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Research Article

Fundamental Solution in the Theory of Thermomicrostretch Elastic Diffusive Solids

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We construct the fundamental solution of system of differential equations in the theory of thermomicrostretch elastic diffusive solids in case of steady oscillations in terms of elementary functions. Some basic properties of the fundamental solution are established. Some special cases are also discussed.

1. Introduction

Eringen [1] developed the theory of micropolar elastic solid with stretch. He derived the equations of motion, constitutive equations, and boundary conditions for the class of micropolar solid which can stretch and contract. This model introduced and explained the motion of certain class of granular and composite materials in which grains and fibres are elastic along the direction of their major axis. This theory is generalization of the theory of micropolar elasticity [2, 3]. Eringen [4] developed a theory of thermomicrostretch elastic solid in which he included microstructural expansions and contractions. Microstretch continuum is a model for Bravais lattice with a basis on the atomic level and a two-phase dipolar solid with a core on the macroscopic level. In the framework of the theory of thermomicrostretch solids, Eringen established a uniqueness theorem for the mixed initial boundary value problem. The theory was illustrated with the solution of one-dimensional waves and compared with lattice dynamical results. The asymptotic behavior of solutions and an existence result were presented by Bofill and Quintanilla [5]. A reciprocal theorem and a representation of Galerkin type were presented by De Cicco and Nappa [6].

In classical theory of thermoelasticity, Fourier's heat conduction theory assumes that the thermal disturbances propagate at infinite speed which is unrealistic from the physical point of view. Lord and Shulman [7] incorporates a flux rate term into Fourier's law of heat conduction and formulates a generalized theory admitting finite speed for thermal signals.

Lord and Shulman [7] theory of generalized thermoelasticity has been further extended to homogeneous anisotropic heat conducting materials recommended by Dhaliwal and Sherief [8]. All these theories predict a finite speed of heat propagation. Chanderashekhariah [9] refers to this wave-like thermal disturbance as second sound. A survey article of various representative theories in the range of generalized thermoelasticity has been brought out by Hetnarski and Ignaczak [10].

Diffusion is defined as the spontaneous movement of the particles from a high-concentration region to the low-concentration region, and it occurs in response to a concentration gradient expressed as the change in the concentration due to change in position. Thermal diffusion utilizes the transfer of heat across a thin liquid or gas to accomplish isotope separation. Today, thermal diffusion remains a practical process to separate isotopes of noble gases (e.g., xenon) and other light isotopes (e.g., carbon) for research purposes. In most of the applications, the concentration is calculated using what is known as Fick's law. This is a simple law which does not take into consideration the mutual interaction between the introduced substance and the medium into which it is introduced or the effect of temperature on this interaction. However, there is a certain degree of coupling with temperature and temperature gradients as temperature speeds up the diffusion process. The thermodiffusion in elastic solids is due to coupling of fields of temperature, mass diffusion and that of strain in addition to heat and mass exchange with the environment.

Nowacki [11–14] developed the theory of thermoelastic diffusion by using coupled thermoelastic model. Dudziak and Kowalski [15] and Olesiak and Pyryev [16], respectively, discussed the theory of thermodiffusion and coupled quasistationary problems of thermal diffusion for an elastic layer. They studied the influence of cross-effects arising from the coupling of the fields of temperature, mass diffusion, and strain due to which the thermal excitation results in additional mass concentration and that generates additional fields of temperature. Uniqueness and reciprocity theorems for the equations of generalized thermoelastic diffusion problem, in isotropic media, were proved by Sherief et al. [17] on the basis of the variational principle equations, under restrictive assumptions on the elastic coefficients. Due to the inherit complexity of the derivation of the variational principle equations, Aouadi [18] proved this theorem in the Laplace transform domain, under the assumption that the functions of the problem are continuous and the inverse Laplace transform of each is also unique. Aouadi [19] derived the uniqueness and reciprocity theorems for the generalized problem in anisotropic media, under the restriction that the elastic, thermal conductivity and diffusion tensors are positive definite.

To investigate the boundary value problems of the theory of elasticity and thermoelasticity by potential method, it is necessary to construct a fundamental solution of systems of partial differential equations and to establish their basic properties, respectively. Hetnarski [20, 21] was the first to study the fundamental solutions in the classical theory of coupled thermoelasticity. The fundamental solutions in the microcontinuum fields theories have been constructed by Svanadze [22], Svanadze and De Cicco [23], and Svanadze and Tracina [24]. The information related to fundamental solutions of differential equations is contained in the books of Hörmander [25, 26].

In this paper, the fundamental solution of system of equations in the case of steady oscillations is considered in terms of elementary functions and basic properties of the fundamental solution are established. Some special cases of interest are also discussed.

2. Basic Equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the point of the Euclidean three-dimensional space E^3: $|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, $\mathbf{D_x} = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ and let t denote the time variable. Following Sherief et al. [17] and Eringen [4], the basic equations for homogeneous isotropic generalized theromicrostretch elastic diffusive solids in the absence of body forces, body couples, body loads, heat and mass diffusion sources are

$$\begin{gathered}
(\mu + K^*)\Delta\overline{\mathbf{u}} + (\lambda + \mu)\,\mathrm{grad}\ \mathrm{div}\,\overline{\mathbf{u}} + K^*\,\mathrm{curl}\ \overline{\boldsymbol{\varphi}} + \chi^*\,\mathrm{grad}\,\overline{\varphi^*} - \beta_1\,\mathrm{grad}\,\overline{T} - \beta_2\,\mathrm{grad}\,\overline{C} = \rho\ddot{\overline{\mathbf{u}}},\\
(f^*\Delta - 2K^*)\overline{\boldsymbol{\varphi}} + (\alpha^* + \beta^*)\,\mathrm{grad}\ \mathrm{div}\,\overline{\boldsymbol{\varphi}} + K^*\,\mathrm{curl}\,\overline{\mathbf{u}} = \rho j\ddot{\overline{\boldsymbol{\varphi}}},\\
(b^*\Delta - c^*)\overline{\varphi^*} - \chi^*\,\mathrm{div}\,\overline{\mathbf{u}} - g^*\overline{T} - h^*\overline{C} = \rho\zeta\ddot{\overline{\varphi^*}},\\
\left(1 + \tau_0\frac{\partial}{\partial t}\right)\left(\beta_1 T_0\,\mathrm{div}\,\dot{\overline{\mathbf{u}}} - g^* T_0\dot{\overline{\varphi^*}} + \rho C_E\dot{\overline{T}} + aT_0\dot{\overline{C}}\right) = K\Delta\overline{T},\\
D\beta_2\Delta\,\mathrm{div}\,\overline{\mathbf{u}} + +Dh^*\Delta\overline{\varphi^*} + Da\Delta\overline{T} - Db\Delta\overline{C} + \dot{\overline{C}} + \tau^0\ddot{\overline{C}} = 0,
\end{gathered} \tag{2.1}$$

