Research Article

Generalized Derivations and Left Ideals in Prime and Semiprime Rings

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Let R be an associative ring, λ a nonzero left ideal of R , $d : R \rightarrow R$ a derivation and $G : R \rightarrow R$ a generalized derivation. In this paper, we study the following situations in prime and semiprime rings: (1) $G(x \circ y) = a(xy \pm yx)$; (2) $G[x, y] = a(xy \pm yx)$; (3) $d(x) \circ d(y) = a(xy \pm yx)$; for all $x, y \in \lambda$ and $a \in \{0, 1, -1\}$.

1. Introduction

Throughout this paper, let R be an associative ring, λ a left ideal of R , d a derivation of R and G a generalized derivation of R . For any two elements $x, y \in R$, $[x, y]$ will denote the commutator element $xy - yx$ and $x \circ y$ denotes $xy + yx$. We use extensively the following basic commutator identities: $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that a ring R is called prime, if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$ and is called semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. An additive mapping $d : R \rightarrow R$ is said to be a derivation of R if for any $x, y \in R$, $d(xy) = d(x)y + xd(y)$ holds. The generalized derivation of R is defined as an additive mapping $G : R \rightarrow R$ such that $G(xy) = G(x)y + xd(y)$ holds for any $x, y \in R$, where d is a derivation of R . So, every derivation is a generalized derivation, but the converse is not true in general. If $d = 0$, then we have $G(xy) = G(x)y$ for all $x, y \in R$, which is called a left multiplier mapping of R . Thus, generalized derivation generalizes both the concepts, derivation as well as left multiplier mapping of R .

In [1], Daif and Bell proved that if R is a semiprime ring with a nonzero ideal I and d is a derivation of R such that $d([x, y]) = \pm[x, y]$ for all $x, y \in I$, then I is central ideal. In particular, if $I = R$, then R is commutative. Recently, Quadri et al. [2] have generalized this result replacing derivation d with a generalized derivation in a prime ring R . More precisely, they obtained the following result.

Let R be a prime ring and I a nonzero ideal of R . If R admits a generalized derivation F associated with a nonzero derivation d such that any one of the following holds: (i) $F([x, y]) = [x, y]$ for all $x, y \in I$; (ii) $F([x, y]) = -[x, y]$ for all $x, y \in I$; (iii) $F(x \circ y) = (x \circ y)$ for all $x, y \in I$; (iv) $F(x \circ y) = -(x \circ y)$ for all $x, y \in I$; then R is commutative.

Recently in [3], the first author of this paper has studied all the results of [2] in semiprime ring. In the present paper, our aim is to discuss similar identities in a left sided ideal of a semiprime rings.

2. Main Results

Theorem 2.1. Let R be a semiprime ring and λ a nonzero left ideal of R . If G is a generalized derivation of R associated with a derivation d of R such that $G(x \circ y) = a(x \pm y)$ for all $x, y \in \lambda$, where $a \in \{0, 1, -1\}$, then $[\lambda, \lambda]d(\lambda) = 0$.

Proof. If $G(\lambda) = 0$, then for any $x, y \in \lambda$,

$$0 = G(xy) = G(x)y + xd(y) = xd(y), \quad (2.1)$$

that is, $\lambda d(\lambda) = 0$. This gives our conclusion. So let $G(\lambda) \neq 0$. Then by our assumption, we have

$$G(x \circ y) = a(x \pm y) \quad (2.2)$$

for all $x, y \in \lambda$. Putting $y = yx$, $x \in \lambda$, we obtain that $G((x \circ y)x) = a((x \pm y)x)$. Since G is a generalized derivation of R , this implies that $G(x \circ y)x + (x \circ y)d(x) = a(x \pm y)x$. This gives by using (2.2) that $(x \circ y)d(x) = 0$ for all $x, y \in \lambda$. Now, we replace y with zy , where $z \in \lambda$, and then we get

