Research Article

Some Families of Two-Step Simultaneous Methods for Determining Zeros of Nonlinear Equations

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We construct two families of two-step simultaneous methods, one of order four and the other of order six, for determining all the distinct zeros of single variable nonlinear equations. The convergence analysis of both the families of methods and the numerical results are also given in order to demonstrate the efficiency and the performance of the new iterative simultaneous methods.

1. Introduction

Determining the zeros of non-linear equations is among the oldest problems in mathematics, whereas the non-linear equations have wide range of applications in science and engineering.

There are numerical methods which find one root at a time, such as Newton's method, and the methods which find all the roots at a time, namely, simultaneous methods, such as Weierstrass method.

The methods for simultaneous finding of all roots of the non-linear equations are very popular as compared to the methods for individual finding of the roots. These methods have a wider region of convergence, are more stable, and can be implemented for parallel computing. More details on simultaneous methods, their convergence properties, computational efficiency, and parallel implementation may be found in [1–5] and references cited therein.

The main goal of this paper is to develop simultaneous methods which have high convergence order and computational efficiency.

2. A Family of Two-Step Fourth-Order Simultaneous Methods for Distinct Roots

In this section, we develop a family of new two-step iterative methods for the simultaneous approximation of all zeros of a non-linear equation using Weierstrass' correction.

There exist many one-step iterative methods for extracting roots of non-linear equations. We consider the following two-step alpha family:

$$\begin{aligned} y_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= y_i - \frac{f(y_i)}{f'(y_i) - \alpha f(y_i)}, \end{aligned} \quad (2.1)$$

where α is any arbitrary real parameter. This family was proposed by Li et al. [6] and has fourth-order convergence. If $\alpha = 0$, then (2.1) reduces to the following iterative method:

$$\begin{aligned} y_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= y_i - \frac{f(y_i)}{f'(y_i)}. \end{aligned} \quad (2.2)$$

This is a well-known two-step Newton's method presented and considered by Traub [7] and has a fourth-order convergence.

Let

$$W_i(x_i) = \frac{f(x_i)}{\prod_{j=1}^n (x_i - x_j)} \quad (\text{Weierstrass' correction}). \quad (2.3)$$

Equation (2.1) can be written as

$$\begin{aligned} y_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= y_i - \frac{f(y_i)/f'(y_i)}{1 - \alpha f(y_i)/f'(y_i)}. \end{aligned} \quad (2.4)$$

Replacing f/f' by W_i in (2.4), we get

$$\begin{aligned} y_i &= x_i - W_i(x_i), \\ z_i &= y_i - \frac{W_i(y_i)}{1 - \alpha W_i(y_i)}, \end{aligned} \quad (2.5)$$

where $W_i(x_i)$ and $W_i(y_i)$ are given by (2.3).

Hence, we get a family of new two-step iterative methods (2.5) abbreviated as MR1, which depends upon a real parameter alpha for extracting all roots of a non-linear equation.

Remark 2.1. Two-step Weierstrass' method is a special case of our family of methods if $\alpha = 0$, in (2.5).

3. Convergence Analysis

In this section, we prove the following theorem on the convergence order of family of two-step simultaneous methods (2.5).

Theorem 3.1. *Let $f(x) = 0$ be a non-linear equation with n number of simple roots $\xi_1, \xi_2, \dots, \xi_n$. If $x_1, x_2, \dots, x_n$ are the initial approximations of the roots, respectively, then, for arbitrary α and sufficiently close initial approximations, the order of convergence of (2.5) equals four.*

Proof. We denote, $e_i = x_i - \xi_i$, $e'_i = y_i - \xi_i$, and $\hat{e}_i = z_i - \xi_i$.

Considering the first equation of (2.5),

$$y_i = x_i - W_i(x_i), \quad (3.1)$$

we have that

$$\begin{aligned} e'_i &= e_i - W_i(x_i), \\ &= e_i(1 - A_i), \end{aligned} \quad (3.2)$$

where

$$A_i = \frac{W_i(x_i)}{e_i} = \prod_{\substack{j \neq i \\ j=1}}^n \left(\frac{x_i - \xi_j}{x_i - x_j} \right). \quad (3.3)$$

Now, if ξ_i is a simple root, then, for small enough ϵ , $|x_i - x_j|$ is bounded away from zero, and so

$$\begin{aligned} \left(\frac{x_i - \xi_j}{x_i - x_j} \right) &= 1 + \left(\frac{x_j - \xi_j}{x_i - x_j} \right) = 1 + O(\epsilon), \\ \prod_{\substack{j \neq i \\ j=1}}^n \frac{(x_i - \xi_j)}{(x_i - x_j)} &= (1 + O(\epsilon))^{n-1} = 1 + (n-1)O(\epsilon) + \dots = 1 + O(\epsilon). \end{aligned} \quad (3.4)$$

