Research Article

Existence Results for Boundary Value Problems of Differential Inclusions with Three-Point Integral Boundary Conditions

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We discuss the existence of solutions for a boundary value problem of second-order differential inclusions with three-point integral boundary conditions involving convex and nonconvex multivalued maps. Our results are based on the nonlinear alternative of Leray-Schauder type and some suitable theorems of fixed point theory.

1. Introduction

Boundary value problems for nonlinear differential equations arise in a variety of areas of applied mathematics, physics, and variational problems of control theory. A point of central importance in the study of nonlinear boundary value problems is to understand how the properties of nonlinearity in a problem influence the nature of the solutions to the boundary value problems. The topic of multipoint nonlocal boundary conditions, initiated by Bicadze and Samarski [1], has been addressed by many authors, for instance, [2–13]. The multi-point boundary conditions appear in certain problems of thermodynamics, elasticity, and wave propagation; see [5] and the references therein. The multi-point boundary conditions may be understood in the sense that the controllers at the end points dissipate or add energy according to sensors located at intermediate positions. However, much of the literature dealing with three-point boundary value problems involves the three-point boundary condition restrictions on the solution or gradient of the solution of the problem.

In this paper, we consider the following second-order differential inclusion with three-point integral boundary conditions:

$$\begin{aligned} -x''(t) &\in F(t, x(t)), \quad 0 < t < 1, \\ x(0) &= 0, \quad x(1) = \alpha \int_0^\eta x(s) ds, \quad 0 < \eta < 1, \end{aligned} \tag{1.1}$$

where $\alpha \in \mathbb{R}$ is such that $\alpha \neq 2/\eta^2$, $F : [0, T] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map, and $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$. We emphasize that the present work is motivated by [14], where the authors discussed the existence of positive solutions for the problem (1.1) with $F(t, x(t))$ as a single-valued map (i.e., $F(t, x(t)) = a(t)f(x(t))$). Note that the three-point boundary condition in (1.1) corresponds to the area under the curve of solutions $x(t)$ from $t = 0$ to $t = \eta$.

Differential inclusions arise in the mathematical modelling of certain problems in economics, optimal control, stochastic analysis, and so forth and are widely studied by many authors, see [15–21] and the references therein.

The aim of our paper is to present existence results for the problem (1.1), when the right-hand side is convex as well as nonconvex valued. The first result relies on the nonlinear alternative of Leray-Schauder type. In the second result, we will combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while, in the third result, we will use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. The methods used are standard; however, their exposition in the framework of problem (1.1) is new.

The paper is organized as follows. In Section 2 we recall some preliminary facts that we need in the sequel, and in Section 3 we prove our main results.

2. Preliminaries

Let us recall some basic definitions on multivalued maps [22, 23].

For a normed space $(X, \|\cdot\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \rightarrow \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \bigcup_{x \in \mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e., $\sup_{x \in \mathbb{B}} \{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X and if, for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, that is, $x_n \rightarrow x_*$, $y_n \rightarrow y_*$, $y_n \in G(x_n)$ imply that $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\text{Fix } G$. A multivalued map $G : [0; 1] \rightarrow P_{cl}(\mathbb{R})$ is said to be measurable if, for every $y \in \mathbb{R}$, the

function

$$t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\} \quad (2.1)$$

is measurable.

Let $C([0, 1])$ denote a Banach space of continuous functions from $[0, 1]$ into $\mathbb{R}$ with the norm $\|x\|_\infty = \sup_{t \in [0, 1]} |x(t)|$. Let $L^1([0, 1], \mathbb{R})$ be the Banach space of measurable functions $x : [0, 1] \rightarrow \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)| dt$.

Definition 2.1. A multivalued map $F : [0, T] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

- (i) $t \mapsto F(t, x)$ is measurable for each $x \in \mathbb{R}$,
- (ii) $x \mapsto F(t, x)$ is upper semicontinuous for almost all $t \in [0, T]$, and further a Carathéodory function F is called L^1 -Carathéodory if
- (iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0, T], \mathbb{R}^+)$ such that

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \leq \varphi_\alpha(t) \quad (2.2)$$

for all $\|x\|_\infty \leq \alpha$ and for a.e. $t \in [0, T]$.

