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Research Article

A Hybrid Iterative Scheme for Mixed Equilibrium Problems, General System of Variational Inequality Problems, and Fixed Point Problems in Hilbert Spaces

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We introduce a new iterative scheme for finding a common element of the set of solutions of a general system of variational inequalities, the set of solutions of a mixed equilibrium problem, and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Using the demiclosedness principle for nonexpansive mappings, we prove that the iterative sequence converges strongly to a common element of the above three sets under some control conditions, and we also give some examples for mappings which satisfy conditions of the main result.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and C a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be *nonexpansive mapping* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The fixed point set of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

Halpern [1] studied the following iteration formula for approximating a fixed point of T. For an arbitrary $v \in C$, let the sequence $\{x_n\}$ be defined by $x_1 \in C$,

$$x_{n+1} = \alpha_n v + (1 - \alpha_n) T x_n, \quad n \geq 1, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence of real numbers in $[0,1]$ that satisfies the following conditions: $C1 : \lim_{n\to\infty}\alpha_n = 0$ and $C2 : \sum_{n=1}^{\infty}\alpha_n = \infty$.

Actually, Halpern studied the special case of (1.1) in which $\alpha_n = n^{-\sigma}$, $\sigma \in (0,1)$, and $v = 0$ and proved that $\{x_n\}$ converges strongly to a fixed point of T. Under a different restriction on the parameter $\{\alpha_n\}$, Lions [2] improved the result of Halpern, still in Hilbert spaces. He proved strong convergence of $\{x_n\}$ to a fixed point of T, where the real sequence $\{\alpha_n\}$ satisfies the following conditions:

$$C1 : \lim_{n\to\infty}\alpha_n = 0, \qquad C2 : \sum_{n=1}^{\infty}\alpha_n = \infty, \qquad C3 : \lim_{n\to\infty}\frac{|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}^2} = 0. \tag{1.2}$$

Reich [3] proved that the result of Halpern remains true when H is uniformly smooth. It was observed that both Halpern's and Lion's conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n+1)$. This was overcome by Wittmann [4] who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

$$C1 : \lim_{n\to\infty}\alpha_n = 0, \qquad C2 : \sum_{n=1}^{\infty}\alpha_n = \infty, \qquad C3 : \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty. \tag{1.3}$$

Recall that the classical variational inequality, denoted by $\mathrm{VI}(C,A)$, is to find an $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The variational inequality, nonconvex variational inequality, and mixed variational inequality have been widely studied in the literature; see, for example, Ceng et al. [5–14], Chang et al. [15], Noor [16–18], Peng and Yao [19–21], Plubtieng and Punpaeng [22], Yao et al. [23], Zeng and Yao [24], Zhao and He [25], and the references therein.

For solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex, a mapping A of C into $\mathbb{R}^n$ is monotone and k-Lipschitz continuous, and $\mathrm{VI}(C,A)$ is nonempty, Korpelevich [26] introduced the following so-called extragradient method:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda A x_n), \\ x_{n+1} &= P_C(x_n - \lambda A y_n), \end{aligned} \tag{1.5}$$

for every $n = 0,1,2,\ldots$, where $\lambda \in (0,1/k)$ and P_C is the projection of $\mathbb{R}^n$ onto C. He showed that the sequences $\{x_n\}$ and $\{y_n\}$ generated by this iterative process converge to the same point $z \in \mathrm{VI}(C,A)$. In 2007, Y. Yao and J. C. Yao [27] introduced a new iterative scheme for finding the common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for α-inverse-strongly monotone mappings in a real Hilbert space.

Let C be a nonempty closed convex subset of a real Hilbert space H.

Let $A, B : C \to H$ be two mappings. In 2008, Ceng et al. [12] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ &\langle \mu B x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{1.6}$$

which is called a general system of variational inequalities, where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then the problem (1.6) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ &\langle \mu A x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{1.7}$$

which was defined by Verma [28] and is called the new system of variational inequalities. Further, if we add up the requirement that $x^* = y^*$, then the problem (1.7) reduces to the classical variational inequality VI(C, A). Ceng et al. [12] introduced and studied a relaxed extragradient method for finding a common element of the set of solutions of the problem (1.6) for the α- and β-inverse-strongly monotone mappings and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Let $x_1 = v \in C$, and $\{x_n\}$, $\{y_n\}$ are given by

$$\begin{aligned} y_n &= P_C(x_n - \mu B x_n), \\ x_{n+1} &= a_n v + b_n x_n + (1 - a_n - b_n) S P_C(y_n - \lambda A y_n), \quad n \geq 1, \end{aligned} \tag{1.8}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, and $\{a_n\}, \{b_n\} \subset [0, 1]$. Then, they proved that the sequence $\{x_n\}$ converges strongly to a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the problem (1.6) under some control conditions.

Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function and F a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Ceng and Yao [13] considered the following mixed equilibrium problem:

$$\text{find } x \in C \text{ such that } F(x, y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1.9}$$

The set of solutions of (1.9) is denoted by MEP(F, φ). It is easy to see that x is a solution of the problem (1.9) impling that $x \in \operatorname{dom} \varphi = \{x \in C \mid \varphi(x) < +\infty\}$.

If $\varphi = 0$, then the mixed equilibrium problem (1.9) becomes the following equilibrium problem:

$$\text{find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{1.10}$$

If $F = 0$, then the mixed equilibrium problem (1.9) reduces to the convex minimization problem

$$\text{find } x \in C \text{ such that } \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1.11}$$

The set of solutions of (1.10) is denoted by $\mathrm{EP}(F)$.

If $\varphi = 0$ and $F(x,y) = \langle Ax, y-x\rangle$ for all $x, y \in C$, where A is a mapping from C into H, then (1.9) reduces to the classical variational inequality and $\mathrm{EP}(F) = \mathrm{VI}(C, A)$.

