Research Article

Varieties of Parametric Classes of Exact Solutions in General Relativity Representing Static Fluid Balls

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Received 4 November 2011; Accepted 18 December 2011

Academic Editor: H. Zhao

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We have presented a method of obtaining parametric classes of spherically symmetric analytic solutions of the general relativistic field equations in canonical coordinates. A number of previously known classes of solutions have been rediscovered which describe perfect fluid balls with infinite central pressure and infinite central density though their ratio is positively finite and less than one. From the solution of one of the newly discovered classes, we have constructed a causal model in which outmarch of pressure and density is positive and monotonically decreasing, and pressure-density ratio is positive and less than one throughout within the balls. Corresponding to this model, we have maximized the Neutron star mass $2.40M_{\odot}$ with the linear dimensions of 28.43 kms and surface red shift of 0.4142.

1. Introduction

Due to nonlinearity of the Einstein's field equations, many attempts have been reported to obtain parametric classes of exact solutions of representing perfect fluid ball in equilibrium such as Tolman [1], Wyman [2], Kuchowich [3], Pant and Sah [4], D.N. Pant and N. Pant [5], Pant [6, 7], Pant [8], Delgaty, Lake [9], Lake [10], Tewari and Pant [11], and Maurya, Gupta [12]. These solutions have four arbitrary constants. The usual boundary conditions determine three arbitrary constants leaving one undetermined; such a solution represents a class of solutions, with undetermined constant being a parameter. The edge of a parametric class of solutions over an ordinary solution lies in the choice of associated parameter which provides us various models of relativistic stars with realistic equation of state. Moreover, by imposing realistic conditions on parameter, one may get physical and significant solutions. Thus in the light of importance of parametric class, in this paper, we present a variety of parametric classes of solutions.

2. Field Equations and Method of Obtaining Parametric Classes of Solutions

We consider the static and spherically symmetric metric in canonical coordinates

$$ds^2 = -e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + e^{\nu} dt^2, \quad (1)$$

where λ and ν are field variables and functions of r only. The field equations of gravitation for a nonempty space-time are

$$R_{ij} - \frac{1}{2} R g_{ij} = -\frac{8\pi G}{c^4} T_{ij}, \quad (2)$$

where R_{ij} is a Ricci tensor, T_{ij} is energy-momentum tensor, and R the scalar curvature. The energy-momentum tensor T_{ij} is defined as

$$T_{ij} = (p + \rho c^2) v_i v_j - p g_{ij}, \quad (3)$$

where p denotes the pressure distribution, ρ the density distribution, and v_i the velocity vector, satisfying the relation

$$g_{ij} v^i v^j = 1. \quad (4)$$

Since the field is static, therefore, only a nonzero component of velocity is v^4

$$v^4 = \frac{1}{\sqrt{g_{44}}}. \quad (5)$$

Thus, under these conditions, the field equations of general relativity for a perfect fluid ball the physical variables $p(r)$ and $\rho(r)$ are (Tolman[1])

$$\frac{8\pi G}{c^4} p = e^{-\lambda} \left(\frac{v'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2}, \quad (6)$$

$$\frac{8\pi G}{c^2} \rho = e^{-\lambda} \left(\frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}, \quad (7)$$

$$\frac{d}{dr} \left(\frac{e^{-\lambda} - 1}{r^2} \right) + \frac{d}{dr} \left(\frac{e^{-\lambda} v'}{2r} \right) + e^{-\lambda-v} \frac{d}{dr} \left(\frac{e^v v'}{2r} \right) = 0, \quad (8)$$

where prime (/) denotes differentiation with respect to r . The problem consists of solving (8) for λ and v and obtaining $p(r)$ and $\rho(r)$ from (6) and (7).