For each $y \in C([0, 1], \mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \left\{ v \in L^1([0, 1], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0, 1] \right\}. \quad (2.3)$$

Let X be a nonempty closed subset of a Banach space E and $G : X \rightarrow \mathcal{P}(E)$ a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E . Let A be a subset of $[0, 1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ -algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0, 1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0, 1], \mathbb{R})$ is decomposable if, for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0, 1] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 2.2. Let Y be a separable metric space and let $N : Y \rightarrow \mathcal{P}(L^1([0, 1], \mathbb{R}))$ be a multivalued operator. One says that N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0, 1] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values.

Define a multivalued operator $\mathcal{F} : C([0, 1] \times \mathbb{R}) \rightarrow \mathcal{P}(L^1([0, 1], \mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \left\{ w \in L^1([0, 1], \mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0, 1] \right\}, \quad (2.4)$$

which is called the Nemytskii operator associated with F .

Definition 2.3. Let $F : [0, 1] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. One says that F is of lower semi-continuous type (l.s.c. type) if its associated

Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Let (X, d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}, \quad (2.5)$$

where $d(A, b) = \inf_{a \in A} d(a, b)$ and $d(a, B) = \inf_{b \in B} d(a, b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [24]).

Definition 2.4. A multivalued operator $N : X \rightarrow P_{cl}(X)$ is called

(a) γ -Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y) \text{ for each } x, y \in X, \quad (2.6)$$

(b) a contraction if and only if it is γ -Lipschitz with $\gamma < 1$.

The following lemmas will be used in the sequel.

Lemma 2.5 (see [25]). *Let X be a Banach space. Let $F : [0, T] \times \mathbb{R} \rightarrow \mathcal{P}_{cp,c}(X)$ be an L^1 -Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0, 1], X)$ to $C([0, 1], X)$. Then the operator*

$$\Theta \circ S_F : C([0, 1], X) \rightarrow P_{cp,c}(C([0, 1], X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x}) \quad (2.7)$$

is a closed graph operator in $C([0, 1], X) \times C([0, 1], X)$.

Lemma 2.6 (see [26]). *Let Y be a separable metric space, and let $N : Y \rightarrow \mathcal{P}(L^1([0, 1], \mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single valued) $g : Y \rightarrow L^1([0, 1], \mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Lemma 2.7 (see [27]). *Let (X, d) be a complete metric space. If $N : X \rightarrow P_{cl}(X)$ is a contraction, then $\text{Fix}N \neq \emptyset$.*

In order to define the solution of (1.1), we consider the following lemma whose proof is given in [14].

Lemma 2.8. *Assume that $\alpha\eta^2 \neq 2$. For a given $y \in C[0, 1]$, the unique solution of the boundary value problem*

$$\begin{aligned} x''(t) + y(t) &= 0, \quad 0 < t < 1, \quad 1 < q \leq 2, \\ x(0) &= 0, \quad x(1) = \alpha \int_0^\eta x(s) \, ds \end{aligned} \quad (2.8)$$

is given by

$$x(t) = \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)y(s)ds - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta-s)^2 y(s)ds - \int_0^t (t-s)y(s)ds. \quad (2.9)$$

Definition 2.9. A function $x \in C^2([0,1], \mathbb{R})$ is a solution of the problem (1.1) if there exists a function $f \in L^1([0,1], \mathbb{R})$ such that $f(t) \in F(t, x(t))$ a.e. on $[0,1]$ and

$$x(t) = \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)f(s)ds - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f(s)ds - \int_0^t (t-s)f(s)ds. \quad (2.10)$$

3. Main Results

Theorem 3.1. Assume that

(H₁) $F : [0,1] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is Carathéodory and has convex values,

