Research Article

The Fundamental Groups of $m$-Quasi-Einstein Manifolds

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In Ricci flow theory, the topology of Ricci soliton is important. We call a metric quasi-Einstein if the $m$-Bakry-Emery Ricci tensor is a constant multiple of the metric tensor. This is a generalization of gradient shrinking Ricci soliton. In this paper, we will prove the finiteness of the fundamental group of $m$-quasi-Einstein with a positive constant multiple.

1. Introduction and Main Results

Ricci flow is introduced in 1982 and developed by Hamilton (cf. [1]):

$$\frac{\partial}{\partial t}g = -2\text{Ric},$$

$$g(0) = g_0.$$  \hspace{1cm} (1.1)

Recently, Perelman supplemented Hamilton’s result and solved the Poincaré Conjecture and the Geometrization Conjecture by using a Ricci flow theory. But in higher dimension greater than 4 classification using Ricci flow is still far-off. Most above all the classification of Ricci solitons, which are singularity models, is not completed. But there exist many properties of Ricci solitons. Here we say $g$ is a Ricci soliton if $(M, g)$ is a Riemannian manifold such that the identity

$$\text{Ric} + L_x g = cg$$  \hspace{1cm} (1.2)
holds for some constant $c$ and some complete vector field $X$ on $M$. If $c > 0$, $c = 0$, or $c < 0$, then we call it shrinking, steady, or expanding. Moreover, if the vector field $X$ appearing in (1.2) is the gradient field of a potential function $(1/2)f$, one has $\text{Ric} + \nabla\nabla f = cg$ and says $g$ is a gradient Ricci soliton. In 2008, López and Rio have shown that if $(M, g)$ is a complete manifold with $\text{Ric} + L_x g \geq cg$ and some positive constant $c$, then $M$ is compact if and only if $||X||$ is bounded. Moreover, under these assumptions if $M$ is compact, then $\pi_1(M)$ is finite. Furthermore, Wylie [2] has shown that under these conditions if $M$ is complete, then $\pi_1(M)$ is finite. Moreover, in 2008, Fang et al. (cf. [3]) have shown that a gradient shrinking Ricci soliton with a bounded scalar curvature has finite topological type. By [4, Proposition 1.5.6], Cao and Zhu have shown that compact steady or expanding Ricci solitons are Einstein manifolds. In addition by [4, Corollary 1.5.9 (ii)] note that compact shrinking Ricci solitons are gradient Ricci solitons. So we are interested in shrinking gradient Ricci solitons. In [6, page 354], Eminenti et al. have shown that compact shrinking Ricci solitons have positive scalar curvatures. In [6] Case et al. have shown that an $m$-quasi-Einstein with $1 \leq m < \infty$ and $c > 0$ has a positive scalar curvature. Let me introduce the definition of $m$-quasi-Einstein.

**Definition 1.1.** The triple $(M, g, f)$ is an $m$-quasi-Einstein manifold if it satisfies the equation

$$\text{Ric} + \text{Hess } f - \frac{1}{m} df \otimes df = cg$$

(1.3)

for some $c \in \mathbb{R}$.

Here $m$-Bakry-Emery Ricci tensor $\text{Ric}_f^m \doteq \text{Ric} + \text{Hess } f - (1/m) df \otimes df$ for $0 < m \leq \infty$ is a natural extension of the Ricci tensor to smooth metric measure spaces (cf. [6, Section 1]). Note that if $m = \infty$, then it reduces to a gradient Ricci soliton. In this paper, we will prove the finiteness of the fundamental group of an $m$-quasi-Einstein with $c > 0$.

**Theorem 1.2.** Let $(M, g, f)$ be a complete manifold with $c > 0$ and $\text{Ric} + \text{Hess } f - (1/m) df \otimes df \geq cg$. Then it has a finite fundamental group.

2. The Proof of Theorem 1.2

The proof of Theorem 1.2 is similar to the proofs of [2, 7].

Proof. We will prove it by dividing into two cases.

