Research Article

Sufficient Conditions and Duality Theorems for Nondifferentiable Minimax Fractional Programming

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Received 20 October 2010; Accepted 28 November 2010

Academic Editors: R. Barrio and S. Zhang

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We consider nondifferentiable minimax fractional programming problems involving B -(p, r)-invex functions with respect to η and b . Sufficient optimality conditions and duality results for a class of nondifferentiable minimax fractional programming problems are obtained under B -(p, r)-invexity assumption on objective and constraint functions. Parametric duality, Mond-Weir duality, and Wolfe duality problems may be formulated, and duality results are derived under B -(p, r)-invex functions.

1. Introduction

Convexity plays an important role in many aspects of mathematical programming including sufficient optimality conditions and duality theorems. In general, we use the invex function to replace convexity on sufficient optimality conditions and duality theorems (see, e.g., [1–6]).

Many authors investigated the optimality conditions and duality theorems for minimax (fractional) programming problems. For details, one can consult [1–14]. In particular, Lai et al. [10] have established the theorems of necessary and sufficient optimality conditions for nondifferentiable minimax fractional problem under the conditions of convexity. In [11], Lai and Lee employed the optimality conditions to construct two parameter-free dual models of nondifferentiable minimax fractional programming problem which involve pseudoconvex and quasiconvex functions, and derived weak and strong duality theorems. In the formulation of the dual models in [11] optimality conditions given in [10] are used. Mishra et al. [4] derived a Kuhn-Tucker-type sufficient optimality condition for an optimal solution to the nondifferentiable minimax fractional programming problem and established weak, strong, and converse duality theorems for the problem and its three

different forms of dual problems under generalized univexity. Mishra et al. [5, 13] considered the nondifferentiable minimax fractional programming problem and obtain optimality and duality results under generalized α -invexity [5] and generalized α -unvexity [13]. Recently, Antczak [15] defined a new class of functions, named $B-(p, r)$ -invex, which is an extension of invex function. In [1], parametric and nonparametric sufficient optimality conditions and several parametric and parameter-free duality models for the generalized fractional minimax programs are obtained under $B-(p, r)$ -invexity assumption on objective and constraint functions.

In this paper, we are inspired to extend the result of Lai et al. [10] to $B-(p, r)$ -invexity and organize this paper as follows. In Section 2 we introduce some basic results. We establish sufficient optimality conditions for nondifferentiable minimax fractional programming problem under $B-(p, r)$ -invex with respect to the same function η and with respect to, not necessarily, the same function b in Section 3. Employing these results, we construct three dual problems in Sections 4–6. Here we investigate weak, strong, and strict converse duality theorems under the framework of $B-(p, r)$ -invex with respect to the same function η and with respect to, not necessarily, the same function b .

2. Some Notations and Preliminary Results

Let $\mathbb{R}^n$ be the n -dimensional Euclidean space and $\mathbb{R}_+^n$ its nonnegative orthant. Throughout the paper, let X be a nonempty open set of $\mathbb{R}^n$.

The following definition can be found in [15].

Definition 2.1 (see [15]). Let p and r be any real numbers. The differentiable function $f : X \rightarrow \mathbb{R}$ is said to be (strictly) $B-(p, r)$ -invex with respect to η and b at $u \in X$ on a nonempty set $X \subset \mathbb{R}^n$ if, there exist a function $\eta : X \times X \rightarrow \mathbb{R}^n$ and a function $b : X \times X \rightarrow \mathbb{R}_+ \setminus \{0\}$ such that, for all $x \in X$, the inequalities

$$\begin{aligned} \frac{1}{r}b(x, u)\left(e^{r(f(x)-f(u))} - 1\right) &\geq \begin{cases} \frac{1}{p}\nabla f(u)(e^{p\eta(x, u)} - \mathbf{1}) & (> \text{ if } x \neq u) \text{ for } p \neq 0, r \neq 0, \\ \nabla f(u)\eta(x, u) & (> \text{ if } x \neq u) \text{ for } p = 0, r \neq 0, \end{cases} \\ b(x, u)(f(x) - f(u)) &\geq \begin{cases} \frac{1}{p}\nabla f(u)(e^{p\eta(x, u)} - \mathbf{1}) & (> \text{ if } x \neq u) \text{ for } p \neq 0, r = 0, \\ \nabla f(u)\eta(x, u) & (> \text{ if } x \neq u) \text{ for } p = 0, r = 0 \end{cases} \end{aligned} \quad (2.1)$$

hold.

f is said to be $B-(p, r)$ -invex (strictly $B-(p, r)$ -invex) with respect to η and b on X if it is $B-(p, r)$ -invex with respect to the same η and b at each $u \in X$.

It should be pointed out that exponentials appearing on the right-hand sides of the above inequalities are understood to be taken componentwise and $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^n$.

We consider the following nondifferentiable minimax fractional programming problem:

$$\begin{aligned} \text{minimize} \quad & F(x) = \sup_{y \in Y} \frac{f(x, y) + (x^\top Bx)^{1/2}}{h(x, y) - (x^\top Dx)^{1/2}} \\ \text{subject to} \quad & g(x) \leq 0, \end{aligned} \quad (P)$$

where Y is a compact subset of $\mathbb{R}^m$, $f(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$, $h(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$, and $g(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}$ are C^1 -functions, B and D are $n \times n$ positive semidefinite matrices, $f(x, y) + (x^\top Bx)^{1/2} \geq 0$, and $h(x, y) - (x^\top Dx)^{1/2} > 0$ for each (x, y) in $X^\circ \times Y$, where X° is the set of feasible solutions of problem (P); that is, $X^\circ = \{x \in X : g(x) \leq 0\}$. This is a nondifferentiable programming problem if either B or D is nonzero. If B and D are null matrices, then problem (P) is a minimax fractional programming problem.

For each $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ define

$$\phi(x, y) = \frac{f(x, y) + (x^\top Bx)^{1/2}}{h(x, y) - (x^\top Dx)^{1/2}}. \quad (2.2)$$

We let

$$\begin{aligned} J &= \{1, 2, \dots, p\}, \\ J(x) &= \{j \in J \mid g_j(x) = 0\}, \\ Y(x) &= \left\{ y \in Y \mid \phi(x, y) = \sup_{z \in Y} \phi(x, z) \right\}, \\ K(x) &= \left\{ (s, t, y) \in \mathbb{N} \times \mathbb{R}_+^s \times \mathbb{R}^{ms} \mid 1 \leq s \leq n+1, t = (t_1, t_2, \dots, t_s) \in \mathbb{R}_+^s \text{ with } \sum_{i=1}^s t_i = 1, \right. \\ &\quad \left. y = (y_1, y_2, \dots, y_s) \text{ with } y_i \in Y(x), i = 1, 2, \dots, s \right\}. \end{aligned} \quad (2.3)$$

Because f and h are continuous differentiable and Y is compact subset of $\mathbb{R}^m$, we see that for each $x_0 \in X^\circ$, $Y(x_0) \neq \emptyset$, and for any $\bar{y}_i \in Y(x_0)$, we have a positive constant

$$k_0 = \phi(x_0, \bar{y}_i) = \frac{f(x_0, \bar{y}_i) + (x_0^\top Bx_0)^{1/2}}{h(x_0, \bar{y}_i) - (x_0^\top Dx_0)^{1/2}}. \quad (2.4)$$

We will use the generalized Schwarz inequality

$$x^\top Bv \leq (x^\top Bx)^{1/2} (v^\top Bv)^{1/2}, \quad \text{for } x, v \in \mathbb{R}^n; \quad (2.5)$$

the equality holds when $Bx = \lambda Bv$, for some $\lambda \geq 0$.

Hence if $v^\top Bv \leq 1$, we have

$$x^\top Bv \leq \left(x^\top Bx\right)^{1/2}. \quad (2.6)$$

In [10] Lai et al. derived the following necessary conditions for optimality (P).