where $\beta_1 = (3\lambda + 2\mu + K^*)\alpha_t$, $\beta_2 = (3\lambda + 2\mu + K^*)\alpha_c$. Here α_t, α_c are the coefficients of linear thermal expansion, and diffusion expansion, respectively; $\overline{\mathbf{u}} = (\overline{u_1}, \overline{u_2}, \overline{u_3})$ is the displacement vector; $\overline{\boldsymbol{\varphi}} = (\overline{\varphi_1}, \overline{\varphi_2}, \overline{\varphi_3})$ is the microrotation vector; φ^* is the microstretch function; ρ, C_E are, respectively, the density and specific heat at constant strain; $\lambda, \mu, K, D, a, b, b^*, c^*, f^*, g^*, h^*, \alpha^*, \beta^*, K^*$, and χ^* are constitutive coefficients; j and ζ are coefficients of microintertia; $\overline{T}$ is the temperature measured from constant temperature T_0 $(T_0 \neq 0)$ and $\overline{C}$ is the concentration; τ^0 is diffusion relaxation time and τ_0 is thermal relaxation time; Δ is the Laplacian operator. Here $\tau_0 = \tau^0 = 0$ for coupled thermoelastic diffusion model.

We define the dimensionless quantities:

$$\begin{gathered}
\mathbf{x}' = \frac{w_1^*\mathbf{x}}{c_1}, \quad \overline{\mathbf{u}}' = \frac{\rho w_1^* c_1\overline{\mathbf{u}}}{\beta_1 T_0}, \quad \overline{\boldsymbol{\varphi}}' = \frac{\rho c_1^2\overline{\boldsymbol{\varphi}}}{\beta_1 T_0}, \quad \overline{\varphi^*}' = \frac{\rho\zeta w_1^{*2}\overline{\varphi^*}}{\beta_1 T_0}, \quad \overline{T}' = \frac{\overline{T}}{T_0}, \quad \overline{C}' = \frac{\beta_2\overline{C}}{\beta_1 T_0},\\
t' = w_1^* t, \quad \tau_0' = w_1^*\tau_0, \quad \tau^{0'} = w_1^*\tau^0, \quad \delta_1 = \frac{\mu + K^*}{\lambda + 2\mu + K^*}, \quad \delta_2 = \frac{\lambda + \mu}{\lambda + 2\mu + K^*},\\
\delta_3 = \frac{K^*}{\lambda + 2\mu + K^*}, \quad \delta_4 = \frac{\chi^*}{\rho\zeta w_1^{*2}}, \quad \delta_5 = \frac{f^* w_1^{*2}}{\rho c_1^4}, \quad \delta_6 = \frac{(\alpha^* + \beta^*)w_1^{*2}}{\rho c_1^4}, \quad \delta_7 = \frac{j w_1^{*2}}{c_1^2},\\
\delta_8 = \frac{b^*}{\zeta(\lambda + 2\mu + K^*)}, \quad \delta_9 = \frac{c^*}{\rho\zeta w_1^{*2}}, \quad \delta_{10} = \frac{\chi^*}{\lambda + 2\mu + K^*}, \quad \delta_{11} = \frac{g^*}{\beta_1}, \quad \delta_{12} = \frac{h^*}{\beta_2},\\
\zeta_1 = \frac{aT_0 c_1^2\beta_1}{w_1^* K\beta_2}, \quad \zeta_2 = \frac{\beta_1^2 T_0}{\rho K w_1^*}, \quad \zeta_3 = \frac{g^*\beta_1 T_0 c_1^2}{\rho\zeta K w_1^{*3}}, \quad q_1^* = \frac{D w_1^*\beta_2^2}{\rho c_1^4},\\
q_2^* = \frac{D w_1^*\beta_2 a}{\beta_1 c_1^2}, \quad q_3^* = \frac{D w_1^* b}{c_1^2}, \quad q_4^* = \frac{D h^*\beta_2}{\rho\zeta w_1^* c_1^2}.
\end{gathered} \tag{2.2}$$

Here $w_1^* = \rho C_E c_1^2/K$ and $c_1 = \sqrt{(\lambda + 2\mu + K^*)/\rho}$ are the characteristic frequency and longitudinal wave velocity in the medium, respectively.

Upon introducing the quantities (2.2) in the basic equations (2.1), after suppressing the primes, we obtain

$$\begin{gathered}
\delta_1 \Delta \overline{\mathbf{u}} + \delta_2 \operatorname{grad} \operatorname{div} \overline{\mathbf{u}} + \delta_3 \operatorname{curl} \overline{\boldsymbol{\varphi}} + \delta_4 \operatorname{grad} \overline{\varphi^*} - \operatorname{grad} \overline{T} - \operatorname{grad} \overline{C} = \ddot{\overline{\mathbf{u}}}, \\
(\delta_5 \Delta - 2\delta_3)\overline{\boldsymbol{\varphi}} + \delta_6 \operatorname{grad} \operatorname{div} \overline{\boldsymbol{\varphi}} + \delta_3 \operatorname{curl} \overline{\mathbf{u}} = \delta_7 \ddot{\overline{\boldsymbol{\varphi}}}, \\
(\delta_8 \Delta - \delta_9)\overline{\varphi^*} - \delta_{10} \operatorname{div} \overline{\mathbf{u}} - \delta_{11} \overline{T} - \delta_{12} \overline{C} = \ddot{\overline{\varphi^*}}, \\
\tau_t^0 \left(\zeta_2 \operatorname{div} \dot{\overline{\mathbf{u}}} - \zeta_3 \dot{\overline{\varphi^*}} + \dot{\overline{T}} + \zeta_1 \dot{\overline{C}} \right) = \Delta \overline{T}, \\
q_1^* \Delta \operatorname{div} \overline{\mathbf{u}} + q_4^* \Delta \overline{\varphi^*} + q_2^* \Delta \overline{T} - q_3^* \Delta \overline{C} + \tau_c^0 \dot{\overline{C}} = 0,
\end{gathered} \tag{2.3}$$

where

$$\tau_t^0 = 1 + \tau_0 \frac{\partial}{\partial t}, \qquad \tau_c^0 = 1 + \tau^0 \frac{\partial}{\partial t}. \tag{2.4}$$

We assume the displacement vector, microrotation, microstretch, temperature change, and concentration functions as

$$\left(\overline{\mathbf{u}}(\mathbf{x}, t), \overline{\boldsymbol{\varphi}}(\mathbf{x}, t), \overline{\varphi^*}(\mathbf{x}, t), \overline{T}(\mathbf{x}, t), \overline{C}(\mathbf{x}, t) \right) = \operatorname{Re}\left[(\mathbf{u}, \boldsymbol{\varphi}, \varphi^*, T, C) e^{-\iota \omega t} \right], \tag{2.5}$$

where ω is oscillation frequency and $\omega > 0$.