This implies that

$$A_i = 1 + O(\epsilon). \quad (3.5)$$

Hence,

$$A_i - 1 = O(\epsilon). \quad (3.6)$$

Thus, (3.2) gives

$$\epsilon'_i = O(\epsilon^2). \quad (3.7)$$

Now, considering second equation of (2.5), we have that

$$\begin{aligned} \hat{\epsilon}_i &= \epsilon'_i - \frac{W_i(y_i)}{1 - \alpha W_i(y_i)}, \\ &= \epsilon'_i \left(1 - \frac{B_i}{1 - \alpha W_i(y_i)} \right), \\ &= \epsilon'_i [1 - B_i (1 - \alpha W_i(y_i))^{-1}], \\ &= \epsilon'_i [1 - B_i (1 + \alpha W_i(y_i) + \cdots)], \\ &= \epsilon'_i [1 - B_i - \alpha W_i(y_i) B_i - \cdots], \end{aligned} \quad (3.8)$$

where

$$B_i = \frac{W_i(y_i)}{\epsilon'_i} = \prod_{\substack{j \neq i \\ j=1}}^n \left(\frac{y_i - \xi_j}{y_i - y_j} \right). \quad (3.9)$$

With the same arguments as in (3.2), we have that

$$B_i = 1 + O(\epsilon^2). \quad (3.10)$$

Now,

$$W_i = \frac{f(y_i)}{\prod_{j \neq i}^n (y_i - y_j)} = \frac{\prod_{j=1}^n (y_i - \xi_j)}{\prod_{\substack{j=1 \\ j \neq i}}^n (y_i - y_j)} = \epsilon'_i \prod_{\substack{j=1 \\ j \neq i}}^n \left(\frac{y_i - \xi_j}{y_i - y_j} \right). \quad (3.11)$$

Using (3.9), this implies that

$$W_i = \epsilon'_i B_i. \quad (3.12)$$

Hence, (3.8) gives

$$\hat{\epsilon}_i = \epsilon'_i [1 - B_i - \alpha \epsilon'_i B_i^2 - \cdots]. \quad (3.13)$$

Now, $\epsilon'_i = O(\epsilon^2)$ and $B_i = 1 + O(\epsilon^2)$. Thus

$$\hat{\epsilon}_i = O(\epsilon^4), \quad (3.14)$$

which proves the theorem. $\square$

4. A Family of Two-Step Sixth-Order Simultaneous Methods for Distinct Roots

Here, we develop a new family of two-step simultaneous iterative methods of order six using the same alpha family (2.1).

Consider

$$N_i(x_i) = \frac{f(x_i)}{f'(x_i)} \quad (\text{Newton's correction}), \quad (4.1)$$

$$W_i(x_i) = \frac{f(x_i)}{\prod_{j=1}^n (x_i - x_j)} \quad (\text{Weierstrass' correction}). \quad (4.2)$$

Taking logarithmic derivatives of (4.2), we have that

$$\frac{W'_i(x_i)}{W_i(x_i)} = \frac{f'(x_i)}{f(x_i)} - \sum_{j=1}^n \frac{1}{(x_i - x_j)}. \quad (4.3)$$

Replacing f'/f by W'_i/W_i in (2.4), we have that

$$\begin{aligned} y_i &= x_i - \frac{1}{W'_i(x_i)/W_i(x_i)}, \\ z_i &= y_i - \frac{1}{(W'_i(y_i)/W_i(y_i)) - \alpha}. \end{aligned} \quad (4.4)$$

Using (4.1) and (4.3) in the above, we have that

$$\begin{aligned} y_i &= x_i - \frac{1}{(1/N_i(x_i)) - \sum_{j=1}^n 1/(x_i - x_j)}, \\ z_i &= y_i - \frac{1}{(1/N_i(y_i)) - \sum_{j=1}^n 1/(y_i - y_j) - \alpha}. \end{aligned} \quad (4.5)$$

Thus, we get a family of new simultaneous iterative methods (4.5), abbreviated as MR2.

Remark 4.1. Two-step Ehrlich-Alberth's method is a special case of our method, if $\alpha = 0$, in (4.5).

5. Convergence Analysis

Here, we prove the following theorem on the convergence order of a family of two-step simultaneous methods (4.5).

