 1$, $s \geq 0$ be integers, and let $1 \leq p \leq \infty$. Let $f$ be $s - 1$ times continuously differentiable, $f^{(s-1)}$ be absolutely continuous on each compact interval, and $f^{(s)} \in L_{p,w}(\mathbb{R})$ (when $s = 0$, these assumptions state merely that $f \in L_{p,w}(\mathbb{R})$). Then, for every integer $n \geq r + s$,

$$
E_{p,n}(w,f) \leq C \left( \frac{a_n}{n} \right)^s \mathcal{K}_{r,p}(w,f^{(s)},\frac{a_n}{n}).
$$

**Theorem 2.1** was shown by using the following Favard-type inequalities (see [3]).

**Theorem 2.2** ([4, Corollary 8]). Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$. Let $f$ be $s - 1$ times continuously differentiable, and let $f^{(s-1)}$ for some integer $s \geq 1$ be absolutely continuous on each compact interval. Let $1 \leq p \leq \infty$ and $f^{(s)} \in L_{p,w}(\mathbb{R})$. Then one has

$$
E_{p,n}(w,f) \leq C \left( \frac{a_n}{n} \right)^s \|w f^{(s)}\|_{L_p(\mathbb{R})},
$$

equivalently,

$$
E_{p,n}(w,f) \leq C \left( \frac{a_n}{n} \right)^s E_{p,n-s}(w,f^{(s)}).
$$

**Remark 2.3.**

1. ([4, Remark 11]) Let $w \in \mathcal{F}(C^2+)$ and let $0 < p < 1$. Then there exists a constant $C_0$ such that for every absolutely continuous function $f$ with $f' \in L_{\infty,w}$ and $w f' \in L_p(\mathbb{R})$, and for every $n \in \mathbb{N}$, we have

$$
E_{p,n}(w,f) \leq C_0 \left( \frac{a_n}{n} \right) \left\{ \|w f'\|_{L_{\infty}(\mathbb{R})} + \|w f'\|_{L_p(\mathbb{R})} \right\}.
$$

2. ([3, Section 4]) As a by-product of the method of the proof for Theorem 2.2, we can obtain the degree of functions which satisfy the Hölder-Lipschitz condition. Let $w \in \mathcal{F}(C^2+), 1 \leq p \leq \infty$ and $1/p < \beta < 1$. Let $g$ be absolutely continuous with
$|g'|^\beta \in L_{p,w}(\mathbb{R})$ (and for $p = \infty$, we require $g$ to be continuous, and $gw$ to vanish at $\pm\infty$). Let $f \in L_{p,w}(\mathbb{R})$ and set

$$f_{g,\beta}^*(x) := \sup_{y \in \mathbb{R}} \frac{|f(x) - f(y)|}{|g(x) - g(y)|^\beta}. \quad (2.6)$$

Then we define

$$F_{g,\beta}(w, p) := \left\{ f \mid f \in L_{p,w}(\mathbb{R}), \|f_{g,\beta}^*\|_{L_\infty(\mathbb{R})} < \infty \right\}. \quad (2.7)$$

Now, we have the following theorem:

**Theorem 2.4** ([3, Theorem 4.2]). Let $1 \leq p \leq \infty$, $1/p \leq \beta \leq 1$, and let $f \in F_{g,\beta}(w, p)$. Then one has

$$E_{p,n}(w; f) \leq C \left( \frac{a_n}{n} \right)^\beta \|w\|_{L_p(\mathbb{R})} \|g'\|_{L_p(\mathbb{R})} \|f_{g,\beta}^*\|_{L_\infty(\mathbb{R})}. \quad (2.8)$$

We define the following class of weights from [5, Definition 1.1].

**Definition 2.5.** Let $w(x) := \exp(-Q(x))$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, continuous, and $Q'$ is positive in $(0, \infty)$. Then one writes $w = \exp(-Q) \in E_1$, if the following are satisfied:

(a) $xQ'(x)$ is strictly increasing in $(0, \infty)$ with $\lim_{x \to 0^+} xQ'(x) = 0$;

(b) the function

$$T(x) := \frac{xQ'(x)}{Q(x)} \quad (2.9)$$

is quasi-increasing in $(C, \infty)$ for some $C > 0$ and $\lim_{x \to \infty} T(x) = \infty$;

(c) assume

$$\frac{yQ'(y)}{xQ'(x)} \leq C_1 \left( \frac{Q(y)}{Q(x)} \right)^{C_3}, \quad y \geq x \geq C_2, \quad (2.10)$$

for some positive constants $C_1, C_2$, and $C_3$.

**Remark 2.6.** Let $w(x) := \exp(-Q(x)) \in F(C^2\uparrow)$ and $T(x)$ is unbounded, then we see $w \in E_1$. In fact, we see this as follows.
From Definition 1.1(e) and (d), we have for $y > x > 0$,

$$\frac{Q'(y)}{Q'(x)} = \exp \left( \int_x^y \frac{Q''(t)}{Q'(t)} \, dt \right),$$

$$\leq \exp \left( C_1 \int_x^y \frac{Q'(t)}{Q(t)} \, dt \right) = \left( \frac{Q(y)}{Q(x)} \right)^{C_1},$$

$$y = \exp \left( \int_x^y \frac{1}{t} \, dt \right) \leq \exp \left( \frac{1}{\Lambda} \int_x^y \frac{Q'(t)}{Q(t)} \, dt \right) = \left( \frac{Q(y)}{Q(x)} \right)^{1/\Lambda}.$$  \hspace{1cm} (2.11)

Therefore we obtain (c) in Definition 2.5 for $C_3 = C_1 + 1/\Lambda$. 