(H₂) there exists a continuous nondecreasing function $\psi : [0, \infty) \rightarrow (0, \infty)$ and a function $p \in L^1([0,1], \mathbb{R}^+)$ such that

$$\|F(t, x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t, x)\} \leq p(t)\psi(\|x\|_\infty) \quad \text{for each } (t, x) \in [0,1] \times \mathbb{R}, \quad (3.1)$$

(H₃) there exists a number $M > 0$ such that

$$\frac{M}{(1 + (2 + |\alpha|\eta^2/|2 - \alpha\eta^2|))\psi(M)\|p\|_{L^1}} > 1. \quad (3.2)$$

Then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Define the operator $\Omega : C([0,1], \mathbb{R}) \rightarrow \mathcal{P}(C([0,1], \mathbb{R}))$ by

$$\begin{aligned} & \Omega(x) \\ &= \left\{ h \in C([0,1], \mathbb{R}) : h(t) = \begin{cases} \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)f(s)ds \\ -\frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f(s)ds - \int_0^t (t-s)f(s)ds, \quad t \in [0,1] \end{cases} \right\} \end{aligned} \quad (3.3)$$

for $f \in S_{F,x}$. We will show that Ω satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that Ω is

convex for each $x \in C([0, 1], \mathbb{R})$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that, for each $t \in [0, 1]$, we have

$$h_i(t) = \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)f_i(s)ds - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f_i(s)ds - \int_0^t (t-s)f_i(s)ds, \quad i = 1, 2. \quad (3.4)$$

Let $0 \leq \omega \leq 1$. Then, for each $t \in [0, 1]$, we have

$$\begin{aligned} [\omega h_1 + (1-\omega)h_2](t) &= \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)[\omega f_1(s) + (1-\omega)f_2(s)]ds \\ &\quad - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta-s)^2 [\omega f_1(s) + (1-\omega)f_2(s)]ds \\ &\quad - \int_0^t (t-s)[\omega f_1(s) + (1-\omega)f_2(s)]ds. \end{aligned} \quad (3.5)$$

Since $S_{F,x}$ is convex (F has convex values), it follows that $\omega h_1 + (1-\omega)h_2 \in \Omega(x)$.

Next, we show that Ω maps bounded sets (balls) into bounded sets in $C([0, 1], \mathbb{R})$. For a positive number r , let $B_r = \{x \in C([0, 1], \mathbb{R}) : \|x\|_\infty \leq r\}$ be a bounded ball in $C([0, 1], \mathbb{R})$. Then, for each $h \in \Omega(x)$, $x \in B_r$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)f(s)ds - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f(s)ds - \int_0^t (t-s)f(s)ds, \\ |h(t)| &\leq \psi(\|x\|_\infty) \left(1 + \frac{2}{|2 - \alpha\eta^2|} \right) \int_0^1 p(s)ds + \frac{|\alpha|\eta^2\psi(\|x\|_\infty)}{|2 - \alpha\eta^2|} \int_0^\eta p(s)ds. \end{aligned} \quad (3.6)$$

Thus,

$$\|h\|_\infty \leq \psi(\|x\|_\infty) \left(1 + \frac{2}{|2 - \alpha\eta^2|} \right) \int_0^1 p(s)ds + \frac{|\alpha|\eta^2\psi(\|x\|_\infty)}{|2 - \alpha\eta^2|} \int_0^\eta p(s)ds. \quad (3.7)$$

Now we show that Ω maps bounded sets into equicontinuous sets of $C([0, 1], \mathbb{R})$. Let $t', t'' \in [0, 1]$ with $t' < t''$ and $x \in B_r$, where B_r is a bounded set of $C([0, 1], \mathbb{R})$. For each $h \in \Omega(x)$, we obtain