Theorem 2.2 ((necessary conditions) see, [10]). *Let x_0 be a (P)-optimal solution and satisfying $x_0^\top Bx_0 > 0$, $x_0^\top Dx_0 > 0$, and $\nabla g_j(x_0)$, $j \in J(x_0)$ is linearly independent. Then there exist $(s, t^*, \bar{y}) \in K(x_0)$, $k_0 \in \mathbb{R}_+$, $w, v \in \mathbb{R}^n$, and $\mu^* \in \mathbb{R}_+^p$ such that*

$$\sum_{i=1}^s t_i^* \{ \nabla f(x_0, \bar{y}_i) + Bw - k_0(\nabla h(x_0, \bar{y}_i) - Dv) \} + \nabla \sum_{j=1}^p \mu_j^* g_j(x_0) = 0, \quad (2.7)$$

$$f(x_0, \bar{y}_i) + \left(x_0^\top Bx_0\right)^{1/2} - k_0 \left(h(x_0, \bar{y}_i) - \left(x_0^\top Dx_0\right)^{1/2} \right) = 0, \quad i = 1, 2, \dots, s, \quad (2.8)$$

$$\sum_{j=1}^p \mu_j^* g_j(x_0) = 0, \quad (2.9)$$

$$t_i^* \geq 0, \quad \sum_{i=1}^s t_i^* = 1, \quad (2.10)$$

$$w^\top Bw \leq 1, \quad v^\top Dv \leq 1,$$

$$x_0^\top Bw = \left(x_0^\top Bx_0\right)^{1/2}, \quad (2.11)$$

$$x_0^\top Dv = \left(x_0^\top Bx_0\right)^{1/2}.$$

It should be noted that both the matrices B and D are positive definite at the solution x_0 in the above theorem. If one of $x_0^\top Bx_0$ and $x_0^\top Dx_0$ is zero, or both B and D are singular at x_0 , then, for $(s, t^*, \bar{y}) \in K(x_0)$, we define a set $Z_{\bar{y}}(x_0)$ by

$$\begin{aligned} & Z_{\bar{y}}(x_0) \\ &= \left\{ z \in \mathbb{R}^n \mid z^\top \nabla g_j(x_0) \leq 0, j \in J(x_0), \text{ with any one of the next conditions (i)–(iii) holds} \right\}. \end{aligned} \quad (2.12)$$

Here conditions (i)–(iii) are given as follows:

(i) if $x_0^\top Bx_0 > 0$ and $x_0^\top Dx_0 = 0$, then

$$z^\top \left(\sum_{i=1}^s t_i^* \left(\nabla f(x_0, \bar{y}_i) + \frac{Bx_0}{\left(x_0^\top Bx_0\right)^{1/2}} - k_0 \nabla h(x_0, \bar{y}_i) \right) \right) + \left(z^\top (k_0^2 D) z \right)^{1/2} < 0, \quad (2.13)$$

(ii) if $x_0^\top Bx_0 = 0$ and $x_0^\top Dx_0 > 0$, then

$$z^\top \left(\sum_{i=1}^s t_i^* \left(\nabla f(x_0, \bar{y}_i) - k_0 \left(\nabla h(x_0, \bar{y}_i) - \frac{Dx_0}{(x_0^\top Dx_0)^{1/2}} \right) \right) \right) + (z^\top Bz)^{1/2} < 0, \quad (2.14)$$

(iii) if $x_0^\top Bx_0 = 0$ and $x_0^\top Dx_0 = 0$, then

$$z^\top \left(\sum_{i=1}^s t_i^* (\nabla f(x_0, \bar{y}_i) - k_0 \nabla h(x_0, \bar{y}_i)) \right) + (z^\top (k_0 D) z)^{1/2} + (z^\top Bz)^{1/2} < 0. \quad (2.15)$$

If we take condition $Z_{\bar{y}}(x_0) = \emptyset$ in Theorem 2.2, then the result of Theorem 2.2 still holds.

3. Optimality Conditions

In this section we derive sufficient conditions for optimality of (P) under the assumption of a particular form of generalized B -(p, r)-invexity. All theorems in this work will be proved only in the case when $p \neq 0, r \neq 0$ (other cases can be dealt with by similarity since the only difference is arising from the form of the inequality defining the class of the B -(p, r)-invex functions with respect to η and b for given p and r). The proofs of the other cases are easier than this one.

We would establish the sufficient conditions under the B -(p, r)-invex function.

Theorem 3.1 (sufficient optimality conditions). *Let $x_0 \in X^\circ$ be a feasible solution of (P) . There exist a positive integer $s, 1 \leq s \leq n+1, t^* \in \mathbb{R}^s, \bar{y}_i \in Y(x_0)$ ($i = 1, 2, \dots, s$), $k_0 \in \mathbb{R}_+, (w, v) \in \mathbb{R}^n \times \mathbb{R}^n$, and $\mu^* \in \mathbb{R}_+^p$ to satisfy relations (2.7)~(2.11). Furthermore suppose that any one of conditions (a) and (b) holds:*

- (a) $Y(\cdot) = \sum_{i=1}^s t_i^* ((f(\cdot, \bar{y}_i) + (\cdot)^\top Bw) - k_0(h(\cdot, \bar{y}_i) - (\cdot)^\top Dv))$ is B -(p, r)-invex with respect to η and b at x_0 , and $\sum_{j=1}^p \mu_j^* g_j(\cdot)$ is B -(p, r)-invex with respect to the same function η and another function b_1 at x_0 on X° , not necessarily, equal to b ,
- (b) $Y_1(\cdot) = \sum_{i=1}^s t_i^* ((f(\cdot, \bar{y}_i) + (\cdot)^\top Bw) - k_0(h(\cdot, \bar{y}_i) - (\cdot)^\top Dv)) + \sum_{j=1}^p \mu_j^* g_j(\cdot)$ is B -(p, r)-invex with respect to η and b at x_0 on X° ,

then x_0 is an optimal solution of (P) .

Proof. Suppose that x_0 is not an optimal solution of (P) . Then there exists a (P) -feasible solution x_1 such that

$$\sup_{y \in Y} \frac{f(x_1, y) + (x_1^\top Bx_1)^{1/2}}{h(x_1, y) - (x_1^\top Dx_1)^{1/2}} < \sup_{y \in Y} \frac{f(x_0, y) + (x_0^\top Bx_0)^{1/2}}{h(x_0, y) - (x_0^\top Dx_0)^{1/2}}. \quad (3.1)$$

We note that

$$\sup_{y \in Y} \frac{f(x_0, y) + (x_0^\top B x_0)^{1/2}}{h(x_0, y) - (x_0^\top D x_0)^{1/2}} = \frac{f(x_0, \bar{y}_i) + (x_0^\top B x_0)^{1/2}}{h(x_0, \bar{y}_i) - (x_0^\top D x_0)^{1/2}} = k_0, \quad (3.2)$$

for $\bar{y}_i \in Y(x_0)$, $i = 1, 2, \dots, s$, and

$$\frac{f(x_1, \bar{y}_i) + (x_1^\top B x_1)^{1/2}}{h(x_1, \bar{y}_i) - (x_1^\top D x_1)^{1/2}} \leq \sup_{y \in Y} \frac{f(x_1, y) + (x_1^\top B x_1)^{1/2}}{h(x_1, y) - (x_1^\top D x_1)^{1/2}}. \quad (3.3)$$

Then, we obtain

$$\frac{f(x_1, \bar{y}_i) + (x_1^\top B x_1)^{1/2}}{h(x_1, \bar{y}_i) - (x_1^\top D x_1)^{1/2}} < k_0, \quad \text{for } i = 1, 2, \dots, s. \quad (3.4)$$

It follows that

$$f(x_1, \bar{y}_i) + (x_1^\top B x_1)^{1/2} - k_0 \left(h(x_1, \bar{y}_i) - (x_1^\top D x_1)^{1/2} \right) < 0, \quad \text{for } i = 1, 2, \dots, s. \quad (3.5)$$