If $f : \mathbb{R} \to \mathbb{R}$, and $h > 0$, then we define the differences of $f$ inductively by the formula

$$\Delta_0^h f(x) = f(x),$$

$$\Delta_1^h f(x) = \Delta_h f(x) = f \left( x + \frac{h}{2} \right) - f \left( x - \frac{h}{2} \right),$$

$$\Delta_k^h f(x) = \Delta_{k-1}^h (\Delta_h f)(x) = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} f \left( x + \frac{k}{2}h - jh \right), \quad k = 2, 3, \ldots, x \in \mathbb{R}. \hspace{1cm} (2.12)$$

We set

$$\sigma(t) := \inf \left\{ a_u : \frac{a_u}{u} \leq t \right\}, \quad t > 0,$$

$$\Phi(x) := \sqrt{1 - \frac{|x|}{\sigma(t)}} + T(\sigma(t))^{-1/2}, \quad x \in \mathbb{R}. \hspace{1cm} (2.13)$$

By [6], when $w = \exp(-Q)$ is the Erdős-type weight we define for $f \in L_{p,w}(\mathbb{R}), 0 < p \leq \infty$,

$$\omega_{r,p}(w, f, t) := \sup_{0 < h \leq t} \left\| w \Delta_{\Phi(x)}^h (f) \right\|_{L_p(|x| \leq \sigma(2t))} + \inf_{R \in P_{r-1}} \|w(x)(f - R)(x)\|_{L_p(|x| \geq \sigma(4t))}. \hspace{1cm} (2.14)$$

If $w = \exp(-Q)$ is the Freud-type weights, then we define

$$\omega_{r,p}(w, f, t) := \sup_{0 < h \leq t} \left\| w \Delta_{\Phi(x)}^h (f) \right\|_{L_p(|x| \leq \sigma(h))} + \inf_{R \in P_{r-1}} \|w(x)(f - R)(x)\|_{L_p(|x| \geq \sigma(t))}. \hspace{1cm} (2.15)$$

The following Jackson-type inequality is known.
**Theorem 2.7** (see [5, Theorem 1.2], [6, Corollary 1.4]). Let \( w \in \mathcal{E}_1 \). Let \( 0 < p \leq \infty \). Then for \( f : \mathbb{R} \to \mathbb{R} \) for which \( f \in L_{p,w}(\mathbb{R}) \) (and for \( p = \infty \), we require \( f \) to be continuous, and \( f w \) to vanish at \( \pm \infty \)), one has for \( n \geq C_5 \),

\[
E_{p,n}(f,w) \leq C_1 \omega_{r,p} \left( w, f, C_2 \frac{a_n}{n} \right),
\]

where \( C_j, j = 1, 2, 3 \), do not depend on \( f \) and \( n \).

We also consider the following class of weights which are called the Freud weights.

**Definition 2.8** ([7, Definition 3.3]). Let \( w = \exp(-Q) \), where \( Q : \mathbb{R} \to \mathbb{R} \) is even, and \( Q' \) exists and is positive on \((0, \infty)\). Moreover, assume that \( x Q'(x) \) is strictly increasing, with right limit 0 at 0, and for some \( \lambda, A, B > 1 \), \( C > 0 \),

\[
A \leq \frac{Q(\lambda x)}{Q(x)} \leq B, \quad x \geq C.
\]

Then we call \( w \) Freud weight, and write \( w \in \mathcal{F}^* \).

**Remark 2.9.** Let \( w = \exp(-Q) \in \mathcal{F}(C^2+) \). Then we see the following.

1. If \( T(x) = (xQ'(x))/Q(x) \) is bounded, then we say that \( w = \exp(-Q) \) is the Freud-type weight, and we write \( w \in \mathcal{F} \). Then we see \( \mathcal{F}^* \cap \mathcal{F}(C^2+) = \mathcal{F} \). In fact, when \( w \in \mathcal{F} \), \( 1 < \nu \) and \( x > 0 \) large enough, by Definition 1.1(e),

\[
\frac{Q'(\nu x)}{Q'(x)} = \exp \left( \int_x^{\nu x} \frac{Q''(u)}{Q'(u)} du \right) \leq \exp \left( C_1 \int_x^{\nu x} \frac{Q'(u)}{Q'(u)} du \right),
\]

and then since \( T(x) \) is bounded, there exists \( C > 0 \) such that

\[
\frac{Q'(\nu x)}{Q'(x)} \leq \exp \left( C_1 C \int_x^{\nu x} \frac{1}{u} du \right) = v^{C_1 C}.
\]

Similarly,

\[
\frac{Q'(\nu x)}{Q'(x)} = \exp \left( \int_x^{\nu x} \frac{Q''(u)}{Q'(u)} du \right) \geq \exp \left( C_2 \int_x^{\nu x} \frac{Q'(u)}{Q'(u)} du \right)
\]

\[
\geq \exp \left( C_2 \Lambda \int_x^{\nu x} \frac{1}{u} du \right) = v^{C_2 \Lambda}.
\]

Therefore, if we take \( \lambda := \nu > 1 \) large enough, then we have (2.17).