$$\begin{aligned}
 |h(t'') - h(t')| &= \left| \frac{2(t'' - t')}{2 - \alpha\eta^2} \int_0^1 (1-s)f(s)ds - \frac{\alpha(t'' - t')}{2 - \alpha\eta^2} \int_0^\eta (\eta - s)^2 f(s)ds \right. \\
 &\quad \left. + \int_0^{t''} (t'' - s)f(s)ds + \int_0^{t'} (t' - s)f(s)ds \right| \\
 &\leq \left| \frac{2(t'' - t')\psi(\|x\|_\infty)}{2 - \alpha\eta^2} \int_0^1 (1-s)p(s)ds \int_0^1 (1-s)ds \right| \\
 &\quad + \left| \frac{\alpha(t'' - t')\psi(\|x\|_\infty)}{2 - \alpha\eta^2} \int_0^\eta (\eta - s)^2 p(s)ds \right| \\
 &\quad + \left| \psi(\|x\|_\infty) \int_0^{t'} (t'' - t')p(s)ds \right| + \left| \psi(\|x\|_\infty) \int_{t'}^{t''} (t'' - s)p(s)ds \right|.
 \end{aligned} \tag{3.8}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $t'' - t' \rightarrow 0$. As Ω satisfies the above three assumptions, it follows by the Ascoli-Arzelà theorem that $\Omega : C([0, 1], \mathbb{R}) \rightarrow \mathcal{P}(C([0, 1], \mathbb{R}))$ is completely continuous.

In our next step, we show that Ω has a closed graph. Let $x_n \rightarrow x_*$, $h_n \in \Omega(x_n)$ and $h_n \rightarrow h_*$. Then we need to show that $h_* \in \Omega(x_*)$. Associated with $h_n \in \Omega(x_n)$, there exists $f_n \in S_{F, x_n}$ such that, for each $t \in [0, 1]$,

$$h_n(t) = \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)f_n(s)ds - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta - s)^2 f_n(s)ds - \int_0^t (t-s)f_n(s)ds. \tag{3.9}$$

Thus we have to show that there exists $f_* \in S_{F, x_*}$ such that, for each $t \in [0, 1]$,

$$h_*(t) = \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)f_*(s)ds - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta - s)^2 f_*(s)ds - \int_0^t (t-s)f_*(s)ds. \tag{3.10}$$

Let us consider the continuous linear operator $\Theta : L^1([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ given by

$$\begin{aligned}
 f \mapsto \Theta(f)(t) &= \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)f(s)ds \\
 &\quad - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta - s)^2 f(s)ds - \int_0^t (t-s)f(s)ds.
 \end{aligned} \tag{3.11}$$

Observe that

$$\begin{aligned} \|h_n(t) - h_*(t)\| = & \left\| \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)(f_n(s) - f_*(s))ds - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta-s)^2(f_n(s) - f_*(s))ds \right. \\ & \left. - \int_0^t (t-s)(f_n(s) - f_*(s))ds \right\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \quad (3.12)$$

Thus, it follows by Lemma 2.5 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \rightarrow x_*$, therefore, we have

$$h_*(t) = \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)f_*(s)ds - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f_*(s)ds - \int_0^t (t-s)f_*(s)ds, \quad (3.13)$$

for some $f_* \in S_{F,x_*}$.

Finally, we discuss a priori bounds on solutions. Let x be a solution of (1.1). Then there exists $f \in L^1([0, 1], \mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in [0, 1]$, we have

$$x(t) = \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)f(s)ds - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f(s)ds - \int_0^t (t-s)f(s)ds. \quad (3.14)$$

In view of (H_2) , for each $t \in [0, 1]$, we obtain

$$\begin{aligned} |x(t)| & \leq \left(1 + \frac{2}{|2 - \alpha\eta^2|}\right) \int_0^1 f(s)ds + \frac{|\alpha|\eta^2}{|2 - \alpha\eta^2|} \int_0^\eta f(s)ds \\ & \leq \left(1 + \left(2 + |\alpha|\eta^2/|2 - \alpha\eta^2|\right)\right) \psi(\|x\|_\infty) \int_0^1 p(s)ds. \end{aligned} \quad (3.15)$$

Consequently, we have

$$\frac{\|x\|_\infty}{(1 + (2 + |\alpha|\eta^2/|2 - \alpha\eta^2|))\psi(\|x\|_\infty)\|p\|_{L^1}} \leq 1. \quad (3.16)$$

In view of (H_3) , there exists M such that $\|x\|_\infty \neq M$. Let us set

$$U = \{x \in C([0, 1], \mathbb{R}) : \|x\|_\infty < M + 1\}. \quad (3.17)$$

Note that the operator $\Omega : \overline{U} \rightarrow \mathcal{P}(C([0, 1], \mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U , there is no $x \in \partial U$ such that $x \in \mu\Omega(x)$ for some $\mu \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [28], we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. $\square$

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray-Schauder type together with the selection theorem of Bressan and Colombo [26] for lower semi-continuous maps with decomposable values.