Using (2.5) into (2.3), we obtain the system of equations of steady oscillations as

$$\begin{gathered}
\left(\delta_1 \Delta + \omega^2 \right) \mathbf{u} + \delta_2 \operatorname{grad} \operatorname{div} \mathbf{u} + \delta_3 \operatorname{curl} \boldsymbol{\varphi} + \delta_4 \operatorname{grad} \varphi^* - \operatorname{grad} T - \operatorname{grad} C = \mathbf{0}, \\
(\delta_5 \Delta + \mu^*) \boldsymbol{\varphi} + \delta_6 \operatorname{grad} \operatorname{div} \boldsymbol{\varphi} + \delta_3 \operatorname{curl} \mathbf{u} = \mathbf{0}, \\
-\delta_{10} \operatorname{div} \mathbf{u} + (\delta_8 \Delta + \zeta^*) \varphi^* - \delta_{11} T - \delta_{12} C = 0, \\
-\tau_t^{10} \left[\zeta_2 \operatorname{div} \mathbf{u} - \zeta_3 \varphi^* + \zeta_1 C \right] + \left(\Delta - \tau_t^{10} \right) T = 0, \\
q_1^* \Delta \operatorname{div} \mathbf{u} + q_4^* \Delta \varphi^* + q_2^* \Delta T - q_3^* \Delta C + \tau_c^{10} C = 0,
\end{gathered} \tag{2.6}$$

where

$$\tau_t^{10} = -\iota \omega (1 - \iota \omega \tau_0), \qquad \tau_c^{10} = -\iota \omega \left(1 - \iota \omega \tau^0 \right), \qquad \mu^* = \delta_7 \omega^2 - 2\delta_3, \qquad \zeta^* = \omega^2 - \delta_9. \tag{2.7}$$

We introduce the matrix differential operator

$$\mathbf{F}(\mathbf{D}_\mathbf{x}) = \left\| F_{gh}(\mathbf{D}_\mathbf{x}) \right\|_{9 \times 9}, \tag{2.8}$$

where

$$F_{mn}(\mathbf{D_x}) = \left[\delta_1\Delta + \omega^2\right]\delta_{mn} + \delta_2\frac{\partial^2}{\partial x_m \partial x_n}, \quad F_{m,n+3}(\mathbf{D_x}) = F_{m+3,n}(\mathbf{D_x}) = \delta_3\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial x_r},$$

$$F_{m7}(\mathbf{D_x}) = \delta_4\frac{\partial}{\partial x_m}, \quad F_{m8}(\mathbf{D_x}) = F_{m9}(\mathbf{D_x}) = -\frac{\partial}{\partial x_m}, \quad F_{m+3,n+3}(\mathbf{D_x}) = \left(\delta_5\Delta + \mu^*\right)\delta_{mn} + \delta_6\frac{\partial^2}{\partial x_m \partial x_n},$$

$$F_{m+3,7}(\mathbf{D_x}) = F_{7,n+3}(\mathbf{D_x}) = F_{m+3,8}(\mathbf{D_x}) = F_{8,n+3}(\mathbf{D_x}) = F_{m+3,9}(\mathbf{D_x}) = F_{9,n+3}(\mathbf{D_x}) = 0,$$

$$F_{7n}(\mathbf{D_x}) = -\delta_{10}\frac{\partial}{\partial x_n}, \quad F_{77}(\mathbf{D_x}) = \delta_8\Delta + \zeta^*, F_{78}(\mathbf{D_x}) = -\delta_{11}, \quad F_{79}(\mathbf{D_x}) = -\delta_{12},$$

$$F_{8n}(\mathbf{D_x}) = -\zeta_2\tau_t^{10}\frac{\partial}{\partial x_n}, \quad F_{87}(\mathbf{D_x}) = \zeta_3\tau_t^{10}, \quad F_{88}(\mathbf{D_x}) = \Delta - \tau_t^{10}, \quad F_{89} = -\zeta_1\tau_t^{10},$$

$$F_{9n}(\mathbf{D_x}) = q_1^*\Delta\frac{\partial}{\partial x_n}, \quad F_{97}(\mathbf{D_x}) = q_4^*\Delta, \quad F_{98}(\mathbf{D_x}) = q_2^*\Delta,$$

$$F_{99}(\mathbf{D_x}) = -q_3^*\Delta + \tau_c^{10}, \quad m,n = 1,2,3. \tag{2.9}$$

Here ε_{mrn} is alternating tensor and δ_{mn} is the Kronecker delta function.

The system of equations (2.6) can be written as

$$\mathbf{F}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}, \tag{2.10}$$

where $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \psi^*, T, C)$ is a nine-component vector function on E^3.

Definition 2.1. The fundamental solution of the system of equations (2.6) (the fundamental matrix of operator $\mathbf{F}$) is the matrix $\mathbf{G}(\mathbf{x}) = \|G_{gh}(\mathbf{x})\|_{9\times 9}$ satisfying condition [25]

$$\mathbf{F}(\mathbf{D_x})\mathbf{G}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x}), \tag{2.11}$$

where δ is the Dirac delta, $\mathbf{I} = \|\delta_{gh}\|_{9\times 9}$ is the unit matrix, and $\mathbf{x} \ \epsilon \ \mathrm{E}^3$.

Now we construct $\mathbf{G}(\mathbf{x})$ in terms of elementary functions.

3. Fundamental Solution of System of Equations of Steady Oscillations

We consider the system of equations

$$\delta_1\Delta\mathbf{u} + \delta_2 \operatorname{grad} \operatorname{div} \mathbf{u} + \delta_3 \operatorname{curl} \boldsymbol{\varphi} - \delta_{10} \operatorname{grad} \psi^* - \zeta_2\tau_t^{10} \operatorname{grad} T + q_1^* \operatorname{grad} C + \omega^2\mathbf{u} = \mathbf{H}', \tag{3.1}$$

$$\left(\delta_5\Delta + \mu^*\right)\boldsymbol{\varphi} + \delta_6 \operatorname{grad} \operatorname{div} \boldsymbol{\varphi} + \delta_3 \operatorname{curl} \mathbf{u} = \mathbf{H}'', \tag{3.2}$$

$$\delta_4 \operatorname{div} \mathbf{u} + (\delta_8\Delta + \zeta^*)\psi^* + \zeta_3\tau_t^{10}T + q_4^*C = Z, \tag{3.3}$$

$$-\operatorname{div} \mathbf{u} - \delta_{11}\psi^* + \left(\Delta - \tau_t^{10}\right)T + q_2^*C = L, \tag{3.4}$$

$$-\Delta \operatorname{div} \mathbf{u} - \delta_{12}\Delta\psi^* - \zeta_1\tau_t^{10}\Delta T - q_3^*\Delta C + \tau_c^{10}C = M, \tag{3.5}$$

where $\mathbf{H}'$ and $\mathbf{H}''$ are three-component vector functions on $E^3$and Z, L, and M are scalar functions on E^3.

