Research Article

Product Property on Generalized Lindelöf Spaces

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We study the product properties of nearly Lindelöf, almost Lindelöf, and weakly Lindelöf spaces. We prove that in weak P -spaces, these topological properties are preserved under finite topological products. We also show that the product of separable spaces is weakly Lindelöf.

1. Introduction

In 1959 Frolík [1] introduced the notion of weakly Lindelöf space that afterward was studied by several authors. In 1982 Balasubramanian [2] introduced and studied the notion of nearly Lindelöf spaces as a generalization of the nearly compact spaces; then in 1986 Mršević et al. [3] gave some characterizations of these spaces. In 1984 Willard and Dissanayake [4] gave the notion of almost Lindelöf spaces. In 1996 Cammaroto and Santoro [5] studied and gave further new results related to these generalizations of Lindelöf spaces, and recently the authors (see [6–8]) studied mappings and semiregular property on these generalizations of Lindelöf spaces. By using the regularly open and regularly closed sets, these structures can also be extended to the bitopological spaces; for more details on regularly pairwise open and closed sets see, for example, [9–12].

It is well known that many of the results on the invariance of covering properties under product are negative, that is, the covering properties are simply not preserved by the product unless one or more of the factors are assumed to be satisfied as additional conditions.

In this work, we discuss the product problem in the sense of generalizations of Lindelöf spaces, namely, nearly Lindelöf, almost Lindelöf, and weakly Lindelöf spaces. We will note that a well-known example shows the properties nearly Lindelöf and almost Lindelöf are not finitely productive. We also give some necessary conditions for these covering properties to be preserved under a finite product.

In this paper, we let (X, τ) be a topological space on which no separation axioms are considered unless explicitly stated. The interior and the closure of any subset A of (X, τ) will be denoted by $\text{Int}(A)$ and $\text{Cl}(A)$, respectively. Recall that a subset $A \subseteq X$ is called regularly open (regularly closed) if $A = \text{Int}(\text{Cl}(A))$ ($A = \text{Cl}(\text{Int}(A))$). The topology generated by regularly open subsets of a space (X, τ) is called the semiregularization of the space (X, τ) and is denoted by (X, τ^*) or simply by X^* . A space (X, τ) is said to be semiregular if the regularly open sets form a base for the topology or equivalently $\tau = \tau^*$. By regularly open cover of X we mean a cover of X by regularly open sets in (X, τ) . Moreover, a space X is called a P -space if every G_δ -set is open in X , and it is called nearly compact [13] if every open cover $\{U_\alpha : \alpha \in \Delta\}$ of X admits a finite subfamily such that $X = \bigcup_{k=1}^n \text{Int}(\text{Cl}(U_k))$ or, equivalently, every regularly open cover of X has a finite subcover.

Recall also that a function f from a topological space X to a topological space Y is called R -map [14] (almost continuous [15]) if $f^{-1}(V)$ is regularly open (open) in X for every regularly open set V in Y . It is called θ -continuous [16] if for every $x \in X$ and every open subset V of Y containing $f(x)$, there exists an open subset U in X containing x such that $f(\text{Cl}(U)) \subseteq \text{Cl}(V)$. Moreover, f is called almost open [15] if $f(U)$ is open in Y for every regularly open subset U in X , and it is called almost closed [15] if $f(C)$ is closed in Y for every regularly closed subset C in X .

2. Preliminaries

It is known that a nonempty product space $\prod_{\alpha \in \Delta} X_\alpha$ is Hausdorff (regular, completely regular, resp.) if and only if each factor space X_α is Hausdorff (regular, completely regular, resp.). A nonempty product space $\prod_{\alpha \in \Delta} X_\alpha$ is compact if and only if each factor space X_α is compact. Moreover, the product of a paracompact space with a compact T_2 -space is paracompact. However, products of normal, paracompact, or Lindelöf spaces often fail to be normal, paracompact, or Lindelöf, respectively. Note that a space X is a P -space if and only if the countable union of closed sets is closed if and only if the countable intersection of open sets is open.