The equilibrium problems and variational inequality problems can be applied for the problems in science and technology, economics, optimizations, and control theory. The problems of finding a solution of the systems of variational inequalities which is also a solution of the mixed equilibrium problems and the problem of finding a solution of the mixed equilibrium problems under the constraints that it is also a solution of the system of variational inequalities are very useful and important for studying problems in science and applied science.

Inspired and motivated by these facts, we introduce a new iteration process for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a nonexpansive mapping, and the set of solutions of a general system of variational inequalities in a real Hilbert space. Start with an arbitrary $v \in C$, and let $x_1 \in C$, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be the sequences generated by

$$\begin{gathered} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \mu B u_n), \\ x_{n+1} = a_n v + (1-a_n) T P_C(y_n - \lambda A y_n), \quad n \geq 1, \end{gathered} \tag{1.12}$$

where $\lambda > 0$ and $\mu > 0$ are two constants, $\{r_n\} \subset (0,\infty)$ and $\{a_n\} \subset [0,1]$. Using the demiclosedness principle for nonexpansive mappings, we will show that the sequence $\{x_n\}$ converges strongly to a common element of the above three sets under some control conditions.

2. Preliminaries

In this section, we recall the well-known results and give some useful lemmas that will be used in the next section.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and C a nonempty closed convex subset of H.

A mapping $A : C \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C. \tag{2.1}$$

A is called *α-strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C. \tag{2.2}$$

This implies that

$$\|Ax - Ay\| \geq \alpha\|x - y\|, \tag{2.3}$$

that is, A is *α-expansive* and, when $\alpha = 1$, it is *expansive*.

We can see easily that the following implications in monotonicity, strong monotonicity, and expansiveness hold:

$$\begin{aligned} &\text{strong monotonicity} \Rightarrow \text{monotonicity} \\ &\Downarrow \\ &\text{expansiveness.} \end{aligned} \tag{2.4}$$

The mapping A is called *L-Lipschitz continuous* (or *Lipschitzian*) if there exists a constant $L \geq 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{2.5}$$

A is called *α-inverse-strongly monotone* (or *α-cocoercive*) if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2.6}$$

It is obvious that every α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous. Also, if A is α-strongly and L-Lipschitz continuous, then A is (α/L^2)-inverse-strongly monotone, but inverse-strongly monotone need not be strongly monotone.

A mapping A is called *relaxed c-cocoercive*, if there exists a constant $c > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-c)\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2.7}$$

A mapping A is called *relaxed (c,d)-cocoercive*, if there exist two constants $c, d > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-c)\|Ax - Ay\|^2 + d\|x - y\|^2, \quad \forall x, y \in C. \tag{2.8}$$

For $c = 0$, A is d-strongly monotone. This class of mappings is more general than the class of strongly monotone mappings. As a result, we have the following implication: d-strong monotonicity $\Rightarrow$ relaxed (c, d)-cocoercivity.

It is known that if the operator is Lipschitz continuous, then the relaxed cocoercivity is strongly monotone, but the strongly monotone does not imply the cocoercivity as shown in the following example.

Example 2.1. Let $H = R$, $C = [1, \infty)$, and $A : C \to H$ be defined by $Ax = x^2$, $x \in C$. For $x, y \in C$, we have

$$\begin{aligned} \langle Ax - Ay, x - y\rangle &= \left(x^2 - y^2\right)(x - y) \\ &= (x + y)|x - y|^2 \\ &\geq 2|x - y|^2. \end{aligned} \tag{2.9}$$

Hence, A is 2-strongly monotone. If A is μ-cocoercive for some $\mu > 0$, then $\langle Ax - Ay, x - y\rangle = (x+y)|x-y|^2 \geq \mu|x^2-y^2|^2$. This implies $x+y \leq 1/\mu$ for all $x, y \in [1, \infty)$ which is a contradiction. Hence, A is not μ-cocoercive for any $\mu > 0$.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.10}$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.11}$$

Obviously, this immediately implies that

$$\|(x - y) - (P_C x - P_C y)\|^2 \leq \|x - y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.12}$$

Recall that $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{gathered} \langle x - P_C x, y - P_C x\rangle \leq 0, \\ \|x - y\|^2 \geq \|x - P_C x\|^2 + \|P_C x - y\|^2, \end{gathered} \tag{2.13}$$

for all $x \in H$ and $y \in C$; see Goebel and Kirk [29] for more details.

For solving the mixed equilibrium problem, let us assume the following assumptions for the bifunction F, φ and the set C:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex;

(A5) for each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z); \tag{2.14}$$

(B2) C is a bounded set.

In the sequel we will need to use the following lemma.

Lemma 2.2 (see [30]). *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5), and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows*

$$T_r(x) = \left\{ z \in C : F(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z),\ \forall y \in C \right\} \tag{2.15}$$

for all $x \in H$. Then the following conclusions hold:

(1) *for each $x \in H$, $T_r(x) \neq \emptyset$;*

(2) *T_r is single valued;*

(3) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{2.16}$$

(4) *$F(T_r) = \mathrm{MEP}(F, \varphi)$;*

(5) *$\mathrm{MEP}(F, \varphi)$ is closed and convex.*

We also need the following lemmas.

Lemma 2.3 (see [31]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + t_n c_n, \quad \forall n \geq 1, \tag{2.17}$$

where $\{t_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy the restrictions

$$\text{(i)} \sum_{n=1}^{\infty} t_n = \infty, \qquad \text{(ii)} \sum_{n=1}^{\infty} b_n < \infty, \qquad \text{(iii)} \limsup_{n\to\infty} c_n \leq 0. \tag{2.18}$$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 (see [32]). *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space. Then, for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, one has*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2. \tag{2.19}$$

Lemma 2.5 (see [29] (demiclosedness principle)). *Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed: that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x \in C$) and the sequence $\{(I - T)x_n\}$ converges strongly to some y (for short, $(I - T)x_n \to y$), it follows that $(I - T)x = y$. Here I is the identity operator of H.*

The following lemma is an immediate consequence of an inner product.