Theorem 3.2. *Assume that (H_2) , (H_3) , and the following conditions hold:*

(H_4) $F : [0, 1] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that

- (a) $(t, x) \mapsto F(t, x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,
- (b) $x \mapsto F(t, x)$ is lower semicontinuous for each $t \in [0, 1]$,

(H_5) for each $\sigma > 0$, there exists $\varphi_\sigma \in L^1([0, 1], \mathbb{R}^+)$ such that

$$\|F(t, x)\| = \sup\{|y| : y \in F(t, x)\} \leq \varphi_\sigma(t) \quad \forall \|x\|_\infty \leq \sigma \quad \text{for a.e. } t \in [0, 1]. \quad (3.18)$$

Then the boundary value problem (1.1) has at least one solution on $[0, 1]$.

Proof. It follows from (H_4) and (H_5) that F is of l.s.c. type. Then, from Lemma 2.6, there exists a continuous function $f : C([0, 1], \mathbb{R}) \rightarrow L^1([0, 1], \mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0, 1], \mathbb{R})$.

Consider the problem

$$\begin{aligned} -x''(t) &= f(x(t)), \quad 0 < t < 1, \\ x(0) &= 0, \quad x(1) = \alpha \int_0^\eta x(s) ds, \quad 0 < \eta < 1, \quad \alpha \neq 2/\eta^2. \end{aligned} \quad (3.19)$$

Observe that, if $x \in C^2([0, 1])$ is a solution of (3.19), then x is a solution to the problem (1.1). In order to transform the problem (3.19) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\overline{\Omega}x(t) = \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)f(x(s))ds - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f(x(s))ds - \int_0^t (t-s)f(x(s))ds. \quad (3.20)$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.1. So we omit it. This completes the proof. $\square$

Now we prove the existence of solutions for the problem (1.1) with a nonconvex-valued right-hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [27].

Theorem 3.3. *Assume that the following conditions hold:*

(H_6) $F : [0, 1] \times \mathbb{R} \rightarrow P_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [0, 1] \rightarrow P_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$.

(H₇) $H_d(F(t, x), F(t, \bar{x})) \leq m(t)|x - \bar{x}|$ for almost all $t \in [0, 1]$ and $x, \bar{x} \in \mathbb{R}$ with $m \in L^1([0, 1], \mathbb{R}^+)$ and $d(0, F(t, 0)) \leq m(t)$ for almost all $t \in [0, 1]$.

Then the boundary value problem (1.1) has at least one solution on $[0, 1]$ if

$$\left(1 + \left(2 + |\alpha|\eta^2 / |2 - \alpha\eta^2|\right)\right) \|m\|_{L^1} < 1. \quad (3.21)$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0, 1], \mathbb{R})$ by the assumption (H₆), so F has a measurable selection (see Theorem III.6 [29]). Now we show that the operator Ω satisfies the assumptions of Lemma 2.7. To show that $\Omega(x) \in P_{cl}(C([0, 1], \mathbb{R}))$ for each $x \in C([0, 1], \mathbb{R})$, let $\{u_n\}_{n \geq 0} \in \Omega(x)$ be such that $u_n \rightarrow u$ ($n \rightarrow \infty$) in $C([0, 1], \mathbb{R})$. Then $u \in C([0, 1], \mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0, 1]$,

$$u_n(t) = \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)v_n(s)ds - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta-s)^2 v_n(s)ds - \int_0^t (t-s)v_n(s)ds. \quad (3.22)$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0, 1], \mathbb{R})$. Thus, $v \in S_{F,x}$ and, for each $t \in [0, 1]$,

$$\begin{aligned} u_n(t) \longrightarrow u(t) &= \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)v(s)ds \\ &\quad - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta-s)^2 v(s)ds - \int_0^t (t-s)v(s)ds. \end{aligned} \quad (3.23)$$

Hence, $u \in \Omega(x)$.