The system of equations (3.1)–(3.5) may be written in the form

$$\mathbf{F}^{\mathrm{tr}}(\mathbf{D}_{\mathbf{x}})\mathbf{U}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}), \tag{3.6}$$

where $\mathbf{F}^{\mathrm{tr}}$ is the transpose of matrix $\mathbf{F}$, $\mathbf{Q} = (\mathbf{H}', \mathbf{H}'', Z, L, M)$, and $\mathbf{x} \ \epsilon \ E^3$.

Applying the operator div to (3.1) and (3.2), we obtain

$$\begin{gathered}
\left(\Delta + \omega^2\right) \operatorname{div} \mathbf{u} - \delta_{10}\Delta\psi^* - \zeta_2\tau_t^{10}\Delta T + q_1^*\Delta C = \operatorname{div} \mathbf{H}', \\
\left(\upsilon^*\Delta + \mu^*\right) \operatorname{div} \boldsymbol{\varphi} = \operatorname{div} \mathbf{H}'', \\
\delta_4 \operatorname{div} \mathbf{u} + (\delta_8\Delta + \zeta^*)\psi^* + \zeta_3\tau_t^{10}T + q_4^*C = Z, \\
-\operatorname{div} \mathbf{u} - \delta_{11}\psi^* + \left(\Delta - \tau_t^{10}\right)T + q_2^*C = L, \\
-\Delta \operatorname{div} \mathbf{u} - \delta_{12}\Delta\psi^* - \zeta_1\tau_t^{10}\Delta T - q_3^*\Delta C + \tau_c^{10}C = M,
\end{gathered} \tag{3.7}$$

where $\upsilon^* = \delta_5 + \delta_6$.

Equations $(3.7)_1$, $(3.7)_3$, $(3.7)_4$, and $(3.7)_5$ may be written in the form

$$\mathbf{N}(\Delta)\mathbf{S} = \overline{\mathbf{Q}}, \tag{3.8}$$

where $\mathbf{S} = (\operatorname{div} \mathbf{u}, \psi^*, T, C)$, $\overline{\mathbf{Q}} = (d_1, d_2, d_3, d_4) = (\operatorname{div} \mathbf{H}', Z, L, M)$, and

$$\mathbf{N}(\Delta) = \|N_{mn}(\Delta)\|_{4\times 4} = \begin{Vmatrix} \Delta + \omega^2 & -\delta_{10}\Delta & -\zeta_2\tau_t^{10}\Delta & q_1^*\Delta \\ \delta_4 & \delta_8\Delta + \zeta^* & \zeta_3\tau_t^{10} & q_4^* \\ -1 & -\delta_{11} & \Delta - \tau_t^{10} & q_2^* \\ -\Delta & -\delta_{12}\Delta & -\zeta_1\tau_t^{10}\Delta & -q_3^*\Delta + \tau_c^{10} \end{Vmatrix}_{4\times 4}. \tag{3.9}$$

Equations $(3.7)_1$, $(3.7)_3$, $(3.7)_4$, and $(3.7)_5$ may be also written as

$$\Gamma_1(\Delta)\mathbf{S} = \boldsymbol{\Psi}, \tag{3.10}$$

where

$$\begin{gathered}
\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3, \Psi_4), \quad \Psi_n = e^* \sum_{m=1}^{4} N_{mn}^* d_m, \\
\Gamma_1(\Delta) = e^* \det \mathbf{N}(\Delta), \quad e^* = -\frac{1}{q_3^*\delta_9}, \quad n = 1, 2, 3, 4,
\end{gathered} \tag{3.11}$$

and N_{mn}^* is the cofactor of the elements N_{mn} of the matrix $\mathbf{N}$.

From (3.9) and (3.11), we see that

$$\Gamma_1(\Delta) = \prod_{m=1}^{4}\left(\Delta + \lambda_m^2\right), \tag{3.12}$$

where λ_m^2, $m = 1, 2, 3, 4$ are the roots of the equation $\Gamma_1(-\kappa) = 0$ (with respect to κ).

From $(3.7)_2$, it follows that

$$\left(\Delta + \lambda_7^2\right) \operatorname{div} \boldsymbol{\varphi} = \frac{1}{\delta^*} \operatorname{div} \mathbf{H}'', \tag{3.13}$$

where $\lambda_7^2 = \mu^*/\upsilon^*$.

Applying the operators $\delta_5\Delta + \mu^*$ and δ_3 curl to (3.1) and (3.2), respectively, we obtain

$$\begin{aligned}(\delta_5\Delta + \mu^*)\left[\delta_1\Delta\mathbf{u} + \delta_2 \operatorname{grad} \operatorname{div} \mathbf{u} + \omega^2\mathbf{u}\right] + \delta_3(\delta_5\Delta + \mu^*) \operatorname{curl} \boldsymbol{\varphi} \\ = (\delta_5\Delta + \mu^*)\left[\mathbf{H}' + \delta_{10} \operatorname{grad} \psi^* + \zeta_2\tau_t^{10} \operatorname{grad} T - q_1^* \operatorname{grad} C\right],\end{aligned} \tag{3.14}$$

$$\delta_3(\delta_5\Delta + \mu^*) \operatorname{curl} \boldsymbol{\varphi} = -\delta_3^2 \operatorname{curl} \operatorname{curl} \mathbf{u} + \delta_3 \operatorname{curl} \mathbf{H}''. \tag{3.15}$$

Now

$$\operatorname{curl} \operatorname{curl} \mathbf{u} = \operatorname{grad} \operatorname{div} \mathbf{u} - \Delta\mathbf{u}. \tag{3.16}$$