**Case 1.** $||\nabla f||$ is bounded. We claim that the bounded $||\nabla f||$ implies the compactness of $M$. Let $q$ be a point in $M$, and consider any geodesic $\gamma : [0, \infty) \to M$ emanating from $q$ and parametrized by arc length $t$. Then we have

$$\int_0^T \text{Ric}(\dot{\gamma}, \dot{\gamma}) \geq cT + \frac{1}{m} \int_0^T (df(\dot{\gamma}))^2 - \int_0^T g(\nabla f, \dot{\gamma}) \geq cT - g(\nabla f, \dot{\gamma})_0^T.$$

(2.1)

Since $g(\nabla f, \dot{\gamma})_0^T$ is bounded we have that $\int_0^\infty \text{Ric}(\dot{\gamma}, \dot{\gamma}) = \infty$. Hence, the claim is followed by the proof of [4, Theorem 1]. Let $(\tilde{M}, \tilde{g})$ be the Riemannian universal cover of $(M, g)$, let $p : (\tilde{M}, \tilde{g}) \to (M, g)$ be a projection map, and let $\tilde{f}$ be a map $f \circ p$. Since $p$ is a local isometry, then the same inequality holds, that is, $\text{Ric}(\tilde{g}) + \text{Hess}_\tilde{g} \tilde{f} - (1/m) d\tilde{f} \otimes d\tilde{f} \geq c\tilde{g}$. Now, since
Note that by \( \| \nabla \tilde{f} \| \) is bounded, it is followed from the above argument that \( \tilde{M} \) is compact. So \( \pi_1(M) \) is finite.

Case 2. \( \| \nabla f \| \) is unbounded. We will prove this case by following the proof of [2]. By Case 1, \( M \) is noncompact. For any \( p \in M \), define

\[
H_p = \max\{0, \sup \{ \text{Ric}_y(v, v) : y \in B(p, 1), \|v\| = 1 \} \}.
\]  
(2.2)

Note that by [7, Lemma 2.2] we have

\[
\int_0^r \text{Ric}(\hat{\gamma}, \hat{\gamma}) ds \leq 2(n - 1) + H_p + H_q.
\]  
(2.3)

Assume that \( d(p, q) > 1 \). On the other hand, from the inequality of Theorem 1.2, we have

\[
\begin{align*}
\int_0^r \text{Ric}(\hat{\gamma}, \hat{\gamma}) ds & \geq cd(p, q) + \frac{1}{m} \int_0^r (df(\hat{\gamma}))^2 - \int_0^r \hat{\gamma}(g(\nabla f, \hat{\gamma})) \\
& \geq cd(p, q) - \| \nabla f \|_p - \| \nabla f \|_q,
\end{align*}
\]  
(2.4)

since \( g(\nabla f, \hat{\gamma}) \leq \| \nabla f \| \| \hat{\gamma} \| \). Hence, we have that for any \( p, q \in M \)

\[
d(p, q) \leq \max \left\{ 1, \frac{1}{c} \left( 2(n - 1) + H_p + H_q + \| \nabla f \|_p + \| \nabla f \|_q \right) \right\}. \tag{2.5}
\]

Now we will apply a similar argument like Case 1. Fix \( \bar{p} \in \tilde{M} \), and let \( h \in \pi_1(M) \) identified as a deck transformation on \( \tilde{M} \). Note that \( B(\bar{p}, 1) \) and \( B(h(\bar{p}), 1) \) are isometric, and thus \( \tilde{H}_\bar{p} = H_{h(\bar{p})} \). Also \( \| \nabla \tilde{f} \|_\bar{p} = \| \nabla \tilde{f} \|_{h(\bar{p})} \). So we conclude that

\[
d(\bar{p}, h(\bar{p})) \leq \max \left\{ 1, \frac{2}{c} \left( 2(n - 1) + H_{\bar{p}} + \| \nabla \tilde{f} \|_{\bar{p}} \right) \right\} \tag{2.6}
\]

for any \( h \in \pi_1(M) \). Since the right-hand side is independent of \( h \), this proves this case. \( \square \)

References


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