Conversely, to show \( \mathcal{F}^* \cap \mathcal{F}(C^2+) \subset \mathcal{F} \) we suppose that there exists \( w = \exp(-Q) \in \mathcal{F}(C^2+) \) such that \( T(x) = (xQ'(x))/Q(x) \) is unbounded. Then since \( T(x) \) is quasi-increasing
in $(0, \infty)$, we see that $T(x) \to \infty$ as $x \to \infty$. So, for any $M > 0$ there exists $L > 0$ (large enough) such that $T(x) > M$ for $x > L$. Therefore, we have for $x > L$ and any $\lambda > 1$,

$$
\frac{Q'(\lambda x)}{Q'(x)} = \exp\left(\int_x^{\lambda x} \frac{Q''(u)}{Q'(u)} \, du\right) \geq \exp\left(C_3 \int_x^{\lambda x} \frac{Q'(u)}{Q(u)} \, du\right) = \exp\left(C_3 \int_x^{\lambda x} \frac{T(u)}{u} \, du\right) = \lambda^{C_3 M},
$$

(2.21)

where $C_3 > 0$ is a constant, that is, (1) does not hold. Hence we have $\mathcal{F}^* \cap \mathcal{F}(C^2+) \subset \mathcal{F}$. Consequently, we have $\mathcal{F} = \mathcal{F}^* \cap \mathcal{F}(C^2+)$.

(2) If $T(x) = (xQ'(x))/Q(x)$ is bounded, then for $x \geq 1$, there exists $c, C > 0$ such that

$$
x^c \leq Q(x) \leq Cx^c.
$$

(2.22)

**Theorem 2.10** ([7, Theorem 3.5]). Let $w \in \mathcal{F}^*$. Let $0 < p \leq \infty$. Then for $f : \mathbb{R} \to \mathbb{R}$ for which $f \in L_{p,w}(\mathbb{R})$ (and for $p = \infty$, one requires $f$ to be continuous, and $fw$ to vanish at $\pm \infty$), one has for $n \geq C_3$,

$$
E_{p,n}(f;w) \leq C_1 \omega_{r,p}\left(w, f, C_2 \frac{a_n}{n}\right),
$$

(2.23)

where $C_j, j = 1, 2, 3$, do not depend on $f$ and $n$.

Damlan [6] introduces the following $K$-functional: let $f \in L_{p,w}(\mathbb{R}), 0 < p \leq \infty$ and $r \geq 1$ be an integer, then we define

$$
\overline{K}_{r,p}(w, f, \delta) := \inf_{P \in P_r}\left\{\|w(f - P)\|_{L_p(\mathbb{R})} + \delta\|wP^{(r)}\|_{L_p(\mathbb{R})}\right\},
$$

(2.24)

where $\delta > 0$ are chosen in advance and

$$
n = n(\delta) := \inf\left\{k : \frac{a_k}{k} \leq \delta\right\}.
$$

(2.25)

Then Damlan gives the following.

**Theorem 2.11** ([6, Theorem 1.3 (b)]). Let $w \in \mathcal{E}_1, r \geq 1, 0 < p \leq \infty$, and let $fw \in L_p(\mathbb{R})$ (and for $p = \infty$, one requires $f$ to be continuous, and $fw$ to vanish at $\pm \infty$). Then one has

$$
\omega_{r,p}(w, f, t) \sim \overline{K}_{r,p}(w, f, \delta).
$$

(2.26)

For the Freud-type weights we have also the followings.
Theorem 2.12 ([7, Theorem 3.9, 3.10]). Let \( w \in \mathcal{F}^* \), \( r \geq 1 \), \( 0 < p \leq \infty \), and let \( f w \in L_p(\mathbb{R}) \) (and for \( p = \infty \), we require \( f \) to be continuous, and \( f w \) to vanish at \( \pm \infty \)). Then we have

\[
\mathcal{K}_{r,p}(w, f, \delta) - \omega_{r,p}(w, f, t) \sim \mathcal{K}_{r,p}(w, f, \delta). \tag{2.27}
\]

For \( w \in \mathcal{F}(C^2^+) \), we see easily that

\[
\mathcal{K}_{r,p}(w, f, \delta) \leq \mathcal{K}_{r,p}(w, f, \delta). \tag{2.28}
\]

So, from Theorem 2.1 we obtain the following corollary.

Corollary 2.13. Let \( w = \exp(-Q) \in \mathcal{F}(C^2^+) \). Let \( r \geq 1 \), \( s \geq 0 \) be integers, and let \( 1 \leq p \leq \infty \). Let \( f \) be \( s - 1 \) times continuously differentiable, \( f^{(s-1)} \) be absolutely continuous, and \( f^{(s)} \in L_{p,w}(\mathbb{R}) \) (when \( s = 0 \), these assumptions state merely that \( f \in L_{p,w}(\mathbb{R}) \)). Then, for every integer \( n \geq r + s \),

\[
E_{p,n}(w; f) \leq C\left( \frac{a_n}{n} \right)^s \mathcal{K}_{r,p}(w, f^{(s)}, \frac{a_n}{n}). \tag{2.29}
\]

The main theme in [3] is to summarize the above theorems. Let \( w \in \mathcal{F}(C^2^+), r \geq 1 \) be an integer, and let \( 1 \leq p \leq \infty \). We have the following succession of the theorems. We use the constant \( C_i > 0 \), \( i = 1, 2, 3, \ldots \) which do not depend on \( f \) and \( n \).