Next we show that there exists $\gamma < 1$ such that

$$H_d(\Omega(x), \Omega(\bar{x})) \leq \gamma \|x - \bar{x}\|_\infty \quad \text{for each } x, \bar{x} \in C([0, 1], \mathbb{R}). \quad (3.24)$$

Let $x, \bar{x} \in C([0, 1], \mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0, 1]$,

$$h_1(t) = \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)v_1(s)ds - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta-s)^2 v_1(s)ds - \int_0^t (t-s)v_1(s)ds. \quad (3.25)$$

By (H₇), we have

$$H_d(F(t, x), F(t, \bar{x})) \leq m(t)|x(t) - \bar{x}(t)|. \quad (3.26)$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|, \quad t \in [0, 1]. \quad (3.27)$$

Define $U : [0, 1] \rightarrow \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|\}. \quad (3.28)$$

Since the multivalued operator $V(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [29]), there exists a function $v_2(t)$ which is a measurable selection for V . So $v_2(t) \in F(t, \bar{x}(t))$, and, for each $t \in [0, 1]$, we have $|v_1(t) - v_2(t)| \leq m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0, 1]$, let us define

$$h_2(t) = \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)v_2(s)ds - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta-s)^2 v_2(s)ds - \int_0^t (t-s)v_2(s)ds. \quad (3.29)$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \left| \frac{2t}{2 - \alpha\eta^2} \right| \int_0^1 |1-s||v_1(s) - v_2(s)|ds \\ &\quad + \left| \frac{\alpha t}{2 - \alpha\eta^2} \right| \int_0^\eta |\eta-s|^2 |v_1(s) - v_2(s)|ds + \int_0^t |t-s||v_1(s) - v_2(s)|ds \\ &\leq \left(1 + \left(2 + |\alpha|\eta^2 / |2 - \alpha\eta^2| \right) \right) \int_0^1 m(s) \|x - \bar{x}\| ds. \end{aligned} \quad (3.30)$$

Hence,

$$\|h_1 - h_2\|_\infty \leq \left(1 + \left(2 + |\alpha|\eta^2 / |2 - \alpha\eta^2| \right) \right) \|m\|_{L^1} \|x - \bar{x}\|_\infty. \quad (3.31)$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$\begin{aligned} H_d(\Omega(x), \Omega(\bar{x})) &\leq \gamma \|x - \bar{x}\|_\infty \\ &\leq \left(1 + \left(2 + |\alpha|\eta^2 / |2 - \alpha\eta^2| \right) \right) \|m\|_{L^1} \|x - \bar{x}\|_\infty. \end{aligned} \quad (3.32)$$

Since Ω is a contraction, it follows by Lemma 2.7 that Ω has a fixed point x which is a solution of (1.1). This completes the proof. $\square$

Remark 3.4. By fixing the functions and parameters involved in the problem (1.1), we obtain some interesting results:

- (i) if we take $F(t, x) = \{f(t, x)\}$, where $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, then our results correspond to the ones for a single-valued problem, which are new results in the present configuration;
- (ii) in the limit $\eta \rightarrow 1$, our results correspond to an inclusion problem with integral boundary condition (see [30]) of the type

$$x(1) = \alpha \int_0^1 x(s)ds. \quad (3.33)$$

In this case, the solution defined by (2.10) takes the form

$$x(t) = \int_0^1 \frac{t(1-s)[2-\alpha(1-s)]}{2-\alpha} f(s) ds - \int_0^t (t-s)f(s) ds. \quad (3.34)$$

- (iii) By fixing $\alpha = 0$ in (1.1), our results reduce to the ones for a second-order inclusion problem with Dirichlet boundary conditions ($x(0) = 0, x(1) = 0$).

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