From relations (2.6), (2.11), (2.8), (2.10), and (3.5), we have

$$\begin{aligned} \Upsilon(x_1) &= \sum_{i=1}^s t_i^* \left(f(x_1, \bar{y}_i) + (x_1^\top B x_1)^{1/2} - k_0 \left(h(x_1, \bar{y}_i) - (x_1^\top D x_1)^{1/2} \right) \right) \\ &\leq \sum_{i=1}^s t_i^* \left(f(x_1, \bar{y}_i) + (x_1^\top B x_1)^{1/2} - k_0 \left(h(x_1, \bar{y}_i) - (x_1^\top D x_1)^{1/2} \right) \right) < 0 \\ &= \sum_{i=1}^s t_i^* \left(f(x_0, \bar{y}_i) + (x_0^\top B x_0)^{1/2} - k_0 \left(h(x_0, \bar{y}_i) - (x_0^\top D x_0)^{1/2} \right) \right) \\ &= \sum_{i=1}^s t_i^* \left(f(x_0, \bar{y}_i) + (x_0^\top B x_0)^{1/2} - k_0 \left(h(x_0, \bar{y}_i) - (x_0^\top D x_0)^{1/2} \right) \right) = \Upsilon(x_0). \end{aligned} \quad (3.6)$$

That is,

$$\Upsilon(x_1) < \Upsilon(x_0). \quad (3.7)$$

From relations (P) and (2.9), we obtain

$$\sum_{j=1}^p \mu_j^* g_j(x_1) \leq 0 = \sum_{j=1}^p \mu_j^* g_j(x_0). \quad (3.8)$$

If hypothesis (a) holds, from the B -(p, r)-invexity with respect to η and b_1 at x_0 of $\sum_{j=1}^p \mu_j^* g_j(\cdot)$, we have

$$\frac{1}{r} b_1(x_1, x_0) \left(e^{r(\sum_{j=1}^p \mu_j^* g_j(x_1) - \sum_{j=1}^p \mu_j^* g_j(x_0))} - 1 \right) \geq \frac{1}{p} \nabla \sum_{j=1}^p \mu_j^* g_j(x_0) \left(e^{p\eta(x_1, x_0)} - 1 \right). \quad (3.9)$$

From the inequalities (3.8) and (3.9), we get

$$\frac{1}{p} \nabla \left(\sum_{j=1}^p \mu_j^* g_j(x_0) \right) \left(e^{p\eta(x_1, x_0)} - 1 \right) \leq 0. \quad (3.10)$$

Now, multiplying equality (2.7) by $(1/p)(e^{p\eta(x_1, x_0)} - 1)$, we know

$$\frac{1}{p} \left(e^{p\eta(x_1, x_0)} - 1 \right) \left(\sum_{i=1}^s t_i^* \{ \nabla f(x_0, \bar{y}_i) + Bw - k_0(\nabla h(x_0, \bar{y}_i) - Dv) \} + \nabla \sum_{j=1}^p \mu_j^* g_j(x_0) \right) = 0. \quad (3.11)$$

From relations (3.10) and (3.11), we have

$$\frac{1}{p} \left(e^{p\eta(x_1, x_0)} - 1 \right) \left(\sum_{i=1}^s t_i^* \{ \nabla f(x_0, \bar{y}_i) + Bw - k_0(\nabla h(x_0, \bar{y}_i) - Dv) \} \right) \geq 0. \quad (3.12)$$

From the B -(p, r)-invexity with respect to the same function η and the function b at x_0 of $\Upsilon(\cdot)$,

$$\frac{1}{r} b(x, u) \left(e^{r(\Upsilon(x_1) - \Upsilon(x_0))} - 1 \right) \geq \frac{1}{p} \nabla \Upsilon(x_0) \left(e^{p\eta(x_1, x_0)} - 1 \right). \quad (3.13)$$

From inequality (3.12) and the above inequality, we obtain

$$\Upsilon(x_1) \geq \Upsilon(x_0), \quad (3.14)$$

which contradicts (3.7), and proves that x_0 is an optimal solution to (P).

If hypothesis (b) holds, from the B -(p, r)-invexity with respect to η and b at x_0 of $\Upsilon_1(\cdot)$, then

$$\frac{1}{r} b(x, u) \left(e^{r(\Upsilon_1(x_1) - \Upsilon_1(x_0))} - 1 \right) \geq \frac{1}{p} \nabla \Upsilon_1(x_0) \left(e^{p\eta(x_1, x_0)} - 1 \right). \quad (3.15)$$

The above inequality along with (2.7) yields

$$\Upsilon_1(x_1) \geq \Upsilon_1(x_0), \quad (3.16)$$

which contradicts (3.7). Hence, the proof is completed. $\square$

4. Parametric Dual-Type Model

We use the optimality conditions of the preceding section and show that the following formation is a dual (D) to the minimax problem (P):

$$\max_{(s,t,y) \in K(z)} \sup_{(z,\mu,k,w,v) \in H_1(s,t,y)} k, \quad (D)$$

where $H_1(s,t,y)$ denotes the set of $(z,\mu,k,w,v) \in \mathbb{R}^n \times \mathbb{R}_+^p \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\sum_{i=1}^s t_i \{ \nabla f(z, y_i) + Bw - k(\nabla h(z, y_i) - Dv) \} + \nabla \sum_{j=1}^p \mu_j g_j(z) = 0, \quad (4.1)$$

$$\sum_{i=1}^s t_i \left\{ f(z, y_i) + z^\top Bw - k(h(z, y_i) - z^\top Dv) \right\} \geq 0, \quad (4.2)$$

$$\sum_{j=1}^p \mu_j g_j(z) \geq 0, \quad (4.3)$$

$$\begin{aligned} w^\top Bw &\leq 1, & v^\top Dv &\leq 1, \\ (s,t,y) &\in K(z). \end{aligned} \quad (4.4)$$

If for a triplet $(s,t,y) \in K(z)$ the set $H_1(s,t,y)$ is empty, then we define the supremum over it to be $-\infty$.

Let Γ denote the set of all feasible points of (D). Moreover, we denote $pr_X \Gamma = \{z \in X \mid (z,\mu,s,t,y,w,v,k) \in \Gamma\}$.

We can derive the following weak duality theorem between (P) and (D).

Theorem 4.1 (weak duality). *Let x and (z,μ,s,t,y,w,v,k) be (P)-feasible and (D)-feasible, respectively. Suppose that any one of the following conditions (a) and (b) holds:*

- (a) $\Upsilon_2(\cdot) = \sum_{i=1}^s t_i ((f(\cdot, y_i) + (\cdot)^\top Bw) - k(h(\cdot, y_i) - (\cdot)^\top Dv))$ is B -(p,r)-invex with respect to η and b at z and $\sum_{j=1}^p \mu_j g_j(\cdot)$ is B -(p,r)-invex with respect to the same function η and another function b_2 at z on $X^\circ \cup pr_X \Gamma$, not necessarily, equal to b ,
- (b) $\Upsilon_3(\cdot) = \sum_{i=1}^s t_i ((f(\cdot, y_i) + (\cdot)^\top Bw) - k(h(\cdot, y_i) - (\cdot)^\top Dv)) + \sum_{j=1}^p \mu_j g_j(\cdot)$ is (p,r) -invex with respect to η and b at z on $X^\circ \cup pr_X \Gamma$.