$$0 = z(x \circ y)d(x) + [x, z]yd(x) = [x, z]yd(x) \quad (2.3)$$

for all $x, y, z \in \lambda$. Since λ is a left ideal, it follows that $[x, z]Ryd(x) = 0$ for all $x, y, z \in \lambda$. Since R is semiprime, it must contain a family $\Omega = \{P_\alpha : \alpha \in \Lambda\}$ of prime ideals such that $\bigcap_{\alpha \in \Lambda} P_\alpha = \{0\}$. If P is typical member of Ω and $x \in \lambda$, we have either $[x, \lambda] \subseteq P$ or $\lambda d(x) \subseteq P$. For fixed P , the sets $T_1 = \{x \in \lambda : [x, \lambda] \subseteq P\}$ and $T_2 = \{x \in \lambda : \lambda d(x) \subseteq P\}$ form two additive subgroups of λ such that $T_1 \cup T_2 = \lambda$. Therefore, either $T_1 = \lambda$ or $T_2 = \lambda$, that is, either $[\lambda, \lambda] \subseteq P$ or $\lambda d(\lambda) \subseteq P$. Both of these two conditions together imply that $[\lambda, \lambda]d(\lambda) \subseteq P$ for any $P \in \Omega$. Therefore, $[\lambda, \lambda]d(\lambda) \subseteq \bigcap_{\alpha \in \Lambda} P_\alpha = 0$. $\square$

Corollary 2.2. Let R be a prime ring and λ be a nonzero left ideal of R . If R admits a generalized derivation G associated with a derivation d such that $G(x \circ y) = a(x \circ y)$ for all $x, y \in \lambda$, where $a \in \{0, 1, -1\}$, then one of the following holds:

- (i) $\lambda d(\lambda) = 0$;
- (ii) R is commutative ring with $\text{char}(R) = 2$;
- (iii) R is commutative ring with $\text{char}(R) \neq 2$ and $G(x) = ax$ for all $x \in \lambda$.

Proof. By Theorem 2.1, we have $[\lambda, \lambda]d(\lambda) = 0$. This gives

$$0 = [\lambda, \lambda^2]d(\lambda) = [\lambda, \lambda]\lambda d(\lambda) = [\lambda, \lambda]R\lambda d(\lambda). \quad (2.4)$$

Since R is prime, either $[\lambda, \lambda] = 0$ or $\lambda d(\lambda) = 0$. Now $\lambda d(\lambda) = 0$ gives our conclusion (i). So, let $[\lambda, \lambda] = 0$ which gives $0 = [\lambda, R\lambda] = [\lambda, R]\lambda$ implying $0 = [\lambda, R]$. Again, this gives $0 = [R\lambda, R] = [R, R]\lambda$. Since left annihilator of a left-sided ideal is zero, $[R, R] = 0$, that is, R is commutative. If $\text{char}(R) = 2$, we obtain conclusion (ii). So assume that $\text{char}(R) \neq 2$. Then our assumption $G(x \circ y) = a(x \circ y)$ gives $2G(xy) = 2a(xy)$ for all $x, y \in \lambda$. Since $\text{char}(R) \neq 2$, then $G(xy) = a(xy)$ for all $x, y \in \lambda$. This gives for all $x, y \in \lambda$,

$$0 = G(xy) - a(xy) = G(x)y + xd(y) - axy = (G(x) - ax)y + xd(y). \quad (2.5)$$

Let $r \in R$. Since R is commutative, $xr \in \lambda$. Put $x = xr$ in last result, we get $0 = (G(x) - ax)ry + xrd(y) + xd(r)y = \{(G(x) - ax)y + xd(y)\}r + d(r)yx$. Since R is commutative, using (2.5), it yields $d(R)\lambda^2 = \{0\}$, implying $d = 0$. Then, (2.5) implies $(G(x) - ax)\lambda = 0$, which yields $G(x) - ax = 0$ for all $x \in \lambda$. $\square$

Theorem 2.3. *Let R be a semiprime ring, λ a nonzero left ideal of R and G a generalized derivation of R associated with a derivation d of R . If $G[x, y] = a(xy \pm yx)$ for all $x, y \in \lambda$, then $[\lambda, \lambda]d(\lambda) = 0$.*

Proof. If $G(\lambda) = 0$, then $0 = G(\lambda^2) = G(\lambda)\lambda + \lambda d(\lambda) = \lambda d(\lambda)$ and hence, $[\lambda, \lambda]d(\lambda) = 0$, which is our conclusion. Assume next that $G(\lambda) \neq 0$. Then by our assumption, we have

$$G[x, y] = a(xy \pm yx) \quad (2.6)$$

for all $x, y \in \lambda$. Put $y = yx$ and get $G([x, y]x) = a(xy \pm yx)x$, that is, $G([x, y])x + [x, y]d(x) = a(xy \pm yx)x$. Now using (2.6), the above relation yields $[x, y]d(x) = 0$ for all $x, y \in \lambda$. Putting $y = zy$, where $z \in \lambda$, we obtain that $[x, z]yd(x) = 0$, which is same as (2.3) in Theorem 2.1. By same argument of Theorem 2.1, we can conclude the result here. $\square$