By using the transformation

$$U = r^{m-1} e^{mv/2}, \quad V = e^{-\lambda}. \quad (9)$$

Equation (8) reduces to the following linear differential equation in V :

$$\begin{aligned} \frac{dV}{dr} - 2 \left\{ \frac{d}{dr} \left[\log \left(\frac{r^{4-1/m} U^{1-1/m}}{rU' + U} \right) \right] - \frac{2mU}{r(rU' + U)} \right\} V \\ = \frac{-2mU}{r(rU' + U)}. \end{aligned} \quad (10)$$

On solving (10), we get

$$V = e^{-\lambda} = \frac{r^{8-2/m} U^{2(1-1/m)}}{(rU' + U)^2} \left[A - 2m \int \frac{(rU' + U) U^{-1+2/m} e^{\int (4mU dr/r(rU' + U))}}{r^{9-2/m}} dr \right] e^{-\int (4mU dr/r(rU' + U))}, \quad (11)$$

where “ A ” is an arbitrary constant. Our aim is to explore the possibilities of choosing U such that the right-hand side of (11) becomes integrable. In this paper, we assume

$$e^{\int (4mU dr/r(rU' + U))} = r^l (rU' + U)^n; \quad (12)$$

l , m , and n are arbitrary constants. Equation (12) results into a second degree homogenous differential equation in U

$$nr^2 U'' + (l + 2n)rU' + (l - 4m)U = 0. \quad (13)$$

The solution is

$$U = C_1 r^{a+b-1} + C_2 r^{a-b-1}, \quad (14)$$

where

$$a = \frac{n-l}{2n}, \quad b = \frac{1}{2n} \sqrt{(n-l)^2 + 16mn}, \quad (15)$$

provided, $n \neq 0$.

C_1 and C_2 are arbitrary constants. Also (11) is simplified into

$$\begin{aligned} V = e^{-\lambda} \\ = \frac{r^{8+n-l-(a-b)(n+2/m)} [A - 2I]}{[(a+b)C_1 r^{2b} + (a-b)C_2]^{n+2} (C_1 r^{2b} + C_2)^{2(-1+1/m)}}, \end{aligned} \quad (16)$$

where

$$\begin{aligned} I \equiv m \int r^{l-n-9+(a-b)(n+2/m)} [(a+b)C_1 r^{2b} + (a-b)C_2]^{n+1} \\ \times [C_1 r^{2b} + C_2]^{-1+2/m} dr. \end{aligned} \quad (17)$$

The solution is complete if (17) is integrated. In the foregoing sections, we shall discuss the method of solving (17), which will result into the variety of classes of solutions.

3. Varieties of Classes of Solutions

To explore the integrability of (17), we substitute

$$r^{2b} = y, \quad (18)$$

$$l - n - 9 + (a-b) \left(n + \frac{2}{m} \right) = 2b\alpha - 1. \quad (19)$$

So that (17) transforms into

$$\begin{aligned} I \equiv \frac{m}{2b} \int y^{\alpha-1} [(a+b)C_1 y + (a-b)C_2]^{n+1} \\ \times [C_1 y + C_2]^{-1+2/m} dy. \end{aligned} \quad (20)$$

It is easy to see that right-hand side of (20) can be integrated by parts with the restriction on the exponents of integrand to be nonnegative integral values. Thus we arrive at varieties of classes of solutions.

3.1. Class I (new class of solutions). We assume $n+1=0$ and $-1+2/m=0$. In view of (18) and (20), the resulting class of solutions is

$$\begin{aligned} e^v = (C_1 r^{a+b-2} + C_2 r^{a-b-2}), \\ e^{-\lambda} = \frac{[Ar^{7-l} - 4/(l-7)] [C_1 r^{2b} + C_2]}{[(a+b)C_1 r^{2b} + (a-b)C_2]}, \end{aligned} \quad (21)$$

where

$$a = \frac{1+l}{2}, \quad b = \frac{\sqrt{(1+l)^2 - 32}}{-2}. \quad (22)$$

We observe that $(e^\nu)_{r=0}$ becomes singular for all values except $\sqrt{32} - 1 \leq l \leq 5$. It may be pointed out here that for $l = 5$, we rediscover Tolman's IV solution which is the only member

of the class giving rise to finite central pressure and central density.