Then

$$\sup_{y \in Y} \frac{f(x, y) + (x^\top Bx)^{1/2}}{h(x, y) - (x^\top Dx)^{1/2}} \geq k. \quad (4.5)$$

Proof. Suppose on the contrary that

$$\sup_{y \in Y} \frac{f(x, y) + (x^\top Bx)^{1/2}}{h(x, y) - (x^\top Dx)^{1/2}} < k. \quad (4.6)$$

Then, we have an inequality

$$f(x, y) + (x^\top Bx)^{1/2} - k(h(x, y) - (x^\top Dx)^{1/2}) < 0 \quad \forall y \in Y. \quad (4.7)$$

It follows that for $t_i \geq 0, i = 1, 2, \dots, s$ with $\sum_{i=1}^s t_i = 1$, we have

$$t_i \left[f(x, y_i) + (x^\top Bx)^{1/2} - k(h(x, y_i) - (x^\top Dx)^{1/2}) \right] \leq 0, \quad (4.8)$$

with at least one strict inequality because $t = (t_1, t_2, \dots, t_s) \neq 0$. From relations (2.6), (4.4), (4.8), and (4.2), we obtain

$$\begin{aligned} \Upsilon_2(x) &= \sum_{i=1}^s t_i \left(f(x, y_i) + x^\top Bw - k(h(x, y_i) - x^\top Dv) \right) \\ &\leq \sum_{i=1}^s t_i \left[f(x, y_i) + (x^\top Bx)^{1/2} - k(h(x, y_i) - (x^\top Dx)^{1/2}) \right] < 0 \\ &\leq \sum_{i=1}^s t_i \left(f(z, y_i) + z^\top Bw - k(h(z, y_i) - z^\top Dv) \right) = \Upsilon_2(z). \end{aligned} \quad (4.9)$$

That is,

$$\Upsilon_2(x) < \Upsilon_2(z). \quad (4.10)$$

From relations (P) and (4.3), we have

$$\sum_{j=1}^p \mu_j g_j(x) \leq 0 \leq \sum_{j=1}^p \mu_j g_j(z), \quad (4.11)$$

If hypothesis (a) holds, from B -(p, r)-invexity with respect to η and b at z of $\Upsilon_2(\cdot)$, we get

$$\frac{1}{r} b(x, z) \left(e^{r(\Upsilon_2(x) - \Upsilon_2(z))} - 1 \right) \geq \frac{1}{p} \nabla \Upsilon_2(z) \left(e^{p\eta(x, z)} - 1 \right). \quad (4.12)$$

From the above inequality together with relation (4.10), we have

$$\frac{1}{p} \nabla \Upsilon_2(z) \left(e^{p\eta(x, z)} - 1 \right) < 0. \quad (4.13)$$

Multiplying (4.1) by $(1/p)(e^{p\eta(x, z)} - 1)$, we obtain

$$\frac{1}{p} \left(e^{p\eta(x, z)} - 1 \right) \left(\sum_{i=1}^s t_i \{ \nabla f(z, y_i) + Bw - k(\nabla h(z, y_i) - Dv) \} + \nabla \sum_{j=1}^p \mu_j g_j(z) \right) = 0. \quad (4.14)$$

From the above equality and inequality (4.13), we get

$$\frac{1}{p} \left(e^{p\eta(x,z)} - 1 \right) \sum_{j=1}^p \mu_j \nabla g_j(z) > 0. \quad (4.15)$$

Using the B -(p, r)-invexity of $\sum_{j=1}^p \mu_j g_j(\cdot)$ with respect to the same function η and the function b_2 at z and inequality (4.15), we get

$$\sum_{j=1}^p \mu_j g_j(x) > \sum_{j=1}^p \mu_j g_j(z). \quad (4.16)$$

which contradicts (4.11) and proves that $\sup_{y \in Y} ((f(x, y) + (x^\top Bx)^{1/2}) / (h(x, y) - (x^\top Dx)^{1/2})) \geq k$.

If hypothesis (b) holds, from the B -(p, r)-invexity with respect to η and b at z of $\Upsilon_3(\cdot)$, then

$$\frac{1}{r} b(x, z) \left(e^{r(\Upsilon_3(x) - \Upsilon_3(z))} - 1 \right) \geq \frac{1}{p} \nabla \Upsilon_3(z) \left(e^{p\eta(x,z)} - 1 \right). \quad (4.17)$$

By the above inequality and equality (4.1), we have

$$\Upsilon_3(x) \geq \Upsilon_3(z). \quad (4.18)$$

From relations (4.10) and (4.11), we obtain

$$\Upsilon_3(x) < \Upsilon_3(z), \quad (4.19)$$

which contradicts inequality (4.18). Thus, the proof is complete. $\square$

Theorem 4.2 (strong duality). *Let x^* be an optimal solution of (P), and let x^* satisfy a constraint qualification for (P). Then there exist $(s^*, t^*, y^*) \in K(x^*)$ and $(x^*, \mu^*, k^*, w^*, v^*) \in H_1(s^*, t^*, y^*)$ such that $(x^*, \mu^*, s^*, t^*, y^*, k^*, w^*, v^*)$ is a feasible solution of (D). If in addition the hypothesis of Theorem 4.1 holds, then $(x^*, \mu^*, s^*, t^*, y^*, k^*, w^*, v^*)$ is an optimal solution of (D) and the two problems (P) and (D) have the same optimal value.*

Proof. By Theorem 2.2, there exist $(s^*, t^*, y^*) \in K(x^*)$ and $(x^*, \mu^*, k^*, w^*, v^*) \in H_1(s^*, t^*, y^*)$ such that $(x^*, \mu^*, s^*, t^*, y^*, k^*, w^*, v^*)$ is feasible for (D), and

$$k^* = \frac{f(x^*, y_i^*) + (x^{*\top} Bx^*)^{1/2}}{h(x^*, y_i^*) - (x^{*\top} Dx^*)^{1/2}}. \quad (4.20)$$

The optimality of this feasible solution for (D) follows from Theorem 4.1. $\square$

Theorem 4.3 (strict converse duality). *Let $\bar{x}$ and $(z^*, \mu^*, s^*, t^*, y^*, k^*, w^*, v^*)$ be optimal solutions of (P) and (D), respectively, and assume that the hypothesis of Theorem 4.2 is fulfilled. Suppose that any one of the following conditions (a) and (b) holds:*

- (a) $\Upsilon_4(\cdot) = \sum_{i=1}^{s^*} t_i^* \{f(\cdot, y_i^*) + (\cdot)^\top Bw^* - k^*(h(\cdot, y_i^*) - (\cdot)^\top Dv^*)\}$ is strictly B -(p, r)-invex with respect to η and b at z^* and $\sum_{j=1}^p \mu_j^* g_j(\cdot)$ is (p, r) -invex with respect to the same function η and another function b_4 at z^* on $X^\circ \cup pr_X \Gamma$, not necessarily, equal to b ,
- (b) $\Upsilon_5(\cdot) = \sum_{i=1}^{s^*} t_i^* \{f(\cdot, y_i^*) + (\cdot)^\top Bw^* - k^*(h(\cdot, y_i^*) - (\cdot)^\top Dv^*)\} + \sum_{j=1}^p \mu_j^* g_j(\cdot)$ is strictly (p, r) -invex with respect to η and b at z^* on $X^\circ \cup pr_X \Gamma$.

Then $\bar{x} = z^*$, that is, z^* solves (P) and $\sup_{y \in Y} ((f(z^*, y) + ((z^*)^\top Bz^*)^{1/2}) / (h(z^*, y) - ((z^*)^\top Dz^*)^{1/2})) = k^*$.

Proof. We shall assume that $\bar{x} \neq z^*$ and reach a contradiction. From Theorem 4.2, we know that there exist $(\bar{s}, \bar{t}, \bar{y}) \in K(\bar{x})$ and $(\bar{x}, \bar{\mu}, \bar{k}, \bar{w}, \bar{v}) \in H_1(\bar{s}, \bar{t}, \bar{y})$ such that $(\bar{x}, \bar{\mu}, \bar{s}, \bar{t}, \bar{y}, \bar{k}, \bar{w}, \bar{v})$ is an optimal solution for (D) with the optimal value

$$\sup_{y \in Y} \frac{f(\bar{x}, y) + (\bar{x}^\top B\bar{x})^{1/2}}{h(\bar{x}, y) - (\bar{x}^\top D\bar{x})^{1/2}} = \bar{k}. \quad (4.21)$$

Now like the proof of Theorem 4.1 by x replacing by $\bar{x}$ and $(z, \mu, s, t, y, w, v, k)$ by $(z^*, \mu^*, s^*, t^*, y^*, k^*, w^*, v^*)$, we obtain

$$\sup_{y \in Y} \frac{f(\bar{x}, y) + (\bar{x}^\top B\bar{x})^{1/2}}{h(\bar{x}, y) - (\bar{x}^\top D\bar{x})^{1/2}} > k^*. \quad (4.22)$$

The above inequality contradicts

$$\sup_{y \in Y} \frac{f(\bar{x}, y) + (\bar{x}^\top B\bar{x})^{1/2}}{h(\bar{x}, y) - (\bar{x}^\top D\bar{x})^{1/2}} = k^* = \bar{k}. \quad (4.23)$$

Therefore, we conclude that $\bar{x} = z^*$. Here, the proof of the theorem is complete. $\square$

Remark 4.4. In Theorem 4.3, if $\sum_{j=1}^p \mu_j^* g_j(\cdot)$ is a strictly B -(p, r)-invex function with respect to η and b_4 and $\Upsilon_4(\cdot)$ is a B -(p, r)-invex function with respect to the same function η and the function b , not necessarily, equal to b_4 , then Theorem 4.3 also holds.