Lemma 2.6. *In a real Hilbert space H, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H. \tag{2.20}$$

In 2009, Kangtunyakarn and Suantai [33] introduced a new mapping called the S-mapping as follows.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. For each $n \in \mathbb{N}$, and $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0,1]$ with $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. Very recently, in 2009, Kangtunyakarn and Suantai [33] introduced the new mapping $S_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,0} &= I, \\
U_{n,1} &= \alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I, \\
U_{n,2} &= \alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I, \\
U_{n,3} &= \alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I, \\
&\vdots \\
U_{n,N-1} &= \alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I, \\
S_n = U_{n,N} &= \alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I.
\end{aligned} \tag{2.21}$$

The mapping S_n is called the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Nonexpansivity of each T_i ensures the nonexpansivity of S_n. They also showed the following useful fact.

Lemma 2.7 (see [33]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\cap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, $j = 1, 2, \ldots, N$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$ for all $j = 1, 2, \ldots, N-1$, $\alpha_1^N \in (0,1]$ and $\alpha_2^j, \alpha_3^j \in [0,1)$ for all $j = 1, 2, \ldots, N$. Let S be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \cap_{i=1}^N F(T_i)$.*

Lemma 2.8 (see [12]). *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the problem* (1.6) *if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C\left[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)\right], \quad \forall x \in C, \tag{2.22}$$

where $y^* = P_C(x^* - \mu Bx^*)$.

Throughout this paper, the set of fixed points of the mapping G is denoted by $\mathrm{GVI}(C, A, B)$.

Next, we prove a lemma which is very useful for our consideration.

Lemma 2.9. *Let $G : C \to C$ be defined by*

$$G(x) = P_C\left[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)\right], \quad \forall x \in C, \tag{2.23}$$

where $\lambda, \mu > 0$ and $A, B : C \to H$ are two mappings. If $I - \lambda A$ and $I - \mu B$ are nonexpansive mappings, then G is nonexpansive.

Proof. For any $x, y \in C$, we have

$$\begin{aligned}
\|G(x) - G(y)\| &= \|P_C\left[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)\right] \\
&\quad - P_C\left[P_C(y - \mu By) - \lambda A P_C(y - \mu By)\right]\|^2 \\
&\le \|P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx) - (P_C(y - \mu By) - \lambda A P_C(y - \mu By))\| \\
&= \|(I - \lambda \mathrm{A}) P_C(I - \mu B)x - (I - \lambda A) P_C(I - \mu B)y\| \\
&\le \|x - y\|.
\end{aligned} \tag{2.24}$$

This shows that G is a nonexpansive mapping. □

3. Main Results

In this section, we prove strong convergence theorems of the iterative scheme (1.12) to a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a nonexpansive mapping, and the set of solutions of a general system of variational inequality in a real Hilbert space.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ a proper lower semi-continuous and convex function. Let $\lambda, \mu > 0$, and let $A, B : C \to H$ be such that $I - \lambda A$ and $I - \mu B$ are nonexpansive mappings. Let T be a nonexpansive self-mapping of C such that $\Omega = F(T) \cap \mathrm{GVI}(C, A, B) \cap \mathrm{MEP}(F, \varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in C. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be defined by (1.12), $\{r_n\} \subset (0, \infty)$ and $\{a_n\} \subset [0, 1]$ such that*

(C1) $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty}(a_{n+1}/a_n) = 1$, *and* $\sum_{n=1}^{\infty} a_n = \infty$,

(C2) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

If $\lim_{n\to\infty}\|Ay_n - Ay^*\| = 0$ and $\lim_{n\to\infty}\|Bu_n - Bx^*\| = 0$ for all $x^* \in \Omega$ and $y^* = P_C(x^* - \mu Bx^*)$, then $\{x_n\}$ converges strongly to $\overline{x} = P_\Omega v$ and $(\overline{x}, \overline{y})$ is a solution of the problem (1.6), where $\overline{y} = P_C(\overline{x} - \mu B\overline{x})$.

Proof. Let $x^* \in \Omega$ and $\{T_{r_n}\}$ a sequence of mappings defined as in Lemma 2.2. It follows from Lemma 2.8 that

$$x^* = P_C\left[P_C(x^* - \mu Bx^*) - \lambda A P_C(x^* - \mu Bx^*)\right]. \tag{3.1}$$

Put $y^* = P_C(x^* - \mu Bx^*)$ and $t_n = P_C(y_n - \lambda Ay_n)$, then $x^* = P_C(y^* - \lambda Ay^*)$ and

$$x_{n+1} = a_n v + (1 - a_n)Tt_n. \tag{3.2}$$

Since $I - \lambda A$, $I - \mu B$, T_{r_n}, and P_C are nonexpansive mappings, we have

$$\begin{aligned}
\|t_n - x^*\|^2 &= \left\|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\right\|^2 \\
&\le \left\|y_n - y^*\right\|^2 \\
&= \left\|P_C(u_n - \mu Bu_n) - P_C(x^* - \mu Bx^*)\right\|^2 \\
&\le \|u_n - x^*\|^2 \\
&= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \\
&\le \|x_n - x^*\|^2,
\end{aligned} \tag{3.3}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|a_n v + (1 - a_n)Tt_n - x^*\| \\
&\le a_n\|v - x^*\| + (1 - a_n)\|t_n - x^*\| \\
&\le a_n\|v - x^*\| + (1 - a_n)\|x_n - x^*\| \\
&\le \max\{\|v - x^*\|, \|x_1 - x^*\|\}.
\end{aligned} \tag{3.4}$$