4. Properties of the New Class of Solutions

In view of (??), we obtain from (6) and (7), the pressure and density distribution, respectively,

$$\begin{aligned} \frac{8\pi G}{c^4} P &= \frac{1}{r^2} \left\{ \frac{[Ar^{7-l} - 4/l - 7] [(a+b-1)C_1 r^{2b} + (a-b-1)C_2]}{[(a+b)C_1 r^{2b} + (a-b)C_2]} - 1 \right\}, \\ \frac{8\pi G}{c^2} \rho &= \frac{1}{r^2} \left[\frac{[(Ar^{7-l} - 4/l - 7) 4b^2 C_1 C_2 r^{2b} + ((l-8)Ar^{7-l} + 4/l - 7) \times (C_1 r^{2b} + C_2)((a+b)C_1 r^{2b} + (a-b)C_2)]}{[(a+b)C_1 r^{2b} + (a-b)C_2]^2} + 1 \right]. \end{aligned} \quad (23)$$

The central values of pressure and density are infinite; however, the limiting value of their ratio is finite and equal to the limiting value of $dp/d\rho$:

$$\begin{aligned} \left(\frac{p}{\rho c^2} \right)_{r \rightarrow 0} &= \frac{1}{c^2} \left(\frac{dp}{d\rho} \right)_{r \rightarrow 0} \\ &= - \frac{[4(a+b-1) + (l-7)(a+b)]}{[4 + (l-7)(a+b)]}. \end{aligned} \quad (24)$$

The causality condition at the centre is valid for all those values of l satisfying the inequality $\sqrt{32} - 1 \leq l \leq 5$.

In addition to the parameter l , the solutions (??) contain three arbitrary constants C_1 , C_2 , and A . These are to be determined by matching the solutions (??) with Schwarzschild exterior solution for a ball of mass M and linear dimension $2r_b$:

$$\begin{aligned} p(r_b) &= 0, \\ e^{-\lambda(r_b)} &= 1 - 2u, \\ e^{\nu(r_b)} &= 1 - 2u, \end{aligned} \quad (25)$$

where

$$u = \frac{GM}{c^2 r_b}. \quad (26)$$

Consequently,

$$C_1 = - \frac{[(a-b-1)(1-2u) - 1]}{2br_b^{a+b-2}}, \quad (27)$$

$$C_2 = \frac{[(a+b-1)(1-2u) - 1]}{2br_b^{a-b-2}}, \quad (28)$$

$$A = r_b^{l-7} \left[\frac{[1-2u][(a+b)C_1 r_b^{2b} + (a-b)C_2]}{[C_1 r_b^{2b} + C_2]} + \frac{4}{l-7} \right]. \quad (29)$$

For e^ν to be definitely positive in the region $0 \leq r \leq r_b$, we must have $C_1, C_2 \geq 0$. Thus in view of (27) and (28), we have

$$\frac{a+b-2}{a+b-1} \leq 2u \leq \frac{a-b-2}{a-b-1}. \quad (30)$$

5. Particular Member of Class I for $l = 4.75$

In this section, we shall present a detail study of the particular solution corresponding to $l = 4.75$.