(a) [Theorem 2.7 with \( r = 1 \) (the Erdős-type case)], [Theorem 2.10 with \( r = 1 \) (the Freud case)]:

let \( f : \mathbb{R} \to \mathbb{R} \) and if \( 1 \leq p < \infty \), assume that \( f w \in L_p(\mathbb{R}) \). If \( p = \infty \), assume in addition that \( f \) is continuous and that \( f w \) has limit 0 at \( \pm \infty \). Then we have

\[
E_{p,n}(w; f) \leq C_1 \omega_{1,p}\left( w, f, C_2 \frac{a_n}{n} \right). \tag{2.30}
\]

(b) [Theorem 2.22]:

let \( f \) be \( s - 1 \) times continuously differentiable, and let for some integer \( s \geq 1 \), \( f^{(s-1)}(x) \) be absolutely continuous on each compact interval. Let \( 1 \leq p \leq \infty \) and \( f^{(s)} \in L_{p,w}(\mathbb{R}) \). Then we have

\[
E_{p,n}(w; f) \leq C_3 \left( \frac{a_n}{n} \right)^s \left\| w f^{(s)} \right\|_{L_p(\mathbb{R})}. \tag{2.31}
\]

equivalently,

\[
E_{p,n}(w; f) \leq C_4 \left( \frac{a_n}{n} \right)^s E_{p,n}(w; f^{(s)}). \tag{2.32}
\]
In this section we consider a subclass of
Example 1.2, in fact, this is true.

The equivalences mentioned in the last of Section 2 give a certain suggestion, that is,

\[ \mathcal{K}_{r,p}(w,f,t) - \omega_{r,p}(w,f,t). \]  

In fact, this is true.
Theorem 3.3. Let $r \geq 2$ be a positive integer, and let $w \in \Phi(C^r+)$. Let $f \in L_{p,w}(\mathbb{R})$. Then one has (3.3). Similarly, when $w \in \Phi(C^2+)$ and $r = 1$, one has (3.3).

To prove Theorem 3.3, we need some lemmas.

Lemma 3.4. Let $r \geq 1$ be an integer. If $f^{(r-1)}(x)$ is absolutely continuous on $\mathbb{R}$, then for $k = 1, 2, \ldots, r$, one has the following representation:

$$\Delta_{\Phi_t(x)}^k f(x) = (\Phi_t(x))^k \left\{ \int_{-h/2}^{h/2} f^{(k)}(x + \Phi_t(x)(u_1 + u_2 + \cdots + u_k)) du_1 du_2 \cdots du_k \right\}. \quad (3.4)$$

Proof.

$$\int_{-h/2}^{h/2} f^{(1)}(x + \Phi_t(x)u_1) du_1 = (\Phi_t(x))^{-1} \int_{x-\Phi_t(x)h/2}^{x+\Phi_t(x)h/2} f^{(1)}(v_1) dv_1$$

$$= (\Phi_t(x))^{-1} \Delta_{\Phi_t(x)} h f(x). \quad (3.5)$$

Therefore, for $r = 1$ we have (3.4). For some $k \geq 1$ we suppose (3.4). Then we have

$$\left\{ \int_{-h/2}^{h/2} \right\}^{k+1} f^{(k+1)}(u) du_1 du_2 \cdots du_{k+1}$$

$$= \int_{-h/2}^{h/2} \left[ \left\{ \int_{-h/2}^{h/2} \right\}^k \left( f^{(k)}(\tilde{u}) du_1 du_2 \cdots du_k \right) \right] du_{k+1}$$

$$= (\Phi_t(x))^{-k} \int_{-h/2}^{h/2} \Delta_{\Phi_t(x)}^k (\Delta_{\Phi_t(x)} h f(x)) du_{k+1}$$

$$= (\Phi_t(x))^{-(k+1)} \Delta_{\Phi_t(x)}^{k+1} (\Delta_{\Phi_t(x)} h f(x))$$

where

$$u = u(x; u_1, \ldots, u_{k+1}) = x + \Phi_t(x)(u_1 + u_2 + \cdots + u_{k+1}),$$

$$\tilde{u} = \tilde{u}(x; u_1, \ldots, u_{k+1}) = x + \Phi_t(x) u_{k+1} + \Phi_t(x)(u_1 + u_2 + \cdots + u_k). \quad (3.7)$$

Hence we have (3.4) for $r = 1, 2, 3, \ldots$.