Thus, $\{x_n\}$ is bounded. Consequently, the sequences $\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{Ay_n\}$, $\{Bu_n\}$, and $\{Tt_n\}$ are also bounded. Also, observe that

$$\begin{aligned}
\|t_{n+1} - t_n\| &= \left\|P_C(y_{n+1} - \lambda Ay_{n+1}) - P_C(y_n - \lambda Ay_n)\right\| \\
&\le \left\|y_{n+1} - y_n\right\| \\
&= \left\|P_C(u_{n+1} - \mu Bu_{n+1}) - P_C(u_n - \mu Bu_n)\right\| \\
&\le \|u_{n+1} - u_n\|.
\end{aligned} \tag{3.5}$$

On the other hand, from $u_n = T_{r_n}x_n \in \operatorname{dom}\varphi$ and $u_{n+1} = T_{r_{n+1}}x_{n+1} \in \operatorname{dom}\varphi$, we have

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \tag{3.6}$$

$$F(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0, \quad \forall y \in C. \tag{3.7}$$

Putting $y = u_{n+1}$ in (3.6) and $y = u_n$ in (3.7), we have

$$\begin{aligned} F(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0, \\ F(u_{n+1}, u_n) + \varphi(u_n) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0. \end{aligned} \tag{3.8}$$

From the monotonicity of F, we obtain that

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0, \tag{3.9}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0. \tag{3.10}$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left(\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right), \end{aligned} \tag{3.11}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\|. \end{aligned} \tag{3.12}$$

It follows from (3.5) and (3.12) that

$$\|t_{n+1} - t_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\|. \tag{3.13}$$

Next, we show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. From (3.13), we have

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &= \|a_{n+1}v + (1 - a_{n+1})Tt_{n+1} - a_n v - (1 - a_n)Tt_n\| \\ &= \|(a_{n+1} - a_{\mathrm{n}})v + (a_n - a_{n+1})Tt_{n+1} + (1 - a_n)(Tt_{n+1} - Tt_n)\| \\ &\leq |a_{n+1} - a_n|(\|v\| + \|Tt_{n+1}\|) + (1 - a_n)\|t_{n+1} - t_n\| \end{aligned}$$

$$\begin{aligned}
&\leq |a_{n+1}-a_n|(\|v\|+\|Tt_{n+1}\|)+(1-a_n)\|x_{n+1}-x_n\| \\
&\quad+\frac{1}{c}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\| \\
&\leq (1-a_n)\|x_{n+1}-x_n\|+\frac{1}{c}|r_{n+1}-r_n|\ M+2M|a_{n+1}-a_n|,
\end{aligned} \tag{3.14}$$

where $M=\max\{\|v\|,\sup_n\|Tt_{n+1}\|,\sup_n\|u_{n+1}-x_{n+1}\|\}$. By the assumptions on $\{a_n\}$ and $\{r_n\}$ and Lemma 2.3, we conclude that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.15}$$

From (C1), (C2), (3.5), (3.12), and (3.15), we also have $\|u_{n+1}-u_n\|\to 0$, $\|t_{n+1}-t_n\|\to 0$, and $\|y_{n+1}-y_n\|\to 0$, as $n\to\infty$.

Since

$$x_{n+1}-x_n=a_n(v-x_n)+(1-a_n)(Tt_n-x_n), \tag{3.16}$$

we have that

$$\|Tt_n-x_n\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{3.17}$$

Next, we prove that $\lim_{n\to\infty}\|x_n-u_n\|=0$. From Lemma 2.2(3), we have

$$\begin{aligned}
\|u_n-x^*\|^2=\|T_{r_n}x_n-T_{r_n}x^*\|^2&\leq\langle T_{r_n}x_n-T_{r_n}x^*,x_n-x^*\rangle \\
&=\langle u_n-x^*,x_n-x^*\rangle=\frac{1}{2}\left(\|u_n-x^*\|^2+\|x_n-x^*\|^2-\|x_n-u_n\|^2\right).
\end{aligned} \tag{3.18}$$

Hence,

$$\|u_n-x^*\|^2\leq\|x_n-x^*\|^2-\|x_n-u_n\|^2. \tag{3.19}$$

From Lemma 2.4, (3.3), and (3.19), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2&\leq a_n\|v-x^*\|^2+(1-a_n)\|t_n-x^*\|^2 \\
&\leq a_n\|v-x^*\|^2+(1-a_n)\|u_n-x^*\|^2 \\
&\leq a_n\|v-x^*\|^2+(1-a_n)\left[\|x_n-x^*\|^2-\|x_n-u_n\|^2\right] \\
&\leq a_n\|v-x^*\|^2+\|x_n-x^*\|^2-(1-a_n)\|x_n-u_n\|^2.
\end{aligned} \tag{3.20}$$

It follows that

$$\begin{aligned}(1-a_n)\|x_n-u_n\|^2 &\le a_n\|v-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\le a_n\|v-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.\end{aligned} \tag{3.21}$$

From conditions (C1) and (3.15), we obtain

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.22}$$

By (3.17) and (3.22), we have

$$\|Tt_n-u_n\|\le\|Tt_n-x_n\|+\|x_n-u_n\|\longrightarrow 0,\quad \text{as } n\longrightarrow\infty. \tag{3.23}$$

Next, we prove that $\|Tt_n-t_n\|\to 0$ as $n\to\infty$. From (2.11) and nonexpansiveness of $I-\mu B$, we get

$$\begin{aligned}\|y_n-y^*\|^2 &= \|P_C(u_n-\mu Bu_n)-P_C(x^*-\mu Bx^*)\|^2\\ &\le \langle (u_n-\mu Bu_n)-(x^*-\mu Bx^*),y_n-y^*\rangle\\ &= \frac{1}{2}\Big[\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)\|^2+\|y_n-y^*\|^2\\ &\qquad -\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)-(y_n-y^*)\|^2\Big]\\ &\le \frac{1}{2}\Big[\|u_n-x^*\|^2+\|y_n-y^*\|^2-\|(u_n-x^*)-(y_n-y^*)\|^2\\ &\qquad +2\mu\langle (u_n-x^*)-(y_n-y^*),Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2\Big].\end{aligned} \tag{3.24}$$