5. Mond-Weir Dual-Type Model

In this section, we formulate the Mond-Weir-type dual model to the problem (P) as follows:

$$\max_{(s,t,y) \in K(z)} \sup_{(z,\mu,w,v) \in H_2(s,t,y)} \mathcal{F}(z), \quad (\text{MWD})$$

where $H_2(s, t, y)$ denotes the set of $(z, \mu, w, v) \in \mathbb{R}^n \times \mathbb{R}_+^p \times \mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\begin{aligned} \sum_{i=1}^s t_i \left\{ \left(h(z, y_i) - (z^\top D z)^{1/2} \right) (\nabla f(z, y_i) + B w) \right. \\ \left. - \left(f(z, y_i) + (z^\top B z)^{1/2} \right) (\nabla h(z, y_i) - D v) \right\} + \sum_{j=1}^p \mu_j \nabla g_j(z) = 0, \end{aligned} \quad (5.1)$$

$$\sum_{j=1}^p \mu_j g_j(z) \geq 0, \quad (5.2)$$

$$w^\top B w \leq 1, \quad v^\top D v \leq 1,$$

$$z B w = (z^\top B z)^{1/2} \quad (5.3)$$

$$z D v = (z^\top B z)^{1/2},$$

where

$$\mathcal{F}(z) = \sup_{y \in Y} \frac{f(z, y) + (z^\top B z)^{1/2}}{h(z, y) - (z^\top D z)^{1/2}} = \sup_{y \in Y} \phi(z, y). \quad (5.4)$$

If for a triplet $(s, t, y) \in K(z)$ the set $H_2(s, t, y)$ is empty, then we define the supremum over it to be $-\infty$.

Let $\tilde{\Gamma}$ denote the set of all feasible points of (MWD). Moreover, we denote $pr_X \tilde{\Gamma} = \{z \in X \mid (z, \mu, s, t, y, w, v) \in \tilde{\Gamma}\}$.

We establish the weak, strong, and strict converse duality theorems for (MWD) with respect to the primal problem (P).

Theorem 5.1 (weak duality). *Let x and (z, μ, s, t, y, w, v) be (P)-feasible and (MWD)-feasible, respectively. Suppose that any one of the following conditions (a) and (b) holds:*

- (a) $\Upsilon_6(\cdot) = \sum_{i=1}^s t_i \{ (h(z, y_i) - z^\top D v) (f(\cdot, y_i) + (\cdot)^\top B w) - (f(z, y_i) + z^\top B w) (h(\cdot, y_i) - (\cdot)^\top D v) \}$ is B -(p, r)-invex, respect to η and b at z and $\sum_{j=1}^p \mu_j g_j(\cdot)$ is B -(p, r)-invex with respect to the same function η and another function b_6 at z on $X^\circ \cup \tilde{\Gamma}$, not necessarily, equal to b ,
- (b) $\Upsilon_7 = \sum_{i=1}^s t_i \{ (h(z, y_i) - z^\top D v) (f(\cdot, y_i) + (\cdot)^\top B w) - (f(z, y_i) + z^\top B w) (h(\cdot, y_i) - (\cdot)^\top D v) \} + \sum_{j=1}^p \mu_j g_j(\cdot)$ is B -(p, r)-invex with respect to η and b at z on $X^\circ \cup \tilde{\Gamma}$.

Then

$$\sup_{y \in Y} \frac{f(x, y) + (x^\top Bx)^{1/2}}{h(x, y) - (x^\top Dx)^{1/2}} \geq \mathcal{F}(z). \quad (5.5)$$

Proof. On the contrary, if possible, suppose that for each $x \in X^\circ$,

$$\sup_{y \in Y} \frac{f(x, y) + (x^\top Bx)^{1/2}}{h(x, y) - (x^\top Dx)^{1/2}} < F(z). \quad (5.6)$$

From the above inequality and $y_i \in Y(z)$, $i = 1, 2, \dots, s$, we obtain

$$\frac{f(x, y_i) + (x^\top Bx)^{1/2}}{h(x, y_i) - (x^\top Dx)^{1/2}} < F(z) = \frac{f(z, y_i) + (z^\top Bz)^{1/2}}{h(z, y_i) - (z^\top Dz)^{1/2}}, \quad i = 1, 2, \dots, s. \quad (5.7)$$

By the above inequality, we know that

$$\begin{aligned} & \left(f(x, y_i) + (x^\top Bx)^{1/2} \right) \left(h(z, y_i) - (z^\top Dz)^{1/2} \right) \\ & - \left(h(x, y_i) - (x^\top Dx)^{1/2} \right) \left(f(z, y_i) + (z^\top Bz)^{1/2} \right) < 0, \end{aligned} \quad (5.8)$$

for all $i = 1, 2, \dots, s$ and $y_i \in Y(z)$.

Multiplying the above inequality by $t_i \in \mathbb{R}_+^s$ with $\sum_{i=1}^s t_i = 1$, we have

$$\begin{aligned} & \sum_{i=1}^s t_i \left\{ \left(f(x, y_i) + (x^\top Bx)^{1/2} \right) \left(h(z, y_i) - (z^\top Dz)^{1/2} \right) \right. \\ & \left. - \left(h(x, y_i) - (x^\top Dx)^{1/2} \right) \left(f(z, y_i) + (z^\top Bz)^{1/2} \right) \right\} < 0. \end{aligned} \quad (5.9)$$

From relations (2.6), (5.3), and (5.9), we get

$$\begin{aligned} \Upsilon_6(x) &= \sum_{i=1}^s t_i \left\{ \left(f(x, y_i) + x^\top Bw \right) \left(h(z, y_i) - z^\top Dv \right) - \left(h(x, y_i) - x^\top Dv \right) \left(f(z, y_i) + z^\top Bw \right) \right\} \\ &\leq \sum_{i=1}^s t_i \left\{ \left(f(x, y_i) + (x^\top Bx)^{1/2} \right) \left(h(z, y_i) - (z^\top Dz)^{1/2} \right) \right. \\ &\quad \left. - \left(h(x, y_i) - (x^\top Dx)^{1/2} \right) \left(f(z, y_i) + (z^\top Bz)^{1/2} \right) \right\} \\ &< 0 = \Upsilon_6(z). \end{aligned} \quad (5.10)$$

By relations (P) and (5.2), we have

$$\sum_{j=1}^p \mu_j g_j(x) \leq 0 \leq \sum_{j=1}^p \mu_j g_j(z). \quad (5.11)$$

Now, if condition (a) holds, from B -(p, r)-invexity with respect to η and b_6 at z of $\sum_{j=1}^p \mu_j g_j(\cdot)$, we get

$$\frac{1}{r} b_1(x_1, x_0) \left(e^{r(\sum_{j=1}^p \mu_j^* g_j(x) - \sum_{j=1}^p \mu_j^* g_j(z))} - 1 \right) \geq \frac{1}{p} \nabla \sum_{j=1}^p \mu_j^* g_j(z) \left(e^{p\eta(x, z)} - 1 \right). \quad (5.12)$$

From the above inequality and inequality (5.11), we obtain

$$\frac{1}{p} \nabla \sum_{j=1}^p (\mu_j g_j(z)) \left(e^{p\eta(x, z)} - 1 \right) \leq 0. \quad (5.13)$$