Using (3.15) and (3.16) in (3.14), we obtain

$$\begin{aligned}(\delta_5\Delta + \mu^*)\left[\delta_1\Delta\mathbf{u} + \delta_2 \operatorname{grad} \operatorname{div} \mathbf{u} + \omega^2\mathbf{u}\right] + \delta_3^2\Delta\mathbf{u} - \delta_3^2 \operatorname{grad} \operatorname{div} \mathbf{u} \\ = (\delta_5\Delta + \mu^*)\left[\mathbf{H}' + \delta_{10} \operatorname{grad} \psi^* + \zeta_2\tau_t^{10} \operatorname{grad} T - q_1^* \operatorname{grad} C\right] - \delta_3 \operatorname{curl} \mathbf{H}''.\end{aligned} \tag{3.17}$$

The above equation can also be written as

$$\begin{aligned}&\left\{\left[(\delta_5\Delta + \mu^*)\delta_1 + \delta_3^2\right]\Delta + (\delta_5\Delta + \mu^*)\omega^2\right\}\mathbf{u} \\ &\quad = -\left[\delta_2(\delta_5\Delta + \mu^*) - \delta_3^2\right] \operatorname{grad} \operatorname{div} \mathbf{u} \\ &\qquad + (\delta_5\Delta + \mu^*)\left[\mathbf{H}' + \delta_{10} \operatorname{grad} \psi^* + \zeta_2\tau_t^{10} \operatorname{grad} T - q_1^* \operatorname{grad} C\right] - \delta_3 \operatorname{curl} \mathbf{H}''.\end{aligned} \tag{3.18}$$

Applying the operator $\Gamma_1(\Delta)$ to the (3.18) and using (3.10), we get

$$\begin{aligned}
&\Gamma_1(\Delta)\left[\delta_5\delta_1\Delta^2+\left(\mu^*\delta_1+\delta_5\omega^2+\delta_3^2\right)\Delta+\mu^*\omega^2\right]\mathbf{u}\\
&\quad=-\left[\delta_2(\delta_5\Delta+\mu^*)-\delta_3^2\right]\operatorname{grad}\Psi_1\\
&\qquad+(\delta_5\Delta+\mu^*)\left[\Gamma_1(\Delta)\mathbf{H}'+\delta_{10}\operatorname{grad}\Psi_2+\zeta_2\tau_t^{10}\operatorname{grad}\Psi_3-q_1^*\operatorname{grad}\Psi_4\right]-\delta_3\Gamma_1(\Delta)\operatorname{curl}\mathbf{H}''.
\end{aligned}\tag{3.19}$$

The above equation may be written in the form

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathbf{u}=\mathbf{\Psi}',\tag{3.20}$$

where

$$\Gamma_2(\Delta)=f^*\det\begin{Vmatrix}\delta_1\Delta+\omega^2 & \delta_3\Delta\\ -\delta_3 & \delta_5\Delta+\mu^*\end{Vmatrix}_{2\times2},\quad f^*=\frac{1}{\delta_1\delta_5},\tag{3.21}$$

$$\begin{aligned}
\mathbf{\Psi}'=f^*\Big\{&-\left[\delta_2(\delta_5\Delta+\mu^*)-\delta_3^2\right]\operatorname{grad}\Psi_1\\
&+(\delta_5\Delta+\mu^*)\left[\Gamma_1(\Delta)\mathbf{H}'+\delta_{10}\operatorname{grad}\ \Psi_2+\zeta_2\tau_t^{10}\operatorname{grad}\Psi_3-q_1^*\operatorname{grad}\Psi_4\right]\\
&-\delta_3\Gamma_1(\Delta)\operatorname{curl}\mathbf{H}''\Big\}.
\end{aligned}\tag{3.22}$$

It can be seen that

$$\Gamma_2(\Delta)=\left(\Delta+\lambda_5^2\right)\left(\Delta+\lambda_6^2\right),\tag{3.23}$$

where λ_5^2, λ_6^2 are the roots of the equation $\Gamma_2(-\kappa)=0$ (with respect to κ).

Applying the operators δ_3 curl and $\delta_1\Delta+\omega^2$ to (3.1) and (3.2), respectively, we obtain

$$\delta_3\left(\delta_1\Delta+\omega^2\right)\operatorname{curl}\mathbf{u}=\delta_3\operatorname{curl}\mathbf{H}'-\delta_3^2\ \operatorname{curl}\ \operatorname{curl}\boldsymbol{\varphi},\tag{3.24}$$

$$\left(\delta_1\Delta+\omega^2\right)(\delta_5\Delta+\mu^*)\boldsymbol{\varphi}+\delta_6\left(\delta_1\Delta+\omega^2\right)\operatorname{grad}\ \operatorname{div}\ \boldsymbol{\varphi}+\delta_3\left(\delta_1\Delta+\omega^2\right)\ \operatorname{curl}\ \mathbf{u}=\left(\delta_1\Delta+\omega^2\right)\mathbf{H}''.\tag{3.25}$$

Now

$$\operatorname{curl}\ \operatorname{curl}\ \boldsymbol{\varphi}=\operatorname{grad}\ \operatorname{div}\ \boldsymbol{\varphi}-\Delta\boldsymbol{\varphi}.\tag{3.26}$$

Using (3.24) and (3.26) in (3.25), we obtain

$$\begin{aligned}&\left(\delta_1\Delta+\omega^2\right)(\delta_5\Delta+\mu^*)\boldsymbol{\varphi}+\delta_6\left(\delta_1\Delta+\omega^2\right)\text{grad div}\,\boldsymbol{\varphi}+\delta_3^2\Delta\boldsymbol{\varphi}-\delta_3^2\,\text{grad div}\,\boldsymbol{\varphi}\\&\quad=(\delta_1\Delta+\omega^2)\mathbf{H}''-\delta_3\,\text{curl}\,\mathbf{H}'\,.\end{aligned}\tag{3.27}$$

The above equation may also be written as

$$\begin{aligned}&\left\{\left[(\delta_5\Delta+\mu^*)\delta_1+\delta_3^2\right]\Delta+(\delta_5\Delta+\mu^*)\omega^2\right\}\boldsymbol{\varphi}\\&\quad=-\left[\delta_6\left(\delta_1\Delta+\omega^2\right)-\delta_3^2\right]\text{grad div}\,\boldsymbol{\varphi}+\left(\delta_1\Delta+\omega^2\right)\mathbf{H}''-\delta_3\,\text{curl}\,\mathbf{H}'.\end{aligned}\tag{3.28}$$