Lemma 3.5 (see [1, Lemma 3.4 (3.17)]). Uniformly for $t > 0$, one has

$$Q'(a_t) \sim \frac{t \sqrt{T(a_t)}}{a_t}. \quad (3.8)$$
Lemma 3.6. Let \( w \in \mathcal{F}(C^r) \) and \( r \geq 2 \) be an integer. Then for any integer \( k, \: 1 \leq k \leq r - 1, \) there exist \( c_1, c_2 > 0 \) and \( A > 0 \) such that for \( |x| \geq A, \)

\[
c_1 \leq (-1)^k \frac{w^{(k)}(x)}{(1 + Q'(x)^2)^{k/2} w(x)} \leq c_2. \tag{3.9}
\]

Also, there exist \( c_3, c_4 > 0 \) and \( B > 0 \) such that for \( |x| \geq B, \)

\[
c_3 \leq (-1)^k \frac{(w^{-1})^{(k)}(x)}{(1 + Q'(x)^2)^{k/2} w^{-1}(x)} \leq c_4. \tag{3.10}
\]

Furthermore, for \( k = r, \) (3.9) and (3.10) hold almost everywhere on \( |x| \geq A \) and \( |x| \geq B, \) respectively. When \( w \in \mathcal{F}(C^2) \) and \( r = 1, \) one also has (3.9) and (3.10).

Proof. First, we will see that for \( \mu = \pm 1 \) the following equations hold:

\[
\lim_{|x| \to \infty} \frac{(w^\mu)^{(k)}(x)}{Q'(x)^k w^\mu(x)} = (-\mu)^k, \quad k = 1, \ldots, r - 1. \tag{3.11}
\]

For \( k = r \) we take \( \tilde{Q}(x) \) as follows;

\[
\tilde{Q}^{(j)}(x) = Q^{(j)}(x), \quad x \in \mathbb{R}, \: j = 1, 2, \ldots, r - 1,
\]

\[
\tilde{Q}^{(r)}(x) = Q^{(r)}(x), \quad \text{a.e.} \: x \in \mathbb{R} \setminus \{0\},
\tag{3.12}
\]

and for all \( j = 1, 2, \ldots, r, \) \( \tilde{Q}(x) \) satisfies (3.1) for all \( x \neq 0. \) Then we obtain that exchanging \( Q \) with \( \tilde{Q}, \) (3.11) also holds.

Let \( \mu = 1, \) and let \( 1 \leq k \leq r - 1. \)

\[
w'(x) = -Q'(x)w(x),
\]

\[
w''(x) = \left(-Q''(x) + Q'(x)^2\right)w(x),
\tag{3.13}
\]

\[
w'''(x) = \left(-Q'''(x) + 3Q'(x)Q''(x) - Q'(x)^3\right)w(x),
\]

and we continue this manner, so

\[
w^{(k)}(x) = \left(\sum_{i_1 + 2i_2 + \cdots + ki_k = k, \atop i_i \leq k} c_{i_1,i_2,...,i_k} Q'(x)^{i_1} Q''(x)^{i_2} \cdots Q^{(k)}(x)^{i_k} + (-1)^k Q'(x)^k\right)w(x), \tag{3.14}
\]
where \( c_{i_1,\ldots,i_k} \) are coefficients. Here, from (3.1), for \( |x| \geq A > 0 \) large enough and \( i_j \neq 0 \), \( 2 \leq j \leq k \),

\[
|Q^{(j)}(x)|^{i_j} \leq C_j \left( \frac{|Q(x)|}{Q(x)} \right)^{i_j} = C_j \left( \frac{|Q'(x)|}{Q(x)^{\frac{1}{2}}} \right)^{i_j}.
\] (3.15)

Hence, from (3.14) and (3.15) we have

\[
\lim_{|x| \to \infty} \frac{w^{(k)}(x)}{|Q(x)|^k w(x)} - (-1)^k \leq C \lim_{|x| \to \infty} \frac{1}{Q(x)} = 0, \quad k = 1, \ldots, r - 1,
\] (3.16)

where \( C \) is a positive constant. Therefore, we have (3.11) with \( \mu = 1 \). Similarly, for \( \mu = -1 \),

\[
\lim_{|x| \to \infty} \frac{(w^{-1})^{(k)}(x)}{|Q(x)|^k w^{-1}(x)} = 1, \quad k = 1, \ldots, r - 1.
\] (3.17)

Therefore, we have (3.11) for \( k = 1, 2, \ldots, r - 1 \), and hence we also have (3.9) and (3.10) for \( |x| \) large enough. If in (3.11), we replace \( Q \) with \( Q \), then repeating the above proof we also obtain (3.11), so for \( k = r \) we conclude (3.9) and (3.10) with a.e. \( x \) large enough, that is,

\[
c_1 \leq (-1)^r \frac{w^{(r)}(x)}{(1 + Q'(x)^2)^{r/2} w(x)} \leq c_2, \quad \text{a.e.} \ |x| \geq A,
\] (3.18)

\[
c_3 \leq (-1)^r \frac{(w^{-1})^{(r)}(x)}{(1 + Q'(x)^2)^{r/2} w^{-1}(x)} \leq c_4, \quad \text{a.e.} \ |x| \geq B.
\]

For \( r = 1 \), the lemma is trivial. \( \square \)