Multiplying (5.1) by $(1/p)(e^{p\eta(x, z)} - 1)$, we have

$$\begin{aligned} \frac{1}{p} \left(e^{p\eta(x, z)} - 1 \right) \left\{ \sum_{i=1}^s t_i \left\{ \left(h(z, y_i) - (z^\top D z)^{1/2} \right) (\nabla f(z, y_i) + B w) \right. \right. \\ \left. \left. - \left(f(z, y_i) + (z^\top B z)^{1/2} \right) (\nabla h(z, y_i) - D v) \right\} + \sum_{j=1}^p \mu_j \nabla g_j(z) \right\} = 0. \end{aligned} \quad (5.14)$$

By the above equality and inequality (5.13), we obtain

$$\begin{aligned} \frac{1}{p} \left(e^{p\eta(x, z)} - 1 \right) \left\{ \sum_{i=1}^s t_i \left\{ \left(h(z, y_i) - (z^\top D z)^{1/2} \right) (\nabla f(z, y_i) + B w) \right. \right. \\ \left. \left. - \left(f(z, y_i) + (z^\top B z)^{1/2} \right) (\nabla h(z, y_i) - D v) \right\} \right\} \geq 0. \end{aligned} \quad (5.15)$$

Using the B -(p, r)-invexity with respect to the same function η and the function b at z of $\Upsilon_6(\cdot)$ and the above inequality, we have

$$\Upsilon_6(x) \geq \Upsilon_6(z), \quad (5.16)$$

which contradicts (5.10) and proves that $\sup_{y \in Y} ((f(x, y) + (x^\top B x)^{1/2}) / (h(x, y) - (x^\top D x)^{1/2})) \geq \Upsilon(z)$.

If hypothesis (b) holds, from the B -(p, r)-invexity with respect to η and b at z of $\Upsilon_7(\cdot)$ and the equality (5.1), then

$$\Upsilon_7(x) \geq \Upsilon_7(z). \quad (5.17)$$

From relations (5.10) and (5.11), we have

$$\begin{aligned} & \sum_{i=1}^s t_i \left\{ \left(h(z, y_i) - z^\top Dv \right) \left(f(x, y_i) + x^\top Bw \right) - \left(f(z, y_i) + z^\top Bw \right) \left(h(x, y_i) - x^\top Dv \right) \right\} \\ & + \sum_{j=1}^p \mu_j g_j(x) \\ & < \sum_{i=1}^s t_i \left\{ \left(h(z, y_i) - z^\top Dv \right) \left(f(z, y_i) + z^\top Bw \right) - \left(f(z, y_i) + z^\top Bw \right) \left(h(z, y_i) - z^\top Dv \right) \right\} \\ & + \sum_{j=1}^p \mu_j g_j(z), \end{aligned} \quad (5.18)$$

which contradicts inequality (5.17). Hence, the proof is complete. $\square$

Similar to the proof of Theorem 4.2, we can establish Theorem 5.2.

Theorem 5.2 (strong duality). *Let x^* be an optimal solution of (P) and let x^* satisfy a constraint qualification for (P). Then there exist $(s^*, t^*, y^*) \in K(x^*)$ and $(x^*, \mu^*, w^*, v^*) \in H_1(s^*, t^*, y^*)$ such that $(x^*, \mu^*, s^*, t^*, y^*, w^*, v^*)$ is a feasible solution of (MWD). If in addition the hypothesis of Theorem 5.1 holds, then $(x^*, \mu^*, s^*, t^*, y^*, w^*, v^*)$ is an optimal solution of (MWD) and the two problems (P) and (MWD) have the same optimal value.*

Theorem 5.3 (strict converse duality). *Let $\bar{x}$ and $(z^*, \mu^*, s^*, t^*, y^*, w^*, v^*)$ be optimal solutions of (P) and (MWD), respectively, and assume that the hypothesis of Theorem 5.2 is fulfilled. Suppose that any one of the following conditions (a) and (b) holds:*

- (a) $\Upsilon_8(\cdot) = \sum_{i=1}^{s^*} t_i^* \{ (h(z^*, y_i^*) - (z^*)^\top Dv) (f(\cdot, y_i^*) + (\cdot)^\top Bw) - (f(z^*, y_i^*) + (z^*)^\top Bw) (h(\cdot, y_i^*) - (\cdot)^\top Dv) \}$ is strictly B -(p, r)-invex with respect to η and b at z^* and $\sum_{j=1}^p \mu_j^* g_j(\cdot)$ is B -(p, r)-invex with respect to the same function η and another function b_8 at z^* on $X^\circ \cup \tilde{\Gamma}$, not necessarily, equal to b ,
- (b) $\Upsilon_9(\cdot) = \sum_{i=1}^{s^*} t_i^* \{ (h(z^*, y_i^*) - (z^*)^\top Dv) (f(\cdot, y_i^*) + (\cdot)^\top Bw) - (f(z^*, y_i^*) + (z^*)^\top Bw) (h(\cdot, y_i^*) - (\cdot)^\top Dv) \} + \sum_{j=1}^p \mu_j^* g_j(\cdot)$ is strictly B -(p, r)-invex with respect to η and b at z^* on $X^\circ \cup \tilde{\Gamma}$.

Then $\bar{x} = z^*$, that is, z^* solves (P) and $\sup_{y \in Y} ((f(z^*, y) + ((z^*)^\top Bz^*)^{1/2}) / (h(z^*, y) - ((z^*)^\top Dz^*)^{1/2})) = \varphi(z^*)$.

Proof. We suppose on the contrary that if $\bar{x} \neq z^*$, then

$$\sup_{y \in Y} \frac{f(\bar{x}, y) + (\bar{x}^\top B \bar{x})^{1/2}}{h(\bar{x}, y) - (\bar{x}^\top D \bar{x})^{1/2}} \leq \mathcal{F}(z^*). \quad (5.19)$$

Since $y_i^* \in Y(z^*)$, for $i = 1, 2, \dots, s^*$,

$$\frac{f(z^*, y_i^*) + ((z^*)^\top B z^*)^{1/2}}{h(z^*, y_i^*) - ((z^*)^\top D z^*)^{1/2}} = \mathcal{F}(z^*), \quad \text{for } i = 1, 2, \dots, s^*. \quad (5.20)$$

It follows that

$$\begin{aligned} & \left(f(\bar{x}, y) + (\bar{x}^\top B \bar{x})^{1/2} \right) \left(h(z^*, y_i^*) - ((z^*)^\top D z^*)^{1/2} \right) \\ & - \left(h(\bar{x}, y) - (\bar{x}^\top D \bar{x})^{1/2} \right) \left(f(z^*, y_i^*) + ((z^*)^\top B z^*)^{1/2} \right) \\ & \leq 0, \quad \text{for } i = 1, 2, \dots, s^*, \quad y \in Y. \end{aligned} \quad (5.21)$$

From $y_i^* \in Y(z^*) \subset Y$ and $t \in \mathbb{R}_+^{s^*}$ with $\sum_{i=1}^{s^*} t_i^* = 1$, we have

$$\begin{aligned} & \sum_{i=1}^{s^*} t_i^* \left\{ \left(h(z^*, y_i^*) - ((z^*)^\top D z^*)^{1/2} \right) \left(f(\bar{x}, y_i^*) + (\bar{x}^\top B \bar{x})^{1/2} \right) \right. \\ & \left. - \left(h(\bar{x}, y_i^*) - (\bar{x}^\top D \bar{x})^{1/2} \right) \left(f(z^*, y_i^*) + ((z^*)^\top B z^*)^{1/2} \right) \right\} \leq 0. \end{aligned} \quad (5.22)$$