By (3.3), we obtain

$$\begin{aligned}\|y_n-y^*\|^2 &\le \|u_n-x^*\|^2-\|(u_n-x^*)-(y_n-y^*)\|^2\\ &\qquad +2\mu\langle (u_n-x^*)-(y_n-y^*),Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2\\ &\le \|x_n-x^*\|^2-\|(u_n-x^*)-(y_n-y^*)\|^2\\ &\qquad +2\mu\langle (u_n-x^*)-(y_n-y^*),Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2.\end{aligned} \tag{3.25}$$

Hence,

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le a_n\|v-x^*\|^2+(1-a_n)\|y_n-y^*\|^2\\
&\le a_n\|v-x^*\|^2+(1-a_n)\\
&\quad\times\Big[\|x_n-x^*\|^2-\|(u_n-x^*)-(y_n-y^*)\|^2\\
&\qquad+2\mu\langle(u_n-x^*)-(y_n-y^*),Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2\Big]\\
&\le a_n\|v-x^*\|^2+\|x_n-x^*\|^2-(1-a_n)\|(u_n-x^*)-(y_n-y^*)\|^2\\
&\quad+(1-a_n)2\mu\|(u_n-x^*)-(y_n-y^*)\|\|Bu_n-Bx^*\|,
\end{aligned}\tag{3.26}$$

which implies that

$$\begin{aligned}
(1-a_n)\|(u_n-x^*)-(y_n-y^*)\|^2 &\le a_n\|v-x^*\|^2\\
&\quad+(1-a_n)2\mu\|(u_n-x^*)-(y_n-y^*)\|\|Bu_n-Bx^*\|\\
&\quad+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.
\end{aligned}\tag{3.27}$$

By this together with (C1), (3.15), and $\lim_{n\to\infty}\|Bu_n-Bx^*\|=0$, we obtain

$$\|(u_n-x^*)-(y_n-y^*)\|\longrightarrow 0\quad\text{as } n\longrightarrow\infty.\tag{3.28}$$

From Lemma 2.6 and (2.12), it follows that

$$\begin{aligned}
&\|(y_n-t_n)+(x^*-y^*)\|^2\\
&\quad=\|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*)-[P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)]+\lambda(Ay_n-Ay^*)\|^2\\
&\quad\le\|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*)-[P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)]\|^2\\
&\qquad+2\lambda\langle Ay_n-Ay^*,(y_n-t_n)+(x^*-y^*)\rangle\\
&\quad\le\|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*)\|^2-\|P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)\|^2\\
&\qquad+2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\|\\
&\quad\le\|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*)\|^2-\|TP_C(y_n-\lambda Ay_n)-TP_C(y^*-\lambda Ay^*)\|^2\\
&\qquad+2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\|\\
&\quad\le\|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*)-(Tt_n-x^*)\|\\
&\qquad\times[\|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*)\|+\|Tt_n-x^*\|]\\
&\qquad+2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\|
\end{aligned}$$

$$
\begin{aligned}
&= \left\| u_n - Tt_n + x^* - y^* - (u_n - y_n) - \lambda(Ay_n - Ay^*) \right\| \\
&\quad \times \left[\left\| (y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) \right\| + \|Tt_n - x^*\| \right] \\
&\quad + 2\lambda \left\| Ay_n - Ay^* \right\| \left\| (y_n - t_n) + (x^* - y^*) \right\|.
\end{aligned} \tag{3.29}
$$

By this together with (3.23), (3.28), and $\lim_{n\to\infty}\|Ay_n - Ay^*\| = 0$, we obtain $\|(y_n - t_n) + (x^* - y^*)\| \to 0$ as $n \to \infty$. This together with (3.17), (3.22), and (3.28), we obtain that

$$
\begin{aligned}
\|Tt_n - t_n\| &\le \|Tt_n - x_n\| + \|x_n - u_n\| + \left\| (u_n - y_n) - (x^* - y^*) \right\| \\
&\quad + \left\| (y_n - t_n) + (x^* - y^*) \right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.30}
$$

Next, we show that

$$
\limsup_{n\to\infty} \langle v - \overline{x}, x_n - \overline{x} \rangle \le 0, \tag{3.31}
$$

where $\overline{x} = P_\Omega v$.

Indeed, since $\{t_n\}$ and $\{Tt_n\}$ are two bounded sequences in C, we can choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \rightharpoonup z \in C$ and

$$
\limsup_{n\to\infty} \langle v - \overline{x}, Tt_n - \overline{x} \rangle = \lim_{i\to\infty} \langle v - \overline{x}, Tt_{n_i} - \overline{x} \rangle. \tag{3.32}
$$

Since $\lim_{n\to\infty}\|Tt_n - t_n\| = 0$, we obtain that $Tt_{n_i} \rightharpoonup z$ as $i \to \infty$.

Next, we show that $z \in \Omega$.

(a) We first show $z \in F(T)$.

Since $t_{n_i} \rightharpoonup z$ and $\|Tt_n - t_n\| \to 0$, we obtain by Lemma 2.5 that $z \in F(T)$.

(b) Now, we show that $z \in \mathrm{GVI}(C, A, B)$.

From (3.30) and (3.17), we have

$$
\|t_n - x_n\| \le \|Tt_n - t_n\| + \|Tt_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.33}
$$

Furthermore, by Lemma 2.9, we have that $G : C \to C$ is nonexpansive. Then, we have

$$
\begin{aligned}
\|t_n - G(t_n)\| &= \left\| P_C(y_n - \lambda Ay_n) - G(t_n) \right\| \\
&= \left\| P_C\left[P(u_n - \mu Bu_n) - \lambda AP(u_n - \mu Bu_n) \right] - G(t_n) \right\| \\
&= \|G(u_n) - G(t_n)\| \le \|u_{\mathrm{n}} - t_n\| \\
&\le \|u_n - x_n\| + \|x_n - t_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.34}
$$

Again by Lemma 2.5, we have $z \in \mathrm{GVI}(C, A, B)$.