**Lemma 3.7.** Let \( r \geq 1 \) be an integer. There exists \( C > 0 \) such that for every integer \( k = 1, 2, \ldots, r \),

\[
|Q'(x)|^k w(x) \to 0, \quad \text{as} \ |x| \to \infty,
\] (3.19)

\[
\frac{w^{-1}(x)}{(1 + Q'(x)^2)^{(k-1)/2}} \int_{|x|}^{\infty} \left( 1 + Q'(t)^2 \right)^{k/2} w(t) dt \leq C, \quad x \in \mathbb{R},
\] (3.20)

and for \( x \in \mathbb{R} \),

\[
\left( 1 + Q'(x)^2 \right)^{k/2} w(x) \int_{|x|}^{\infty} \left( 1 + Q'(t)^2 \right)^{-(k-1)/2} w^{-1}(t) dt \leq C, \quad x \in \mathbb{R}.
\] (3.21)
Proof. From Definition 1.1(e) we see that there exist \( C > 0 \) and \( \lambda > 0 \) such that

\[
|Q'(x)| \leq C Q(x)^\lambda, \quad x \in \mathbb{R}, \tag{3.22}
\]

so we have

\[
|Q'(x)|^k w(x) \leq C^k Q(x)^{1k} w(x) \to 0, \quad \text{as } |x| \to \infty. \tag{3.23}
\]

Hence, we have (3.19). From (3.9) with \( k - 1 \), we see that for \( |x| \geq A > 0 \) and constants \( c_1, c_2, c_3 > 0 \),

\[
\frac{w^{-1}(x)}{(1 + Q'(x)^2)^{(k-1)/2}} \int_{|x|}^{\infty} \left(1 + Q'(t)^2\right)^{k/2} w(t) dt \leq c_1 \frac{w^{-1}(x)}{(1 + Q'(x)^2)^{(k-1)/2}} \int_{|x|}^{\infty} (-1)^k w^{(k)}(t) dt \leq c_2 \frac{w^{-1}(x)}{(1 + Q'(x)^2)^{(k-1)/2}} (-1)^{k-1} w^{(k-1)}(x) \leq c_3. \tag{3.24}
\]

Especially, if we use (3.24) with \( |x| = A \), then we have

\[
\frac{w^{-1}(A)}{(1 + Q'(A)^2)^{(k-1)/2}} \int_{A}^{\infty} \left(1 + Q'(t)^2\right)^{k/2} w(t) dt \leq c_3. \tag{3.25}
\]

Then, for \( |x| \leq A \) there exists a constant \( C = C(A) \) such that

\[
\frac{w^{-1}(x)}{(1 + Q'(x)^2)^{(k-1)/2}} \leq C \frac{w^{-1}(A)}{(1 + Q'(A)^2)^{(k-1)/2}}. \tag{3.26}
\]

In fact, for \( |x| \leq A \) we have

\[
1 + Q'(A)^2 \leq \left(1 + Q'(A)^2\right)\left(1 + Q'(x)^2\right), \tag{3.27}
\]

that is,

\[
\left(1 + Q'(A)^2\right)^{(k-1)/2} \leq C(A)\left(1 + Q'(x)^2\right)^{(k-1)/2}, \tag{3.28}
\]

where \( C(A) = (1 + Q'(A)^2)^{(k-1)/2} \). Hence we have (3.26).
When $|x| < A$, we see easily that
\[
\frac{w^{-1}(x)}{(1 + Q'(x)^2)^{(k-1)/2}} \int_{|x|}^{A} \left(1 + Q'(t)^2\right)^{k/2} w(t) dt \leq c_6(A). \tag{3.30}
\]

Hence, with (3.29) we have for $|x| \leq A$,
\[
\frac{w^{-1}(x)}{(1 + Q'(x)^2)^{(k-1)/2}} \int_{|x|}^{A} \left(1 + Q'(t)^2\right)^{k/2} w(t) dt \leq c_6(A). \tag{3.31}
\]

Consequently, from (3.24) and (3.31) we have (3.20). We need to show (3.21). For $|x|$ large enough, we see
\[
Q''(x) - \frac{Q'(x)^2}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R}, \tag{3.32}
\]
(see Definition 1.1(e)) so we can select $|x| \geq A > 0$ large enough such that
\[
Q''(x) < \frac{1}{2k} Q'(x)^2, \quad \text{a.e. } x \in \mathbb{R}. \tag{3.33}
\]

We show (3.21) for $\int_{A}^{\infty} |x|$, $|x| \geq A > 0$. For $|x| \geq A$, we have by (3.33),
\[
I := \int_{A}^{\infty} \frac{w^{-1}(t)}{Q'(t)^{k+1}} dt = \int_{A}^{\infty} \frac{Q'(t)w^{-1}(t)}{Q'(t)^{k}} dt \\
\leq \frac{w^{-1}(x)}{Q'(|x|)^k} + k \int_{A}^{\infty} w^{-1}(t) \frac{Q''(t)}{Q'(t)^{k+1}} dt \\
\leq \frac{w^{-1}(x)}{Q'(|x|)^k} + \frac{1}{2} I. \tag{3.34}
\]