Theorem 5.1. *Let $f(x) = 0$ be a non-linear equation with n number of simple roots $\xi_1, \xi_2, \dots, \xi_n$. If $x_1, x_2, \dots, x_n$ are the initial approximations of the roots, respectively, then, for arbitrary α and sufficiently close initial approximations, the order of convergence of family (4.5) is six.*

Proof. Consider the Newton correction

$$N_i(x_i) = \frac{f(x_i)}{f'(x_i)}. \quad (5.1)$$

This implies that

$$\frac{1}{N_i(x_i)} = \frac{f'(x_i)}{f(x_i)} = \sum_{j=1}^n \frac{1}{(x_i - \xi_j)}, \quad (5.2)$$

where ξ_i is the exact root and x_i is its approximation. This gives

$$\frac{1}{N_i(x_i)} = \frac{1}{x_i - \xi_i} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{(x_i - \xi_j)}. \quad (5.3)$$

Using (5.3) in (4.5), we have that

$$y_i = x_i - \frac{1}{1/(x_i - \xi_i) + \sum_{j=1}^n 1/(x_i - \xi_j) - \sum_{j=1}^n 1/(x_i - x_j)}. \quad (5.4)$$

Let

$$e_i = x_i - \xi_i, \quad e'_i = y_i - \xi_i. \quad (5.5)$$

Thus, from (5.4), we have that

$$\begin{aligned} e'_i &= e_i - \frac{1}{1/e_i + \sum_{j=1}^n 1/(x_i - \xi_j) - \sum_{j=1}^n 1/(x_i - x_j)}, \\ &= e_i - \frac{e_i}{1 + e_i \sum_{j=1}^n [(x_i - x_j - x_i - \xi_j)/(x_i - \xi_j)(x_i - x_j)]} \\ &= e_i - \frac{e_i}{1 + e_i \sum_{j=1}^n [-(x_j - \xi_j)/(x_i - \xi_j)(x_i - x_j)]} \\ &= e_i - \frac{e_i}{1 + e_i \sum_{j=1}^n e_j A_{ij}}, \end{aligned} \quad (5.6)$$

where

$$A_{ij} = \frac{-1}{(x_i - \xi_j)(x_i - x_j)}. \quad (5.7)$$

On simplification, we have that

$$\epsilon'_i = \frac{\epsilon_i^2 \sum_{j=1}^n \epsilon_j A_{ij}}{1 + \epsilon_i \sum_{j=1}^n \epsilon_j A_{ij}}. \quad (5.8)$$

If we assume that absolute values of all errors ϵ_j ($j = 1, \dots, n$) are of the same order as, say $|\epsilon_j| = O(|\epsilon|)$, then

$$\epsilon'_i = \frac{\epsilon_i^2 \sum_{j=1}^n \epsilon_j A_{ij}}{1 + \epsilon_i \sum_{j=1}^n \epsilon_j A_{ij}} = O(\epsilon^3). \quad (5.9)$$

Now considering the second equation of (4.5), we have that

$$z_i = y_i - \frac{1}{(1/N_i) - \sum_{j=1}^n \frac{1}{(y_i - y_j)} - \alpha}. \quad (5.10)$$

This gives

$$z_i = y_i - \frac{1}{1/(y_i - \xi_i) + \sum_{j=1}^n \frac{1}{(y_i - \xi_j)} - \sum_{j=1}^n \frac{1}{(y_i - y_j)} - \alpha}. \quad (5.11)$$

This implies that

$$\begin{aligned} \hat{\epsilon}_i &= \epsilon'_i - \frac{1}{1/\epsilon'_i + \sum_{j=1}^n \frac{1}{(y_i - \xi_j)} - \sum_{j=1}^n \frac{1}{(y_i - y_j)} - \alpha} \\ &= \epsilon'_i - \frac{\epsilon'_i}{1 + \epsilon'_i \left[\sum_{j=1}^n (y_i - y_j - y_i + \xi_j) / (y_i - \xi_j)(y_i - y_j) \right] - \alpha \epsilon'_i} \\ &= \epsilon'_i - \frac{\epsilon'_i}{1 + \epsilon'_i \left[\sum_{j=1}^n -(y_j - \xi_j) / (y_i - \xi_j)(y_i - y_j) \right] - \alpha \epsilon'_i}. \end{aligned} \quad (5.12)$$

Let

$$B_{ij} = \frac{-1}{(y_i - \xi_j)(y_i - y_j)}. \quad (5.13)$$

Hence, the above equation becomes

$$\hat{e}_i = e'_i - \frac{e'_i}{1 + e'_i \left[\sum_{j=1}^n e'_j B_{ij} \right] - \alpha e'_i}. \quad (5.14)$$

On simplification, we have that

$$\hat{e}_i = \frac{(e'_i)^2 \left[\sum_{j=1}^n e'_j B_{ij} - \alpha \right]}{1 + e'_i \left[\sum_{j=1}^n e'_j B_{ij} - \alpha \right]}. \quad (5.15)$$

Since $e'_i = O(\epsilon^3)$, from (5.9),

$$\hat{e}_i = \frac{O(\epsilon^3)^2 \left[\sum_{j=1}^n e'_j B_{ij} - \alpha \right]}{1 + e'_i \left[\sum_{j=1}^n e'_j B_{ij} - \alpha \right]} = O(\epsilon^6). \quad (5.16)$$

This shows that our family of two-step simultaneous methods (4.5) has sixth-order convergence. $\square$

6. Numerical Results

We consider here some numerical examples in order to demonstrate the performance of our family of fourth- and sixth-order two-step simultaneous methods, namely MR1 (2.5) and MR2 (4.5). We compare our family of methods with Zhang et al. method of fifth-order convergence and use the abbreviations as ZPH [8].