Applying the operator $\Delta+\lambda_7^2$ to the (3.28) and using (3.13), we get

$$\begin{aligned}&\left(\Delta+\lambda_7^2\right)\left[\delta_5\delta_1\Delta^2+\left(\mu^*\delta_1+\delta_5\omega^2+\delta_3^2\right)\Delta+\mu^*\omega^2\right]\boldsymbol{\varphi}\\&\quad=-\delta_3\left(\Delta+\lambda_7^2\right)\text{curl}\,\mathbf{H}'+\left(\Delta+\lambda_7^2\right)\left(\delta_1\Delta+\omega^2\right)\mathbf{H}''-\frac{1}{\upsilon^*}\left[\delta_6\left(\delta_1\Delta+\omega^2\right)-\delta_3^2\right]\text{grad div}\,\mathbf{H}''.\end{aligned}\tag{3.29}$$

The above equation may also be rewritten in the form

$$\Gamma_2(\Delta)\left(\Delta+\lambda_7^2\right)\boldsymbol{\varphi}=\boldsymbol{\Psi}'',\tag{3.30}$$

where

$$\begin{aligned}\boldsymbol{\Psi}''=f^*\Big\{&-\delta_3\left(\Delta+\lambda_7^2\right)\text{curl}\,\mathbf{H}'+\left(\Delta+\lambda_7^2\right)\left(\delta_1\Delta+\omega^2\right)\mathbf{H}''\\&-\frac{1}{\upsilon^*}\left[\delta_6\left(\delta_1\Delta+\omega^2\right)-\delta_3^2\right]\text{grad div}\,\mathbf{H}''\Big\}.\end{aligned}\tag{3.31}$$

From (3.10), (3.20), and (3.30), we obtain

$$\boldsymbol{\Theta}(\Delta)\mathbf{U}(\mathbf{x})=\widehat{\boldsymbol{\Psi}}(\mathbf{x}),\tag{3.32}$$

where $\widehat{\mathbf{\Psi}} = (\mathbf{\Psi}', \mathbf{\Psi}'', \Psi_2, \Psi_3, \Psi_4)$

$$\begin{gathered}
\mathbf{\Theta}(\Delta) = \left\|\Theta_{gh}(\Delta)\right\|_{9\times 9}, \\
\Theta_{mm}(\Delta) = \Gamma_1(\Delta)\Gamma_2(\Delta) = \prod_{q=1}^{6}\left(\Delta + \lambda_q^2\right), \\
\Theta_{m+3,n+3}(\Delta) = \Gamma_2(\Delta)\left(\Delta + \lambda_7^2\right) = \prod_{q=5}^{7}\left(\Delta + \lambda_q^2\right), \\
\Theta_{gh}(\Delta) = 0, \qquad \Theta_{77}(\Delta) = \Theta_{88}(\Delta) = \Theta_{99}(\Delta) = \Gamma_1(\Delta), \\
m = 1,2,3, \quad g,h = 1,2,\ldots,9, \quad g \neq h.
\end{gathered} \tag{3.33}$$

Equations (3.11), (3.22), and (3.31) can be rewritten in the form

$$\begin{aligned}
\mathbf{\Psi}' &= \left[f^*(\delta_5\Delta + \mu^*)\Gamma_1(\Delta)\mathbf{J} + q_{11}(\Delta)\,\mathrm{grad}\ \mathrm{div}\right]\mathbf{H}' + q_{21}(\Delta)\ \mathrm{curl}\ \mathbf{H}'' \\
&\quad + q_{31}(\Delta)\,\mathrm{grad}\,Z + q_{41}(\Delta)\,\mathrm{grad}\,L + q_{51}(\Delta)\,\mathrm{grad}\,M, \\
\mathbf{\Psi}'' &= q_{12}(\Delta)\ \mathrm{curl}\ \mathbf{H}' + \left\{f^*\left(\Delta + \lambda_7^2\right)\left(\delta_1\Delta + \omega^2\right)\mathbf{J} + q_{22}(\Delta)\,\mathrm{grad}\ \mathrm{div}\right\}\mathbf{H}'', \\
\Psi_2 &= q_{13}(\Delta)\ \mathrm{div}\ \mathbf{H}' + q_{33}(\Delta)Z + q_{43}(\Delta)L + q_{53}(\Delta)M, \\
\Psi_3 &= q_{14}(\Delta)\ \mathrm{div}\ \mathbf{H}' + q_{34}(\Delta)Z + q_{44}(\Delta)L + q_{54}(\Delta)M, \\
\Psi_4 &= q_{15}(\Delta)\ \mathrm{div}\ \mathbf{H}' + q_{35}(\Delta)Z + q_{45}(\Delta)L + q_{55}(\Delta)M,
\end{aligned} \tag{3.34}$$

where $\mathbf{J} = \|\delta_{gh}\|_{3\times 3}$ is the unit matrix.

In (3.34), we have used the following notations:

$$\begin{gathered}
q_{m1}(\Delta) = f^*e^*\left\{(\delta_5\Delta + \mu^*)\left[\delta_{10}N_{m2}^* + \zeta_2\tau_t^{10}N_{m3}^* - q_1^*N_{m4}^*\right] - \left(\delta_2(\delta_5\Delta + \mu^*) - \delta_3^2\right)N_{m1}^*\right\}, \\
q_{21}(\Delta) = -f^*\delta_3\Gamma_1(\Delta), \qquad q_{12}(\Delta) = -f^*\delta_3\left(\Delta + \lambda_7^2\right), \qquad q_{22}(\Delta) = -\frac{f^*}{\upsilon^*}\left[\delta_6\left(\delta_1\Delta + \omega^2\right) - \delta_3^2\right], \\
q_{mn}(\Delta) = e^*N_{mn}^*, \quad m = 1,3,4,5, \quad n = 3,4,5.
\end{gathered} \tag{3.35}$$

Now from (3.34), we have that

$$\widehat{\mathbf{\Psi}}(\mathbf{x}) = \mathbf{R}^{\mathrm{tr}}(\mathbf{D}_{\mathbf{x}})\mathbf{Q}(\mathbf{x}), \tag{3.36}$$

where

$$\begin{gathered}
\mathbf{R} = \left\| R_{gh} \right\|_{9\times 9}, \\
R_{mn}(\mathbf{D_x}) = f^*\left(\delta_5\Delta + \mu^*\right)\Gamma_1(\Delta)\delta_{mn} + q_{11}(\Delta)\frac{\partial^2}{\partial x_m \partial x_n}, \\
R_{m,n+3}(\mathbf{D_x}) = q_{12}(\Delta)\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial x_r}, \qquad R_{mp}(\mathbf{D_x}) = q_{1,p-4}(\Delta)\frac{\partial}{\partial x_m}, \\
R_{m+3,n}(\mathbf{D_x}) = q_{21}(\Delta)\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial x_r}, \\
R_{m+3,n+3}(\mathbf{D_x}) = f^*\left(\Delta + \lambda_7^2\right)\left(\delta_1\Delta + \omega^2\right)\delta_{mn} + q_{22}(\Delta)\frac{\partial^2}{\partial x_m \partial x_n}, \\
R_{m+3,p}(\mathbf{D_x}) = R_{p,m+3}(\mathbf{D_x}) = 0, \\
R_{pm}(\mathbf{D_x}) = q_{p-4,1}(\Delta)\frac{\partial}{\partial x_n}, \qquad R_{ps}(\mathbf{D_x}) = q_{p-4,s-4}(\Delta), \quad m = 1,2,3, \quad p,s = 7,8,9.
\end{gathered} \tag{3.37}$$