(c) We show that $z \in \mathrm{MEP}(F,\varphi)$. Since $t_{n_i} \rightharpoonup z$ and $\|x_n - t_n\| \to 0$, we obtain that $x_{n_i} \rightharpoonup z$. From $\|u_n - x_n\| \to 0$, we also obtain that $u_{n_i} \rightharpoonup z$. By using the same argument as that in the proof of [30, Theorem 3.1, page 1825], we can show that $z \in \mathrm{MEP}(F,\varphi)$. Therefore, there holds $z \in \Omega$.

On the other hand, it follows from (2.13), (3.17), and $Tt_{n_i} \rightharpoonup z$ as $i \to \infty$ that

$$\begin{aligned}\limsup_{n\to\infty}\langle v-\overline{x}, x_n-\overline{x}\rangle &= \limsup_{n\to\infty}\langle v-\overline{x}, Tt_n-\overline{x}\rangle = \lim_{i\to\infty}\langle v-\overline{x}, Tt_{n_i}-\overline{x}\rangle \\ &= \langle v-\overline{x}, z-\overline{x}\rangle \le 0.\end{aligned} \tag{3.35}$$

Hence, we have

$$\begin{aligned}\|x_{n+1}-\overline{x}\|^2 &= \langle a_n v + (1-a_n)Tt_n - \overline{x}, x_{n+1}-\overline{x}\rangle \\ &= a_n\langle v-\overline{x}, x_{n+1}-\overline{x}\rangle + (1-a_n)\langle Tt_n-\overline{x}, x_{n+1}-\overline{x}\rangle \\ &\le a_n\langle v-\overline{x}, x_{n+1}-\overline{x}\rangle + \frac{1}{2}(1-a_n)\left(\|t_n-\overline{x}\|^2 + \|x_{n+1}-\overline{x}\|^2\right) \\ &\le a_n\langle v-\overline{x}, x_{n+1}-\overline{x}\rangle + \frac{1}{2}(1-a_n)\left(\|x_n-\overline{x}\|^2 + \|x_{n+1}-\overline{x}\|^2\right),\end{aligned} \tag{3.36}$$

which implies that

$$\|x_{n+1}-\overline{x}\|^2 \le (1-a_n)\|x_n-\overline{x}\|^2 + 2a_n\langle v-\overline{x}, x_{n+1}-\overline{x}\rangle. \tag{3.37}$$

By this together with (C1) and (3.35), we have by Lemma 2.3 that $\{x_n\}$ converges strongly to $\overline{x}$. This completes the proof. □

The following examples provide mappings A and B which satisfy those conditions in Theorem 3.1.

Example 3.2. Let $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. If $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$, then we have

(1) $I - \lambda A$ and $I - \mu B$ are nonexpansive,

(2) $\|Ay_n - Ay^*\| \to 0$ and $\|Bu_n - Bx^*\| \to 0$ as $n \to \infty$ for all $x^* \in \Omega$ and $y^* = P_C(x^* - \mu Bx^*)$, where $\{y_n\}$ and $\{u_n\}$ are sequences defined as in Theorem 3.1.

Proof. (1) For any $x, y \in C$, we have

$$\begin{aligned}\|(I-\lambda A)x - (I-\lambda A)y\|^2 &= \|(x-y) - \lambda(Ax-Ay)\|^2 \\ &= \|x-y\|^2 + \lambda^2\|Ax-Ay\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle \\ &\le \|x-y\|^2 + \lambda^2\|Ax-Ay\|^2 - 2\lambda\alpha\|Ax-Ay\|^2 \\ &= \|x-y\|^2 + \lambda(\lambda-2\alpha)\|Ax-Ay\|^2 \\ &\le \|x-y\|^2,\end{aligned} \tag{3.38}$$

hence, $I - \lambda A$ is nonexpansive. Similarly, we can show that $I - \mu B$ is nonexpansive.

(2) Let $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be the sequences defined as in Theorem 3.1. From (3.3), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le a_n\|v-x^*\|^2+(1-a_n)\|t_n-x^*\|^2\\
&= a_n\|v-x^*\|^2+(1-a_n)\left\|P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)\right\|^2\\
&\le a_n\|v-x^*\|^2+(1-a_n)\left\|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*)\right\|^2\\
&\le a_n\|v-x^*\|^2+(1-a_n)\left[\left\|y_n-y^*\right\|^2+\lambda(\lambda-2\alpha)\left\|Ay_n-Ay^*\right\|^2\right]\\
&\le a_n\|v-x^*\|^2+\|x_n-x^*\|^2+(1-a_n)\lambda(\lambda-2\alpha)\left\|Ay_n-Ay^*\right\|^2,\\
\|x_{n+1}-x^*\|^2 &\le a_n\|v-x^*\|^2+(1-a_n)\|t_n-x^*\|^2\\
&\le a_n\|v-x^*\|^2+(1-a_n)\left\|y_n-y^*\right\|^2\\
&\le a_n\|v-x^*\|^2+(1-a_n)\left\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)\right\|^2\\
&\le a_n\|v-x^*\|^2+(1-a_n)\left[\|u_n-x^*\|^2+\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2\right]\\
&\le a_n\|v-x^*\|^2+\|x_n-x^*\|^2+(1-a_n)\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2.
\end{aligned} \tag{3.39}$$

Therefore, we have

$$\begin{aligned}
-(1-a_n)\lambda(\lambda-2\alpha)\left\|Ay_n-Ay^*\right\|^2 &\le a_n\|v-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|,\\
-(1-a_n)\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2 &\le a_n\|v-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.
\end{aligned} \tag{3.40}$$

From (3.15) and (C1), we obtain

$$\left\|Ay_n-Ay^*\right\|\longrightarrow 0,\qquad \|Bu_n-Bx^*\|\longrightarrow 0\quad \text{as } n\longrightarrow\infty. \tag{3.41}$$

□

Example 3.3. Let $A:C\to H$ be an L_A-Lipschitzian and relaxed (c,d)-cocoercive mapping and $B:C\to H$ an L_B-Lipschitzian and relaxed (c',d')-cocoercive mapping. If $0<\lambda<(2(d-cL_A^2))/L_A^2$ and $0<\mu<(2(d'-c'L_B^2))/L_B^2$, then we have

(1) $I-\lambda A$ and $I-\mu B$ are nonexpansive,

(2) $\|Ay_n-Ay^*\|\to 0$ and $\|Bu_n-Bx^*\|\to 0$ as $n\to\infty$ for all $x^*\in\Omega$ and $y^*=P_C(x^*-\mu Bx^*)$, where $\{y_n\}$ and $\{u_n\}$ are sequences defined as in Theorem 3.1.