Corollary 2.4. *Let R be a prime ring and λ a nonzero left ideal of R . If G is a generalized derivation of R associated with a derivation d of R such that $G[x, y] = a(xy \pm yx)$ for all $x, y \in \lambda$, where $a = \{0, 1, -1\}$, then either R is commutative or $\lambda d(\lambda) = 0$ and one of the following holds:*

$$(i) \lambda[\lambda, \lambda] = 0;$$

$$(ii) G(x) = \mp ax \text{ for all } x \in \lambda. \text{ In case } G(x) = -ax \text{ for all } x \in \lambda, \text{ with } a \neq 0, \text{ then } \text{char}(R) = 2.$$

Proof. By the Theorem 2.3, we may conclude that $[\lambda, \lambda]d(\lambda) = 0$. Then by same argument as given in Corollary 2.2, we obtain that either R is commutative or $\lambda d(\lambda) = 0$. Let R be noncommutative, then for any $x, y \in \lambda$, we have $G(xy) = G(x)y + xd(y) = G(x)y$, that is, G acts as a left multiplier map on λ . Then for any $x, y, z \in \lambda$, replacing y with yz in our hypothesis $G[x, y] = a(xy \pm yx)$, we have

$$G([x, y]z + y[x, z]) = a\{(xy \pm yx)z \mp y[x, z]\} \quad (2.7)$$

for all $x, y \in \lambda$. Since G acts as a left multiplier map on λ , this implies

$$G([x, y])z + G(y)[x, z] = a(xy \pm yx)z \mp ay[x, z]. \quad (2.8)$$

By using $G[x, y] = a(xy \pm yx)$, it gives $G(y)[x, z] = \mp ay[x, z]$, that is, $(G(y) \pm ay)[x, z] = 0$ for all $x, y \in \lambda$. Replacing y with yu , where $u \in \lambda$, we find that $(G(y) \pm ay)u[x, z] = 0$, which gives $(G(y) \pm ay)R\lambda[x, z] = 0$. Since R is prime, either $\lambda[\lambda, \lambda] = 0$ or $G(y) = \mp ay$ for all $y \in \lambda$. When $G(y) = -ay$, our assumption $G[x, y] = a(xy + yx)$ implies $-a[x, y] = a(xy + yx)$ for all $x, y \in \lambda$. This implies $2axy = 0$, that is, $2aR\lambda^2 = 0$ implies $\text{char}(R) = 2$, unless $a = 0$. $\square$

Theorem 2.5. *Let R be a semiprime ring, λ a nonzero left ideal of R and d a derivation of R . If $d(x) \circ d(y) = a(xy \pm yx)$ for all $x, y \in \lambda$, where $a \in \{0, 1, -1\}$, then $\lambda[x, d(x)]_2 = 0$. In case $\lambda = R$ and R is 2-torsion free, d maps R into its center.*

Proof. We have for all $x, y \in \lambda$,

$$d(x)d(y) + d(y)d(x) = a(xy \pm yx). \quad (2.9)$$

Putting $y = yx$, we get

$$d(x)(d(y)x + yd(x)) + (d(y)x + yd(x))d(x) = a(xy \pm yx)x. \quad (2.10)$$

Using (2.9), we have

$$d(x)y d(x) + d(y)[x, d(x)] + y d(x)^2 = 0. \quad (2.11)$$

Put $y = xy$ in (2.11), and get

$$d(x)xy d(x) + \{xd(y) + d(x)y\}[x, d(x)] + xy d(x)^2 = 0. \quad (2.12)$$

Left multiplying (2.11) by x and then subtracting from (2.12) yields

$$[d(x), x]y d(x) + d(x)y[x, d(x)] = 0 \quad (2.13)$$

for all $x, y \in \lambda$. Put $y = d(x)y$ in (2.13), we have

$$[d(x), x]d(x)y d(x) + d(x)^2 y[x, d(x)] = 0. \quad (2.14)$$

Left multiplying (2.13) by $d(x)$ and then subtracting from (2.14), we get $[x, d(x)]_2 y d(x) = 0$ for all $x, y \in \lambda$. This implies $[x, d(x)]_2 y[x, d(x)]_2 = 0$ and hence, $y[x, d(x)]_2 R y[x, d(x)]_2 = 0$ for all $x, y \in \lambda$. Since R is semiprime, $\lambda[x, d(x)]_2 = 0$. In case $\lambda = R$, $[x, d(x)]_2 = 0$ for all $x \in R$, and then by [4], $d(R) \subseteq Z(R)$. $\square$

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