The solution is

$$\begin{aligned} e^\nu &= (C_1 r^{0.35962} + C_2 r^{1.39}), \\ e^{-\lambda} &= \frac{[Ar^{2.25} + 1.78][C_1 + C_2 r^{1.03}]}{[2.359C_1 + 3.39C_2 r^{1.03}]}, \\ \frac{8\pi G}{c^4} P &= \frac{1}{r^2} \left[\frac{[Ar^{2.25} + 1.78][1.359C_1 + 2.39C_2 r^{1.03}]}{[2.359C_1 + 3.39C_2 r^{1.03}]} - 1 \right], \\ \frac{8\pi G}{c^2} \rho &= \frac{1}{r^2} \left[\frac{[(Ar^{2.25} + 1.78)1.06C_1 C_2 r^{1.03} + (-3.25Ar^{2.25} - 1.78) \times (C_1 + C_2 r^{1.03})(2.359C_1 + 3.39C_2 r^{1.03})]}{[2.359C_1 + 3.39C_2 r^{1.03}]^2} + 1 \right]. \end{aligned} \quad (31)$$

TABLE 1: The march of pressure, density, pressure-density ratio, and square of adiabatic sound speed within the ball corresponding to $l = 4.75$ with $u = 0.25$.

r/r_b	$(8\pi G/c^4)pr_b^2$	$(8\pi G/c^2)pr_b^2$	$p/\rho c^2$	$(1/c^2)(dp/d\rho)$
0	∞	∞	0.09879	0.09879
0.1	6.58927	32.3632	0.20360	0.18684
0.2	2.29996	9.40575	0.24452	0.23125
0.3	1.18659	4.63777	0.25585	0.26815
0.4	0.70315	2.84393	0.24724	0.29885
0.5	0.44151	1.97301	0.22377	0.32653
0.6	0.28097	1.48486	0.18922	0.36966
0.7	0.17397	1.18389	0.14694	0.39023
0.8	0.09830	0.98631	0.09966	0.42714
0.9	0.04233	0.84997	0.049801	0.46666
1.0	0	0.75238	0	0.45254

In view of (30) e^v to be definitely positive in the region $0 \leq r \leq r_b$, then

$$0.13225 \leq u \leq 0.29079. \quad (32)$$

Corresponding to $u = 0.25$ and in view of (27), (28), and (29), the constants are

$$C_1 r_b^{0.35962} = 0.1891, \quad C_2 r_b^{1.39} = 0.3106, \quad (33)$$

$$Ar_b^{2.25} = -0.2808.$$

In Table 1 the march of pressure, density, pressure-density ratio, and square of adiabatic sound speed $dp/d\rho$ is given for $u = 0.25$. We observe that pressure and density decrease monotonically with the increase of radial coordinate, pressure-density ratio, and square of adiabatic sound speed, which is positive and less than 1 throughout within the ball.

We now present here a model of Neutron star based on the particular solution discussed above. The Neutron star is supposed to have a surface density: $\rho_b = 2 \times 10^{14} \text{ gm-cm}^{-3}$ suggested by Brecher and Caporaso [13]. The resulting causal model has the mass $M = 2.40M_\odot$ and the linear dimension $2r_b \approx 28.43 \text{ km}$. The surface red shift $Z_b = [(1 - 2u)^{-0.5} - 1] \approx 0.4142$.

6. Some More Parametric Classes of Solutions

By assigning the different nonnegative integral values to the exponents of (20), we obtain the following different parametric classes of solutions.

6.1. *Class II.* If we assume $\alpha - 1 = 0$ and $-1 + 2/m = 0$. In view of (17), (18), and (20), the resulting class of solutions is (see Neeraj Pant [6])

$$e^v = (C_1 r^{a+b-2} + C_2 r^{a-b-2}),$$

$$e^{-\lambda} = \left[\frac{Ar^{-2b}}{[(a+b)C_1 r^{2b} + (a-b)C_2]^{n+2}} - \frac{r^{-2b}}{2b(n+2)(a+b)C_1} \right] [C_1 r^{2b} + C_2]. \quad (34)$$

6.2. *Class III.* If we assume $\alpha - 1 = 0$ and $n + 1 = 0$. In view of (17), (18), and (20), the resulting class of solutions is (see Pant [8])

$$e^v = (C_1 r^{a+b-m} + C_2 r^{a-b-m})^{2/m},$$

$$e^{-\lambda} = \left[\frac{Ar^{-2b} - m^2 r^{-2b}/2b C_1 (C_1 r^{2b} + C_2)^{2/m}}{[(a+b)C_1 r^{2b} + (a-b)C_2](C_1 r^{2b} + C_2)^{2(-1+1/m)}} \right]. \quad (35)$$