Hence, we have for a certain constant $c_7 > 0$,
\[
\left(1 + Q'(x)^2\right)^{k/2} w(x) \int_{A}^{\infty} \frac{w^{-1}(t)}{\left(1 + Q'(t)^2\right)^{(k-1)/2}} dt \leq c_7. \tag{3.35}
\]
Lemma 3.8. For $r = 1$, one lets $w \in \mathcal{F}(C^2 +)$, and for integer $r \geq 2$ one lets $w \in \mathcal{F}(C^r +)$. Let $1 \leq p \leq \infty$, and $1 \leq k \leq r$ be an integer. If $g : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, $g(0) = 0$, and $|Q(x)|^{k-1}g' \in L_{p,w}(\mathbb{R})$, then

$$
\left\| \left( 1 + Q'(x)^2 \right)^{k/2} w g \right\|_{L_p(\mathbb{R})} \leq C \left\| \left( 1 + Q'(x)^2 \right)^{(k-1)/2} w g' \right\|_{L_p(\mathbb{R})}.
$$

(3.37)

Proof. We will prove (3.37) for $p = 1$ and $p = \infty$, and then we use the Riesz-Thorin interpolation theorem. Let

$$
\psi(t) := \left( 1 + Q'(t)^2 \right)^{(k-1)/2} \omega(t) g'(t), \quad t \in \mathbb{R}.
$$

(3.38)

Then for almost all $x \geq 0$,

$$
|g(x)| + |g(-x)| \leq \int_0^x \frac{\omega^{-1}(t)}{\left( 1 + Q'(t)^2 \right)^{(k-1)/2}} \{ |\psi(t)| + |\psi(-t)| \} dt.
$$

(3.39)

Denoting

$$
\Psi(t) := |\psi(t)| + |\psi(-t)|,
$$

(3.40)

we get

$$
\left\| \left( 1 + Q^2 \right)^{k/2} w \right\|_{L_p(\mathbb{R})} = \int_0^\infty \left\| \left( 1 + Q'(x)^2 \right)^{k/2} w(x) \left( |g(x)| + |g(-x)| \right) \right\|_p dx
$$

$$
\leq \int_0^\infty \left( 1 + Q'(x)^2 \right)^{k/2} w(x) \int_0^x \frac{\omega^{-1}(t)}{\left( 1 + Q'(t)^2 \right)^{(k-1)/2}} \Psi(t) dt \, dx
$$

$$
= \int_0^\infty \left\{ \frac{\omega^{-1}(x)}{\left( 1 + Q'(x)^2 \right)^{(k-1)/2}} \int_0^x \left( 1 + Q'(t)^2 \right)^{k/2} \omega(t) dt \right\} \Psi(x) dx,
$$

(3.41)
by changing of the integral order. Hence, from (3.20) we have
\[ \left\| (1 + Q')^{k/2} w g \right\|_{L_1(\mathbb{R})} \leq C \int_0^\infty \Psi(x) dx = \| \Psi \|_{L_1(\mathbb{R})}. \tag{3.42} \]

By the definition of (3.38), we have (3.37) for \( p = 1 \).
Next, we show (3.37) for \( p = \infty \). From (3.39) we see that
\[
\left| (1 + Q'(x)^2)^{k/2} w(x) g(x) \right| \leq \left( 1 + Q'(x)^2 \right)^{k/2} w(x) \left\{ |g(x)| + |g(-x)| \right\} \\
\leq \left( 1 + Q'(x)^2 \right)^{k/2} w(x) \int_0^{\|x\|} \frac{\omega^{-1}(t)}{(1 + Q'(t)^2)^{(k-1)/2}} \left\{ |\psi(t)| + |\psi(-t)| \right\} dt \\
\leq 2\|\psi\|_{L_1(\mathbb{R})} \left( 1 + Q'(x)^2 \right)^{k/2} w(x) \int_0^{\|x\|} \omega^{-1}(t) \left( 1 + Q'(t)^2 \right)^{(k-1)/2} dt \\
\leq C\|\psi\|_{L_1(\mathbb{R})}, \tag{3.43}
\]
by (3.21). So we have (3.37) for \( p = \infty \). Let \( \phi \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}) \subset L_p(\mathbb{R}), \ 1 \leq p \leq \infty \), then we set
\[ g(x) := \int_0^x \frac{\phi(t)}{(1 + Q'(t)^2)^{(k-1)/2} w(t)} dt. \tag{3.44} \]
So we have
\[
\left( 1 + Q'(x)^2 \right)^{(k-1)/2} w(x) g'(x) = \phi(x) \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}) \subset L_p(\mathbb{R}), \ 1 \leq p \leq \infty. \tag{3.45}
\]
Since \( L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}) \) is dense in \( L_p(\mathbb{R}), \ 1 \leq p \leq \infty \), using the Riesz-Thorin interpolation theorem for the linear operator,
\[ \phi \rightarrow \left( 1 + Q'(x)^2 \right)^{k/2} w(x) \int_0^x \frac{\omega^{-1}(t)}{(1 + Q'(t)^2)^{(k-1)/2}} \phi(t) dt, \ \ \phi \in L_p(\mathbb{R}), \tag{3.46} \]
we have the result.

\[ \square \]