From relations (2.6), (5.3), and inequality (5.22), we obtain that

$$\begin{aligned} \Upsilon_8(\bar{x}) &= \sum_{i=1}^{s^*} t_i^* \left\{ \left(h(z^*, y_i^*) - (z^*)^\top D v \right) \left(f(\bar{x}, y_i^*) + \bar{x}^\top B w \right) \right. \\ & \quad \left. - \left(f(z^*, y_i^*) + (z^*)^\top B w \right) \left(h(\bar{x}, y_i^*) - \bar{x}^\top D v \right) \right\} \\ &\leq \sum_{i=1}^{s^*} t_i^* \left\{ \left(h(z^*, y_i^*) - (\bar{x}^\top D \bar{x})^{1/2} \right) \left(f(z^*, y_i^*) + ((z^*)^\top B z^*)^{1/2} \right) \right. \\ & \quad \left. - \left(h(\bar{x}, y_i^*) - (\bar{x}^\top D \bar{x})^{1/2} \right) \left(f(z^*, y_i^*) + ((z^*)^\top B z^*)^{1/2} \right) \right\} \\ &\leq 0 = \Upsilon_8(z^*). \end{aligned} \quad (5.23)$$

By relations (P) and (5.2), we get

$$\sum_{j=1}^p \mu_j^* g_j(\bar{x}) \leq 0 \leq \sum_{j=1}^p \mu_j^* g_j(z^*). \quad (5.24)$$

Now, if condition (a) holds, from B -(p, r)-invexity with respect to η and b_8 at z of $\sum_{j=1}^p \mu_j^* g_j(\cdot)$, then

$$\frac{1}{r} b_1(\bar{x}, z^*) \left(e^{r(\sum_{j=1}^p \mu_j^* g_j(\bar{x}) - \sum_{j=1}^p \mu_j^* g_j(z^*))} - 1 \right) \geq \frac{1}{p} \nabla \sum_{j=1}^p \mu_j^* g_j(z^*) \left(e^{p\eta(\bar{x}, z^*)} - 1 \right). \quad (5.25)$$

From the above inequality and inequality (5.24), we know that

$$\frac{1}{p} \nabla \sum_{j=1}^p \mu_j^* g_j(z^*) \left(e^{p\eta(\bar{x}, z^*)} - 1 \right) \leq 0. \quad (5.26)$$

Multiplying (MWD) by $(1/p)(e^{p\eta(\bar{x}, z^*)} - 1)$, we have

$$\begin{aligned} \frac{1}{p} \left(e^{p\eta(\bar{x}, z^*)} - 1 \right) \left\{ \sum_{i=1}^{s^*} t_i^* \left\{ \left(h(z^*, y_i^*) - \left((z^*)^\top D z^* \right)^{1/2} \right) (\nabla f(z^*, y_i^*) + B w^*) \right. \right. \\ \left. \left. - \left(f(z^*, y_i^*) + \left((z^*)^\top B z^* \right)^{1/2} \right) (\nabla h(z^*, y_i^*) - D v) \right\} \right. \\ \left. + \sum_{j=1}^p \mu_j^* \nabla g_j(z^*) \right\} = 0. \end{aligned} \quad (5.27)$$

Basing on the above inequality and (5.26), we get the inequality

$$\begin{aligned} \frac{1}{p} \left(e^{p\eta(\bar{x}, z^*)} - 1 \right) \left\{ \sum_{i=1}^{s^*} t_i^* \left\{ \left(h(z^*, y_i^*) - \left((z^*)^\top D z^* \right)^{1/2} \right) (\nabla f(z^*, y_i^*) + B w^*) \right. \right. \\ \left. \left. - \left(f(z^*, y_i^*) + \left((z^*)^\top B z^* \right)^{1/2} \right) (\nabla h(z^*, y_i^*) - D v) \right\} \right\} \geq 0. \end{aligned} \quad (5.28)$$

Since $\Upsilon_8(\cdot)$ is strictly B -(p, r)-invex with respect to the same function η and the function b at z^* and the above inequality, we obtain

$$\Upsilon_8(z^*) < \Upsilon_8(\bar{x}), \quad (5.29)$$

which contradicts relations (5.23). Hence $\bar{x} = z^*$ and z^* is an optimal solution of (P).

If hypothesis (b) holds, from the strict $B(p, r)$ -invexity with respect to η and b at z^* of $\Upsilon_9(\cdot)$ and equality (5.1), then

$$\Upsilon_9(\bar{x}) > \Upsilon_9(z^*). \quad (5.30)$$

Relations (5.23) together with (5.24) yield

$$\begin{aligned} & \sum_{i=1}^{s^*} t_i^* \left\{ \left(h(z^*, y_i^*) - (z^*)^\top Dv \right) \left(f(\bar{x}, y_i^*) + \bar{x}^\top Bw \right) \right. \\ & \quad \left. - \left(f(z^*, y_i^*) + z^\top Bw \right) \left(h(\bar{x}, y_i^*) - \bar{x}^\top Dv \right) \right\} + \sum_{j=1}^p \mu_j^* g_j(\bar{x}) \\ & \leq \sum_{i=1}^{s^*} t_i^* \left\{ \left(h(z^*, y_i^*) - (z^*)^\top Dv \right) \left(f(z^*, y_i^*) + (z^*)^\top Bw \right) \right. \\ & \quad \left. - \left(f(z^*, y_i^*) + (z^*)^\top Bw \right) \left(h(z^*, y_i^*) - (z^*)^\top Dv \right) \right\} + \sum_{j=1}^p \mu_j^* g_j(z^*), \end{aligned} \quad (5.31)$$

which contradicts (5.30). Hence $z^* = \bar{x}$ is also an optimal solution of (P). The proof is complete. $\square$

Remark 5.4. In Theorem 5.3, if $\sum_{j=1}^p \mu_j^* g_j(\cdot)$ is a strictly $B(p, r)$ -invex function with respect to η and b_8 and $\Upsilon_8(\cdot)$ is a $B(p, r)$ -invex function with respect to the same function η and the function b , not necessarily, equal to b_8 , then Theorem 5.3 also holds.

6. Wolfe Dual-Type Model

Based on the result of Theorem 2.2, we can rewrite Theorem 2.2 as follows.

Theorem 6.1 ((necessary conditions) Lai and Lee [11, Theorem 4]). *Let x_0 be a (P)-optimal solution, and let $\nabla g_j(x_0)$, $j \in J(x_0)$ be linearly independent. Then there exist $(s^*, t^*, y^*) \in K(x_0)$, $(w, v) \in \mathbb{R}^n \times \mathbb{R}^n$, and $\mu^* \in \mathbb{R}_+^p$ such that*

$$\begin{aligned} & \nabla \left(\frac{\sum_{i=1}^{s^*} t_i^* f(x_0, y_i^*) + x_0^\top Bw + \sum_{j=1}^p \mu_j^* g_j(x_0)}{\sum_{i=1}^{s^*} t_i^* (h(x_0, y_i^*) - x_0^\top Dv)} \right) = 0, \\ & \sum_{j=1}^p \mu_j^* g_j(x_0) = 0, \\ & t_i^* \geq 0, \quad \sum_{i=1}^{s^*} t_i^* = 1, \\ & w^\top Bw \leq 1, \quad v^\top Dv \leq 1, \\ & x_0^\top Bw = \left(x_0^\top Bx_0 \right)^{1/2}, \\ & x_0^\top Dv = \left(x_0^\top Bx_0 \right)^{1/2}. \end{aligned} \quad (6.1)$$

In this section, we present the Wolfe dual (WD) to the minimax program (P):

$$\max_{(s,t,y) \in K(z)} \sup_{(z,\mu,w,v) \in H_3(s,t,y)} \mathcal{A}(z), \quad (WD)$$

where $H_3(s,t,y)$ denotes the set of $(z,\mu,w,v) \in \mathbb{R}^n \times \mathbb{R}_+^p \times \mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\nabla \left(\frac{\sum_{i=1}^s t_i (f(z, y_i) + z^\top Bw) + \sum_{j=1}^p \mu_j g_j(z)}{\sum_{i=1}^s t_i (h(z, y_i) - z^\top Dv)} \right) = 0, \quad (s,t,y) \in K, \quad (6.2)$$

$$w^\top Bw \leq 1, \quad v^\top Dv \leq 1,$$

$$z^\top Bw = \left(z^\top Bz \right)^{1/2}, \quad (6.3)$$

$$z^\top Dv = \left(z^\top Dz \right)^{1/2},$$

$$\mathcal{A}(z) = \frac{\sum_{i=1}^s t_i (f(z, y_i) + z^\top Bw) + \sum_{j=1}^p \mu_j g_j(z)}{\sum_{i=1}^s t_i (h(z, y_i) - z^\top Dv)}. \quad (6.4)$$

If for a triplet $(s,t,y) \in K(z)$ the set $H_3(s,t,y)$ is empty, then we define the supremum over it to be $-\infty$.