6.3. *Class IV.* If we assume $n + 1 = 0$ and $-1 + 2/m = 1$. In view of (17), (18), and (20), the resulting class of solutions is (see Pant [8])

$$e^v = (C_1 r^{a+b-1} + C_2 r^{a-b-1})^2,$$

$$e^{-\lambda} = \frac{r^{-2b\alpha} [A - r^{2b\alpha} \{ \alpha C_1 r^{2b} + (\alpha + 1)C_2 \} / b\alpha(\alpha + 1)]}{[(a+b)C_1 r^{2b} + (a-b)C_2]}, \quad (36)$$

where

$$\alpha = \frac{l - 7 + a - b}{2b}. \quad (37)$$

6.4. *Class V.* If we assume $\alpha - 1 = 1$ and $-1 + 2/m = 0$. In view of (17), (18), and (20), the resulting class of solutions is (New Class of Solutions)

$$e^v = (C_1 r^{a+b-2} + C_2 r^{a-b-2}),$$

$$e^{-\lambda} = \frac{Ar^{-4b} - 4Br^{-4b}\mathfrak{A}\mathfrak{B}}{[(a+b)C_1 r^{2b} + (a-b)C_2]^{n+2}(C_1 r^{2b} + C_2)^{-1}}, \quad (38)$$

where $\mathfrak{A}$ denotes $[(a+b)C_1 r^{2b} + (a-b)C_2]^{n+2}$ and $\mathfrak{B}$ denotes $[(a+b)(n+2)C_1 r^{2b} - (a-b)C_2]$,

$$B = \frac{1}{b(a+b)^2(n+2)(n+3)C_1^2}. \quad (39)$$

6.5. *Class VI.* If we assume $\alpha - 1 = 1$ and $n+1 = 0$. In view of (17), (18), and (20), the resulting class of solutions is (New Class of Solutions)

$$e^v = (C_1 r^{a+b-m} + C_2 r^{a-b-m})^{2/m},$$

$$e^{-\lambda} = \left[\frac{\mathfrak{D}(2C_1 r^{2b} - mC_2)}{[(a+b)C_1 r^{2b} + (a-b)C_2](C_1 r^{2b} + C_2)^{2(-1+1/m)}} \right], \quad (40)$$

where $\mathfrak{D}$ denotes $Ar^{-4b} - 2m^2 r^{-4b}/4bC_1^2(C_1 r^{2b} + C_2)^{2/m}$.

6.6. *Class VII.* If we assume $\alpha - 1 = 0$ and $-1 + 2/m = 1$. In view of (17), (18), and (20), the resulting class of solutions is (see Tewari and Pant [11])

$$e^v = (C_1 r^{a+b-1} + C_2 r^{a-b-1})^2,$$

$$e^{-\lambda} = \frac{Ar^{-2b}}{[(a+b)C_1 r^{2b} + (a-b)C_2]^{n+2}} - B(C_1' + C_2' r^{-2b}). \quad (41)$$

7. Conclusion

By assigning different nonnegative integral values to the any two pairs of exponents of integrand of (17), that is, $\alpha - 1$, $n + 1$, $-1 + 2/m$, we arrive at different parametric classes of solutions. All these classes have of the form $e^v = (C_1 r^\beta + C_2 r^\beta)^\gamma$. For meaningful solutions γ , β to be nonnegative, thus restriction on parameter ranges.

Acknowledgments

The author expresses his gratitude to Professor O. P. Shukla, Principal NDA, for his encouragement. He is also grateful to the anonymous referees for pointing out the errors in the original paper and making constructive and relevant suggestions.

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