**Corollary 3.9.** For \( r = 1 \), one lets \( w \in \mathcal{F}(C^2), \) and for integer \( r \geq 2 \) one lets \( w \in \mathcal{F}(C^r) \). Let \( 1 \leq p \leq \infty, \) and \( 1 \leq k \) be an integer. If \( g : \mathbb{R} \rightarrow \mathbb{R} \) is absolutely continuous on \( \mathbb{R}, \ g^{(j)}(0) = 0, \ j = 0, \ldots, k-1, \) and \( g^{(k)} \in L_{p,w}(\mathbb{R}), \) then
\[ \left\| \left( 1 + Q'(x)^2 \right)^{k/2} w g \right\|_{L_p(\mathbb{R})} \leq C \left\| w g^{(k)} \right\|_{L_p(\mathbb{R})}. \tag{3.47} \]
Lemma 3.10 ([4, Lemma 7]). Let \( w \in \mathcal{F}(C^2+) \). For a certain constant \( C_1 > 0 \), let \( u \) satisfy
\[
0 < C_1 \leq u, \quad t = \frac{a_u}{u}.
\]
Then there exists a constant \( C_2 > 1 \) such that for every \( x \) and \( y \) which satisfy \( |x| \leq \sigma(2t) \) and \( |x - y| \leq k\Phi(x)/2 \), one has
\[
\frac{1}{C_2} w(y) \leq w(x) \leq C_2 w(y).
\]

Proof of Theorem 3.3. For a given \( \varepsilon > 0 \), we can select \( g \), where \( g^{(r-1)}(x) \) is absolutely continuous on \( \mathbb{R} \), and \( g^{(r)} \in L_{p,w}(\mathbb{R}) \) such that
\[
\|w(f - g)\|_{L_p(\mathbb{R})} + t' \|w g^{(r)}\|_{L_p(\mathbb{R})} - \varepsilon \leq \mathcal{K}_{r,p}(w, f, t).
\]
Let \( f = h + g \), where \( h \in L_{p,w}(\mathbb{R}) \). Then we have
\[
\omega_{r,p}(h, t) \leq C \|wh\|_{L_p(\mathbb{R})}.
\]
Let \( 0 < s \leq t/r \). From Lemma 3.4, 3.10 and the Hölder-Minkowski inequality, we have
\[
\begin{align*}
\|w \Delta_{s,\Phi}^{r'} g(x)\|_{L_p(|x| \leq \sigma(2t))} &
\leq C \left\| w \left\{ \int_{-s/2}^{s/2} \left| g^{(r)}(u) \right| du_1 du_2 \cdots du_r \right\} \right\|_{L_p(|x| \leq \sigma(2t))} \\
& \leq C \left\| \int_{-s/2}^{s/2} \left| \left( w g^{(r)}(u) \right) \right| du_1 du_2 \cdots du_r \right\|_{L_p(|x| \leq \sigma(2t))} \\
& \leq C \left\| \int_{-s/2}^{s/2} \left| w g^{(r)}(u) \right| du_1 du_2 \cdots du_r \right\|_{L_p(|x| \leq \sigma(2t))} \\
& \leq C s'^r \|w g^{(r)}\|_{L_p(\mathbb{R})} \leq C t' \|w g^{(r)}\|_{L_p(\mathbb{R})},
\end{align*}
\]
where \( u = x + \Phi(t)(u_1 + u_2 + \cdots + u_r) \). We estimate
\[
\inf_{P \in P_{x,1}} \| (g(x) - P(x)) w(x) \|_{L_p(|x| \geq \sigma(2t))}.
\]
Then we may suppose
\[
g^{(j)}(0) - P^{(j)}(0) = 0, \quad j = 0, 1, \ldots, r - 1.
\]
Using Lemma 3.5, we see

\[ |Q'(\sigma(4\ell))|^{-1} \leq Ct, \quad (3.55) \]

(see [7, page 12]). By Corollary 3.9 with \( k = r \) and (3.55) we have

\[
\|w(g - P)\|_{L^p(|x| \geq \sigma(4\ell))} \\
\leq |Q'(\sigma(4\ell))|^{-r} \left( (1 + Q'(x))^2 \right)^{r/2} \|w(g - P)\|_{L^p(|x| \geq \sigma(4\ell))} \\
\leq Ct' \|w\|^{(r)}_{L^p(\mathbb{R})}. \tag{3.56}
\]

Hence, from (3.51), (3.52), and (3.56) we have

\[
\omega_{r,p}(w, f, t) - C\varepsilon \leq C \left( \|w\|_{L^p(\mathbb{R})} + t' \|w\|^{(r)}_{L^p(\mathbb{R})} - \varepsilon \right) \\
\leq C\mathcal{K}_{r,p}(w, f, t). \tag{3.57}
\]

Consequently, we have

\[
\omega_{r,p}(w, f, t) \leq C\mathcal{K}_{r,p}(w, f, t). \tag{3.58}
\]

Therefore, from (3.2) we have (3.3).

**Corollary 3.11.** For \( r = 1 \), one lets \( w \in \mathcal{F}(C^2) \), and for integer \( r \geq 2 \) one lets \( w \in \mathcal{F}(C^r) \). Let \( f \in L^p,w(\mathbb{R}) \). Then one has

\[
\omega_{r,p}(w, f, t) - \mathcal{K}_{r,p}(w, f, t) - \bar{\mathcal{K}}_{r,p}(w, f, t). \tag{3.59}
\]

**Acknowledgments**

The authors thank the referees for many kind suggestions and comments. H. S. Jung was supported by SEOK CHUN Research Fund, Sungkyunkwan University, 2010.

**References**


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