Let $\bar{\Gamma}$ denote the set of all feasible points of (WD). Moreover, we denote $pr_X \bar{\Gamma} = \{z \in X \mid (z,\mu,s,t,y,w,v) \in \bar{\Gamma}\}$.

We assume throughout this section that $\sum_{i=1}^s t_i (f(z, y_i) + z^\top Bw) + \sum_{j=1}^p \mu_j g_j(z) \geq 0$ and $\sum_{i=1}^s t_i (h(z, y_i) - z^\top Dv) > 0$.

We establish the weak, strong, and strict converse duality theorems for (WD) with respect to the primal problem (P).

Theorem 6.2 (weak duality). *Let x and (z,μ,s,t,y,w,v) be (P)-feasible and (WD)-feasible, respectively, and assume that*

$$\begin{aligned} \Upsilon_{10}(\cdot) = & \left[\sum_{i=1}^s t_i (h(z, y_i) - z^\top Dv) \right] \left[\sum_{i=1}^s t_i (f(\cdot, y_i) + (\cdot)^\top Bw) + \sum_{j=1}^p \mu_j g_j(\cdot) \right] \\ & - \left[\sum_{i=1}^s t_i (f(z, y_i) + z^\top Bw) + \sum_{j=1}^p \mu_j g_j(z) \right] \left[\sum_{i=1}^s t_i (h(\cdot, y_i) - (\cdot)^\top Dv) \right] \end{aligned} \quad (6.5)$$

is B -(p,r)-invex with respect to η and b at z on $X^\circ \cup pr_X \bar{\Gamma}$; then

$$\sup_{y \in Y} \frac{f(x, y) + (x^\top Bx)^{1/2}}{h(x, y) - (x^\top Dx)^{1/2}} \geq \mathcal{A}(z). \quad (6.6)$$

Proof. Suppose on the contrary that

$$\sup_{y \in Y} \frac{f(x, y) + (x^\top Bx)^{1/2}}{h(x, y) - (x^\top Dx)^{1/2}} < A(z). \quad (6.7)$$

Hence, we have an inequality

$$\frac{f(x, y) + (x^\top Bx)^{1/2}}{h(x, y) - (x^\top Dx)^{1/2}} < A(z) \quad \forall y \in Y. \quad (6.8)$$

Then, we get

$$\begin{aligned} & \left(f(x, y) + (x^\top Bx)^{1/2} \right) \left\{ \sum_{i=1}^s t_i (h(z, y_i) - z^\top Dv) \right\} \\ & - \left(h(x, y) - (x^\top Dx)^{1/2} \right) \left\{ \sum_{i=1}^s t_i (f(z, y_i) + z^\top Bw) + \sum_{j=1}^p \mu_j g_j(z) \right\} < 0, \end{aligned} \quad (6.9)$$

for all $y \in Y$.

If y is replaced by y_i in the above inequality and is multiplied by t_i , then summing up, we get

$$\begin{aligned} & \sum_{i=1}^s t_i \left(f(x, y_i) + (x^\top Bx)^{1/2} \right) \left\{ \sum_{i=1}^s t_i (h(z, y_i) - z^\top Dv) \right\} \\ & - \left\{ \sum_{i=1}^s t_i (h(x, y_i) - (x^\top Dx)^{1/2}) \right\} \left\{ \sum_{i=1}^s t_i (f(z, y_i) + z^\top Bw) + \sum_{j=1}^p \mu_j g_j(z) \right\} < 0. \end{aligned} \quad (6.10)$$

By (2.6) and (6.3), we have

$$\begin{aligned} \Upsilon_{10}(x) & \leq \left\{ \sum_{i=1}^s t_i (f(x, y_i) + (x^\top Bx)^{1/2}) + \sum_{j=1}^p \mu_j g_j(x) \right\} \left\{ \sum_{i=1}^s t_i (h(z, y_i) - z^\top Dv) \right\} \\ & - \left\{ \sum_{i=1}^s t_i (h(x, y_i) - (x^\top Dx)^{1/2}) \right\} \left\{ \sum_{i=1}^s t_i (f(z, y_i) + z^\top Bw) + \sum_{j=1}^p \mu_j g_j(z) \right\} \\ & < \sum_{j=1}^p \mu_j g_j(x) \left\{ \sum_{i=1}^s t_i (h(z, y_i) - z^\top Dv) \right\}. \end{aligned} \quad (6.11)$$

Since $\sum_{i=1}^s t_i (h(z, y_i) - z^\top Dv) > 0$ and $\sum_{j=1}^p \mu_j g_j(x) \leq 0$, it follows that

$$\Upsilon_{10}(x) < 0 = \Upsilon_{10}(z). \quad (6.12)$$

Using the B -(p, r)-invexity with respect to η and b at z of $Y_{10}(\cdot)$, we have the inequality

$$\frac{1}{p} \nabla Y_{10}(z) \left(e^{p\eta(x,z)} - 1 \right) < 0. \quad (6.13)$$

This contradicts the equality of (6.2) $\times (1/p)(e^{p\eta(x,z)} - 1)$. Here the proof is complete. $\square$

As a consequence of Theorems 6.1 and 6.2, we obtain Theorem 6.3. By a similar way, we can prove the strong duality and strictly converse duality theorems with respect to (P) and (WD) which we state as follows.

Theorem 6.3 (strong duality). *Let x^* be an optimal solution of (P) satisfying the hypothesis of Theorem 6.2. Then there exist $(s^*, t^*, y^*) \in K(x^*)$ and $(x^*, \mu^*, w^*, v^*) \in H_3(s^*, t^*, y^*)$ such that $(x^*, \mu^*, s^*, t^*, y^*, w^*, v^*)$ is feasible for (WD). If any of the conditions of Theorem 6.2 hold, then $(x^*, \mu^*, s^*, t^*, y^*, w^*, v^*)$ is an optimal solution of (WD) and the two problems (P) and (WD) have the same extremal values.*

Theorem 6.4 (strict converse duality). *Let $\bar{x}$ and $(z^*, \mu^*, s^*, t^*, y^*, w^*, v^*)$ be optimal solutions of (P) and (WD), respectively. Suppose that the assumptions of Theorem 6.3 are fulfilled and*

$$\begin{aligned} Y_{11}(\cdot) = & \left[\sum_{i=1}^{s^*} t_i^* \left(h(z^*, y_i^*) - (z^*)^\top Dv^* \right) \right] \left[\sum_{i=1}^{s^*} t_i^* \left(f(\cdot, y_i^*) + (\cdot)^\top Bw^* \right) + \sum_{j=1}^p \mu_j^* g_j(\cdot) \right] \\ & - \left[\sum_{i=1}^{s^*} t_i^* \left(f(z^*, y_i^*) + (z^*)^\top Bw^* \right) + \sum_{j=1}^p \mu_j^* g_j(z^*) \right] \left[\sum_{i=1}^s t_i^* \left(h(\cdot, y_i^*) - (\cdot)^\top Dv^* \right) \right] \end{aligned} \quad (6.14)$$

is strictly B -(p, r)-invex with respect to η and b at z^ on $X^\circ \cup pr_X \bar{\Gamma}$.*

Then $\bar{x} = z^*$, that is, z^* is an optimal solution of (P) and $\sup_{y \in Y} ((f(z^*, y) + ((z^*)^\top Bz^*)^{1/2}) / (h(z^*, y) - ((z^*)^\top Dv^*)^{1/2})) = \mathcal{A}(z^*)$.

Acknowledgment

The author is partly supported by NSC, Taiwan

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