Proof. (1) For any $x, y \in C$, we have

$$\begin{aligned}
\|(I-\mu B)x-(I-\mu B)y\|^2 &= \|(x-y)-\mu(Bx-By)\|^2 \\
&= \|x-y\|^2+\mu^2\|Bx-By\|^2-2\mu\langle x-y, Bx-By\rangle \\
&\le \|x-y\|^2+\mu^2\|Bx-By\|^2 \\
&\quad -2\mu\left[-c'\|Bx-By\|^2+d'\|x-y\|^2\right] \\
&= \|x-y\|^2+\mu^2\|Bx-By\|^2 \\
&\quad +2\mu c'\|Bx-By\|^2-2\mu d'\|x-y\|^2 \\
&\le \|x-y\|^2+\mu^2L_B^2\|x-y\|^2 \\
&\quad +2\mu c'L_B^2\|x-y\|^2-2\mu d'\|x-y\|^2 \\
&= \left(1+2\mu c'L_B^2-2\mu d'+\mu^2L_B^2\right)\|x-y\|^2 \\
&\le \|x-y\|^2,
\end{aligned} \tag{3.42}$$

hence, $I-\mu B$ is nonexpansive. Similarly, we can show that $I-\lambda A$ is nonexpansive.

(2) Let $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be the sequences defined as in Theorem 3.1. From (3.3), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le a_n\|v-x^*\|^2+(1-a_n)\|t_n-x^*\|^2 \\
&\le a_n\|v-x^*\|^2+(1-a_n)\|y_n-y^*\|^2 \\
&\le a_n\|v-x^*\|^2+(1-a_n)\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)\|^2 \\
&\le a_n\|v-x^*\|^2+(1-a_n) \\
&\quad \times\left[\|u_n-x^*\|^2-2\mu\langle u_n-x^*, Bu_n-Bx^*\rangle+\mu^2\|Bu_n-Bx^*\|^2\right] \\
&\le a_n\|v-x^*\|^2+(1-a_n) \\
&\quad \times\left[\|u_n-x^*\|^2+2\mu c'\|Bu_n-Bx^*\|^2-2\mu d'\|u_n-x^*\|^2+\mu^2\|Bu_n-Bx^*\|^2\right] \\
&\le a_n\|v-x^*\|^2+(1-a_n)\left[\|x_n-x^*\|^2+\left(2\mu c'+\mu^2-\frac{2\mu d'}{L_B^2}\right)\|Bu_n-Bx^*\|^2\right] \\
&\le a_n\|v-x^*\|^2+\|x_n-x^*\|^2+(1-a_n)\left(2\mu c'+\mu^2-\frac{2\mu d'}{L_B^2}\right)\|Bu_n-Bx^*\|^2,
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le a_n\|v-x^*\|^2+(1-a_n)\|t_n-x^*\|^2\\
&= a_n\|v-x^*\|^2+(1-a_n)\left\|P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)\right\|^2\\
&\le a_n\|v-x^*\|^2+(1-a_n)\left\|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*)\right\|^2\\
&\le a_n\|v-x^*\|^2+(1-a_n)\left[\|x_n-x^*\|^2+\left(2\lambda c+\lambda^2-\frac{2\lambda d}{L_A^2}\right)\|Ay_n-Ay^*\|^2\right]\\
&\le a_n\|v-x^*\|^2+\|x_n-x^*\|^2+(1-a_n)\left(2\lambda c+\lambda^2-\frac{2\lambda d}{L_A^2}\right)\|Ay_n-Ay^*\|^2.
\end{aligned} \tag{3.43}$$

Therefore, we have

$$\begin{aligned}
&-(1-a_n)\left(2\mu c'+\mu^2-\frac{2\mu d'}{L_B^2}\right)\|Bu_n-Bx^*\|^2\\
&\qquad\le a_n\|v-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|,\\
&-(1-a_n)\left(2\lambda c+\lambda^2-\frac{2\lambda d}{L_A^2}\right)\|Ay_n-Ay^*\|^2\\
&\qquad\le a_n\|v-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.
\end{aligned} \tag{3.44}$$

From (3.15) and (C1), we obtain

$$\|Ay_n-Ay^*\|\longrightarrow 0,\qquad \|Bu_n-Bx^*\|\longrightarrow 0\quad \text{as } n\longrightarrow\infty. \tag{3.45}$$

□

By using the same proof as in Examples 3.2 and 3.3, we can obtain the following example.

Example 3.4. Let A be an α-inverse-strongly monotone mapping of C into H and B be a L-Lipschitzian and relaxed (c,d)-cocoercive mapping of C into H. If $\lambda\in(0,2\alpha)$ and $0<\mu<(2(d-cL^2))/L^2$, then we have

(1) $I-\lambda A$ and $I-\mu B$ are nonexpansive,

(2) $\|Ay_n-Ay^*\|\to 0$ and $\|Bu_n-Bx^*\|\to 0$ as $n\to\infty$ for all $x^*\in\Omega$ and $y^*=P_C(x^*-\mu Bx^*)$, where $\{y_n\}$ and $\{u_n\}$ are sequences defined as in Theorem 3.1.