From (3.6), (3.32), and (3.36), we obtain

$$\mathbf{\Theta U} = \mathbf{R}^{\mathrm{tr}}\mathbf{F}^{\mathrm{tr}}\mathbf{U}. \tag{3.38}$$

It implies that

$$\begin{gathered}
\mathbf{R}^{\mathrm{tr}}\mathbf{F}^{\mathrm{tr}} = \mathbf{\Theta}, \\
\mathbf{F}(\mathbf{D_x})\mathbf{R}(\mathbf{D_x}) = \mathbf{\Theta}(\Delta).
\end{gathered} \tag{3.39}$$

We assume that

$$\lambda_m^2 \neq \lambda_n^2 \neq 0, \quad m,n = 1,2,3,4,5,6,7 \quad m \neq n. \tag{3.40}$$

Let

$$\begin{gathered}
\mathbf{Y}(\mathbf{x}) = \left\| Y_{rs}(\mathbf{x}) \right\|_{9\times 9}, \qquad Y_{mm}(\mathbf{x}) = \sum_{n=1}^{6} r_{1n}\varsigma_n(\mathbf{x}), \qquad Y_{m+3,m+3}(\mathbf{x}) = \sum_{n=5}^{7} r_{2n}\varsigma_n(\mathbf{x}), \\
Y_{77}(\mathbf{x}) = Y_{88}(\mathbf{x}) = Y_{99}(\mathbf{x}) = \sum_{n=1}^{4} r_{3n}\varsigma_n(\mathbf{x}), \\
Y_{vw}(\mathbf{x}) = 0, \quad m = 1,2,3, \quad v,w = 1,2,\ldots,9, \quad v \neq w,
\end{gathered} \tag{3.41}$$

where

$$\begin{aligned}
\varsigma_n(\mathbf{x}) &= -\frac{1}{4\pi|\mathbf{x}|}\exp(\iota\lambda_n|\mathbf{x}|), \quad n = 1,2,\ldots,7,\\
r_{1l} &= \prod_{m=1,m\neq l}^{6}\left(\lambda_m^2-\lambda_l^2\right)^{-1}, \quad l = 1,2,3,4,5,6,\\
r_{2v} &= \prod_{m=5,m\neq v}^{7}\left(\lambda_m^2-\lambda_v^2\right)^{-1}, \quad v = 5,6,7,\\
r_{3w} &= \prod_{m=1,m\neq w}^{4}\left(\lambda_m^2-\lambda_w^2\right)^{-1}, \quad w = 1,2,3,4.
\end{aligned} \tag{3.42}$$

We will prove the following lemma.

Lemma 3.1. *The matrix* $\mathbf{Y}$ *defined above is the fundamental matrix of operator* $\mathbf{\Theta}(\Delta)$*, that is*

$$\mathbf{\Theta}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x}). \tag{3.43}$$

Proof. To prove the lemma, it is sufficient to prove that

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\Upsilon_{11}(\mathbf{x}) = \delta(\mathbf{x}), \qquad \Gamma_2(\Delta)\left(\Delta+\lambda_7^2\right)\Upsilon_{44}(\mathbf{x}) = \delta(\mathbf{x}), \qquad \Gamma_1(\Delta)\Upsilon_{77}(\mathbf{x}) = \delta(\mathbf{x}). \tag{3.44}$$

We find that

$$\begin{aligned}
&r_{11}+r_{12}+r_{13}+r_{14}+r_{15}+r_{16} = 0,\\
&r_{12}\left(\lambda_1^2-\lambda_2^2\right)+r_{13}\left(\lambda_1^2-\lambda_3^2\right)+r_{14}\left(\lambda_1^2-\lambda_4^2\right)+r_{15}\left(\lambda_1^2-\lambda_5^2\right)+r_{16}\left(\lambda_1^2-\lambda_6^2\right) = 0,\\
&r_{13}\left(\lambda_1^2-\lambda_3^2\right)\left(\lambda_2^2-\lambda_3^2\right)+r_{14}\left(\lambda_1^2-\lambda_4^2\right)\left(\lambda_2^2-\lambda_4^2\right)+r_{15}\left(\lambda_1^2-\lambda_5^2\right)\left(\lambda_2^2-\lambda_5^2\right)\\
&\qquad +r_{16}\left(\lambda_1^2-\lambda_6^2\right)\left(\lambda_2^2-\lambda_6^2\right) = 0,\\
&r_{14}\left(\lambda_1^2-\lambda_4^2\right)\left(\lambda_2^2-\lambda_4^2\right)\left(\lambda_3^2-\lambda_4^2\right)+r_{15}\left(\lambda_1^2-\lambda_5^2\right)\left(\lambda_2^2-\lambda_5^2\right)\left(\lambda_3^2-\lambda_5^2\right)\\
&\qquad +r_{16}\left(\lambda_1^2-\lambda_6^2\right)\left(\lambda_2^2-\lambda_6^2\right)\left(\lambda_3^2-\lambda_6^2\right) = 0,\\
&r_{15}\left(\lambda_1^2-\lambda_5^2\right)\left(\lambda_2^2-\lambda_5^2\right)\left(\lambda_3^2-\lambda_5^2\right)\left(\lambda_4^2-\lambda_5^2\right)+r_{16}\left(\lambda_1^2-\lambda_6^2\right)\left(\lambda_2^2-\lambda_6^2\right)\left(\lambda_3^2-\lambda_6^2\right)\left(\lambda_4^2-\lambda_6^2\right) = 0,\\
&r_{16}\left(\lambda_1^2-\lambda_6^2\right)\left(\lambda_2^2-\lambda_6^2\right)\left(\lambda_3^2-\lambda_6^2\right)\left(\lambda_4^2-\lambda_5^2\right)\left(\lambda_5^2-\lambda_6^2\right) = 1,\\
&\left(\Delta+\lambda_m^2\right)\varsigma_n(\mathbf{x}) = \delta(\mathbf{x})+\left(\lambda_m^2-\lambda_n^2\right)\varsigma_n(\mathbf{x}), \quad m,n = 1,2,3,4,5,6.
\end{aligned} \tag{3.45}$$