All the computations are performed using Maple 7.0, using 64 digits floating point arithmetic. We take $\epsilon_i = 10^{-30}$ as tolerance and use the following stopping criteria for estimating the zeros:

$$e_i = \left| z_i^{(n+1)} - z_i^{(n)} \right| < \epsilon_i, \quad \text{for each } i, \quad (6.1)$$

and e_i represents the absolute error.

In all the examples for MR1 and MR2, we have taken $\alpha = 0.5$.

Table 1

Methods	Iterations	e_1	e_2	e_3	e_4
MR1	5	$0.240035e-71$	$0.946203e-62$	$0.218698e-65$	$0.617862e-64$
ZPH	4	$0.100000e-17$	0.0	$0.154240e-18$	$0.100000e-18$
MR2	4	$0.495683e-67$	$0.152442e-73$	$0.159123e-96$	$0.161738e-78$

Example 6.1. Consider

$$f(z) = z^4 - 1 \quad (6.2)$$

with the exact zeros

$$\xi_1 = 1, \quad \xi_2 = -1, \quad \xi_3 = i, \quad \xi_4 = -i. \quad (6.3)$$

The initial approximations have been taken as:

$$z_1^{(0)} = 0.5 + 0.5i, \quad z_2^{(0)} = -1.36 + 0.42i, \quad z_3^{(0)} = -0.25 + 1.28i, \quad z_4^{(0)} = 0.46 - 1.37i. \quad (6.4)$$

The numerical comparison is given in Table 1.

Example 6.2. Consider

$$f(z) = z^7 + z^5 - 10z^4 - z^3 - z + 10 \quad (6.5)$$

with the exact zeros

$$\xi_1 = 2, \quad \xi_2 = 1, \quad \xi_3 = -1, \quad \xi_4 = i, \quad \xi_5 = -i, \quad \xi_6 = -1 + 2i, \quad \xi_7 = -1 - 2i. \quad (6.6)$$

The initial approximations have been taken as

$$\begin{aligned} z_1^{(0)} &= 1.66 + 0.23i, & z_2^{(0)} &= 1.36 - 0.31i, & z_3^{(0)} &= -0.76 + 0.18i, & z_4^{(0)} &= -0.35 + 1.17i, \\ z_5^{(0)} &= 0.29 - 1.37i, & z_6^{(0)} &= -0.75 + 2.36i, & z_7^{(0)} &= -1.27 - 1.62i. \end{aligned} \quad (6.7)$$

The numerical comparison is shown in Table 2.

Table 2

Methods	It	e_1	e_2	e_3	e_4	e_5	e_6	e_7
MR1	5	$0.567890e-65$	$0.720873e-65$	$0.336459e-76$	0.0	$0.125000e-64$	0.0	0.0
ZPH	3	$0.163337e-5$	$0.232441e-5$	0.0	$0.509990e-10$	$0.454933e-9$	0.0	0.0
MR2	4	$0.867278e-62$	$0.411711e-62$	$0.812855e-90$	0.0	$0.1e-63$	0.0	0.0

Table 3

Methods	Iterations	e_1	e_2	e_3
MR1	6	0.0	0.0	0.0
ZPH	8	0.0	$0.1e-8$	$0.9e-8$
MR2	4	$0.111391e-41$	$0.903529e-43$	$0.194093e-16$

Example 6.3. Consider

$$f(z) = z^3 + 5z^2 - 4z - 20 + \cos(z^3 + 5z^2 - 4z - 20) - 1 \quad (6.8)$$

with the exact zeros

$$\xi_1 = -5, \quad \xi_2 = -2, \quad \xi_3 = 2. \quad (6.9)$$

The initial approximations have been taken as:

$$z_1^{(0)} = -5.1, \quad z_2^{(0)} = -1.8, \quad z_3^{(0)} = 1.9. \quad (6.10)$$

The numerical comparison is shown in Table 3.

7. Conclusions

We have developed here two families of two-step simultaneous iterative methods of order four and six for determination of all the distinct zeros of non-linear equations. Weierstrass' two-step and Ehrlich-Alberth's two-step simultaneous methods are special cases of our family of methods. From Tables 1–3, we observe that our methods are very effective and efficient as compared to fifth-order simultaneous method of Zhang et al. [8]. Our results can be considered as an improvement and generalization of the previously known results in the existing literature.

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