Let $\mathcal{A}$ be the class of all α-inverse-strongly monotone mappings from C into H, $\mathcal{B}$ the class of all β-inverse-strongly monotone mappings from C into H, $\mathcal{C}$ the class of all L-Lipschitzian and relaxed (c,d)-cocoercive mappings from C into H, and $\mathfrak{D}$ the class of all L_1-Lipschitzian and relaxed (c',d')-cocoercive mappings from C into H.

Theorem 3.5. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a function from $C\times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ a proper lower semicontinuous and convex function. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of C such that $\Omega = \cap_{i=1}^N F(T_i) \cap \mathrm{GVI}(C,A,B) \cap \mathrm{MEP}(F,\varphi) \neq \emptyset$. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, $j = 1,2,\ldots,N$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$ for all $j = 1,2,\ldots,N-1$, $\alpha_1^N \in (0,1]$ and $\alpha_2^j, \alpha_3^j \in [0,1)$ for all $j = 1,2,\ldots,N$. Let S be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in C. Let $x_1 \in C$ and $\{x_n\}, \{u_n\}, \{y_n\}$ the sequences generated by*

$$\begin{gathered} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \mu B u_n), \\ x_{n+1} = a_n v + (1 - a_n) S P_C(y_n - \lambda A y_n), \quad n \geq 1. \end{gathered} \tag{3.46}$$

If one of the following conditions is satisfied:

(1) $A \in \mathcal{A}$, $B \in \mathcal{B}$, $\lambda \in (0, 2\alpha)$, *and* $\mu \in (0, 2\beta)$,

(2) $A \in \mathcal{C}$, $B \in \mathfrak{D}$, $0 < \lambda < (2(d - cL^2))/L^2$, *and* $0 < \mu < (2(d' - c'L_1^2))/L_1^2$,

(3) $A \in \mathcal{A}$, $B \in \mathcal{C}$, $\lambda \in (0, 2\alpha)$, *and* $0 < \lambda < (2(d - cL^2))/L^2$,

and the sequences $\{r_n\}$ and $\{a_n\}$ are as in Theorem 3.1, then $\{x_n\}$ converges strongly to $\overline{x} = P_\Omega v$ and $(\overline{x}, \overline{y})$ is a solution of the problem (1.6), *where $\overline{y} = P_C(\overline{x} - \mu B\overline{x})$.*

Proof. By Lemma 2.7, we obtain that S is nonexpansive and $F(S) = \cap_{i=1}^N F(T_i)$. Hence, the result is obtained directly from Theorem 3.1 and Examples 3.2–3.4. □

From Theorem 3.1 and Examples 3.2–3.4, we obtain the following result.

Corollary 3.6. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a function from $C\times C$ to $\mathbb{R}$ satisfying (A1)–(A5) and $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ a proper lower semicontinuous and convex function. Let T be a nonexpansive self-mapping of C such that $\Omega = F(T) \cap \mathrm{GVI}(C,A,B) \cap \mathrm{MEP}(F,\varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in C. If one of the following conditions is satisfied:*

(1) $A \in \mathcal{A}$, $B \in \mathcal{B}$, $\lambda \in (0, 2\alpha)$, *and* $\mu \in (0, 2\beta)$,

(2) $A \in \mathcal{C}$, $B \in \mathfrak{D}$, $0 < \lambda < (2(d - cL^2))/L^2$, *and* $0 < \mu < (2(d' - c'L_1^2))/L_1^2$,

(3) $A \in \mathcal{A}$, $B \in \mathcal{C}$, $\lambda \in (0, 2\alpha)$, *and* $0 < \lambda < (2(d - cL^2))/L^2$,

and the sequences $\{r_n\}$, $\{a_n\}$ are as in Theorem 3.1, then the sequence $\{x_n\}$ generated by (1.12) *converges strongly to $\overline{x} = P_\Omega v$ and $(\overline{x}, \overline{y})$ is a solution of the problem* (1.6), *where $\overline{y} = P_C(\overline{x} - \mu B\overline{x})$.*

Let $\varphi = 0$ in Theorem 3.1. From Theorem 3.1 and Examples 3.2–3.4, we obtain the following result.

Corollary 3.7. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A5). Let T be a nonexpansive self-mapping of C such that $\Omega = F(T) \cap \mathrm{GVI}(C, A, B) \cap \mathrm{EP}(F) \neq \emptyset$. Let $x_1, v \in C$ and $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ be the sequences generated by*

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \mu B u_n), \\ x_{n+1} = a_n v + (1 - a_n) T P_C(y_n - \lambda A y_n), \quad n \geq 1. \end{gathered} \tag{3.47}$$

If one of the following conditions is satisfied:

(1) $A \in \mathcal{A}$, $B \in \mathcal{B}$, $\lambda \in (0, 2\alpha)$, *and* $\mu \in (0, 2\beta)$,

(2) $A \in \mathcal{C}$, $B \in \mathfrak{D}$, $0 < \lambda < (2(d - cL^2))/L^2$, *and* $0 < \mu < (2(d' - c'L_1^2))/L_1^2$,

(3) $A \in \mathcal{A}$, $B \in \mathcal{C}$, $\lambda \in (0, 2\alpha)$, *and* $0 < \lambda < (2(d - cL^2))/L^2$,

and the sequences $\{r_n\}$, $\{a_n\}$ are as in Theorem 3.1, then $\{x_n\}$ converges strongly to $\overline{x} = P_\Omega v$ and $(\overline{x}, \overline{y})$ is a solution of the problem (1.6), *where $\overline{y} = P_C(\overline{x} - \mu B\overline{x})$.*

Remark 3.8. In Theorem 3.5, if $F \equiv 0$, then the sequence $\{x_n\}$ generated by (3.46) converges strongly to a solution of the minimization problem which is also a solution of a system of variational inequalities.

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