Now consider

$$\begin{aligned}
\Gamma_1(\Delta)\Gamma_2(\Delta)Y_{11}(\mathbf{x}) &= \left(\Delta+\lambda_2^2\right)\left(\Delta+\lambda_3^2\right)\left(\Delta+\lambda_4^2\right)\left(\Delta+\lambda_5^2\right)\left(\Delta+\lambda_6^2\right)\sum_{n=1}^{6} r_{1n}\left[\delta+\left(\lambda_1^2-\lambda_n^2\right)\varsigma_n\right] \\
&= \left(\Delta+\lambda_2^2\right)\left(\Delta+\lambda_3^2\right)\left(\Delta+\lambda_4^2\right)\left(\Delta+\lambda_5^2\right)\left(\Delta+\lambda_6^2\right)\sum_{n=2}^{6} r_{1n}\left(\lambda_1^2-\lambda_n^2\right)\varsigma_n \\
&= \left(\Delta+\lambda_3^2\right)\left(\Delta+\lambda_4^2\right)\left(\Delta+\lambda_5^2\right)\left(\Delta+\lambda_6^2\right)\sum_{n=2}^{6} r_{1n}\left(\lambda_1^2-\lambda_n^2\right)\left[\delta+\left(\lambda_2^2-\lambda_n^2\right)\varsigma_n\right] \\
&= \left(\Delta+\lambda_3^2\right)\left(\Delta+\lambda_4^2\right)\left(\Delta+\lambda_5^2\right)\left(\Delta+\lambda_6^2\right)\sum_{n=3}^{6} r_{1n}\left(\lambda_1^2-\lambda_n^2\right)\left(\lambda_2^2-\lambda_n^2\right)\varsigma_n \\
&= \left(\Delta+\lambda_4^2\right)\left(\Delta+\lambda_5^2\right)\left(\Delta+\lambda_6^2\right)\sum_{n=3}^{6} r_{1n}\left(\lambda_1^2-\lambda_n^2\right)\left(\lambda_2^2-\lambda_n^2\right)\left[\delta+\left(\lambda_3^2-\lambda_n^2\right)\varsigma_n\right] \\
&= \left(\Delta+\lambda_4^2\right)\left(\Delta+\lambda_5^2\right)\left(\Delta+\lambda_6^2\right)\sum_{n=4}^{6} r_{1n}\left(\lambda_1^2-\lambda_n^2\right)\left(\lambda_2^2-\lambda_n^2\right)\left(\lambda_3^2-\lambda_n^2\right)\varsigma_n \\
&= \left(\Delta+\lambda_5^2\right)\left(\Delta+\lambda_6^2\right)\sum_{n=4}^{6} r_{1n}\left(\lambda_1^2-\lambda_n^2\right)\left(\lambda_2^2-\lambda_n^2\right)\left(\lambda_3^2-\lambda_n^2\right)\left[\delta+\left(\lambda_4^2-\lambda_n^2\right)\varsigma_n\right] \\
&= \left(\Delta+\lambda_6^2\right)\sum_{n=5}^{6} r_{1n}\left(\lambda_1^2-\lambda_n^2\right)\left(\lambda_2^2-\lambda_n^2\right)\left(\lambda_3^2-\lambda_n^2\right)\left(\lambda_4^2-\lambda_n^2\right)\left[\delta+\left(\lambda_5^2-\lambda_n^2\right)\varsigma_n\right] \\
&= \left(\Delta+\lambda_6^2\right)\varsigma_6 = \delta.
\end{aligned} \tag{3.46}$$

Similarly, $(3.44)_2$ and $(3.44)_3$ can be proved.

We introduce the matrix

$$\mathbf{G}(\mathbf{x}) = \mathbf{R}(\mathbf{D}_\mathbf{x})\mathbf{Y}(\mathbf{x}). \tag{3.47}$$

From (3.39), (3.43), and (3.47), we obtain

$$\mathbf{F}(\mathbf{D}_\mathbf{x})\mathbf{G}(\mathbf{x}) = \mathbf{F}(\mathbf{D}_\mathbf{x})\mathbf{R}(\mathbf{D}_\mathbf{x})\mathbf{Y}(\mathbf{x}) = \mathbf{\Theta}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x}). \tag{3.48}$$

Hence, $\mathbf{G}(\mathbf{x})$ is a solution to (2.11). □

Therefore we have proved the following theorem.

Theorem 3.2. *The matrix* $\mathbf{G}(\mathbf{x})$ *defined by* (3.47) *is the fundamental solution of system of equations* (2.6).

4. Basic Properties of the Matrix G(x)

Property 1. Each column of the matrix $\mathbf{G}(\mathbf{x})$ is the solution of the system of equations (2.6) at every point $\mathbf{x} \ \epsilon \ \mathrm{E}^3$ except the origin.

Property 2. The matrix $\mathbf{G}(\mathbf{x})$ can be written in the form

$$\begin{gathered}
\mathbf{G} = \left\| G_{gh} \right\|_{9\times 9}, \\
\mathbf{G}_{mn}(\mathbf{x}) = \mathbf{R}_{mn}(\mathbf{D}_{\mathbf{x}})Y_{11}(\mathbf{x}), \\
\mathbf{G}_{m,n+3}(\mathbf{x}) = \mathbf{R}_{m,n+3}(\mathbf{D}_{\mathbf{x}})Y_{44}(\mathbf{x}), \\
\mathbf{G}_{mp}(\mathbf{x}) = \mathbf{R}_{mp}(\mathbf{D}_{\mathbf{x}})Y_{77}(\mathbf{x}), \quad m = 1,2,\ldots,9, \quad n = 1,2,3, \quad p = 7,8,9.
\end{gathered} \tag{4.1}$$

5. Special Cases

(i) If we neglect the diffusion effect, we obtain the same results for fundamental solution as discussed by Svanadze and De Cicco [23] by changing the dimensionless quantities into physical quantities in case of coupled theory of thermoelasticity.

(ii) If we neglect the thermal and diffusion effects, we obtain the same results for fundamental solution as discussed by Svanadze [22] by changing the dimensionless quantities into physical quantities.

(iii) If we neglect both micropolar and microstretch effects, the same results for fundamental solution can be obtained as discussed by Kumar and Kansal [27] in case of the Lord-Shulman theory of thermoelastic diffusion.

6. Conclusions

The fundamental solution $\mathbf{G}(\mathbf{x})$ of the system of equations (2.6) makes it possible to investigate three-dimensional boundary value problems of generalized theory of thermomicrostretch elastic diffusive solids by potential method [28].

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