It is well known that the product of two Lindelöf spaces is not necessarily Lindelöf since the Sorgenfrey line E is Lindelöf but the Sorgenfrey plane $E \times E$ is not Lindelöf. In 1972, Misra [17] proved that, in T_1 -spaces, finite product of P -spaces is a P -space and no infinite product of P -spaces with more than one point is a P -space. We note that Misra's result for T_1 -spaces, that finite product of P -spaces is a P -space, also holds for arbitrary spaces. Misra [17] also proved that the product of two Lindelöf P -spaces is a Lindelöf P -space. Thus, in P -spaces, finite product of Lindelöf spaces is Lindelöf. The following proposition shows that it is sufficient that one of the two spaces is a P -space, to ensure that their product is Lindelöf. In fact, for T_1 -spaces, this result is an immediate corollary to Misra's Theorem 2.1 and Proposition 4.2(g) in [17].

Proposition 2.1. *Let (X, τ) be a Lindelöf P -space and (Y, σ) a Lindelöf space. Then $(X \times Y, \tau \times \sigma)$ is Lindelöf.*

Proof. Let $\mathcal{U} = \{T_\alpha \times U_\alpha : \alpha \in \Delta\}$ be an open cover of $X \times Y$; Let $x \in X$; then for each $y \in Y$, there exists $T_{\alpha(x)_y} \in \tau$ and $U_{\alpha(x)_y} \in \sigma$ such that $(x, y) \in T_{\alpha(x)_y} \times U_{\alpha(x)_y}$. The subspace $S_x = \{x\} \times Y$ of $X \times Y$ is a homeomorphic copy of the Lindelöf space (Y, σ) , thus it is Lindelöf. Now $\mathcal{U}$ is an open cover of $X \times Y$; hence $\{\{x\} \times U_{\alpha(x)_y} : U_{\alpha(x)_y} \in \sigma\}$ is an open cover of the Lindelöf space $\{x\} \times Y$. Thus it has a countable subcollection $\{\{x\} \times U_{\alpha(x)_{y_n}} : n \in \mathbb{N}\}$ which covers S_x . So the

countable family $\{T_{\alpha(x)_{y_n}} \times U_{\alpha(x)_{y_n}} : n \in \mathbb{N}\}$ covers $S_x = \{x\} \times Y$. Now define $T_x = \bigcap_{n \in \mathbb{N}} T_{\alpha(x)_{y_n}}$. Since X is a P -space, T_x is an open subset in X . Thus $\{T_x \times U_{\alpha(x)_{y_n}} : n \in \mathbb{N}\}$ is a countable open cover of the slab $R_x = T_x \times Y$, and hence all the more $\{T_{\alpha(x)_{y_n}} \times U_{\alpha(x)_{y_n}} : n \in \mathbb{N}\}$ is a countable open cover of the slab R_x .

Now the collection of sets $\{T_x : x \in X\}$ is an open cover of the Lindelöf space X . So it has a countable set of points $\{x_k : k \in \mathbb{N}\}$ such that $X \subseteq \bigcup_{k \in \mathbb{N}} T_{x_k}$. Thus $X \times Y \subseteq (\bigcup_{k \in \mathbb{N}} T_{x_k}) \times Y = \bigcup_{k \in \mathbb{N}} (T_{x_k} \times Y) = \bigcup_{k \in \mathbb{N}} R_{x_k}$.

Thus we have a countable collection $\{R_{x_k} : k \in \mathbb{N}\}$ of slabs covering $X \times Y$, and for each $k \in \mathbb{N}$, R_{x_k} is covered by the countable collection $\{T_{\alpha(x_k)_{y_n}} \times U_{\alpha(x_k)_{y_n}} : n \in \mathbb{N}\}$. So $\{T_{\alpha(x_k)_{y_n}} \times U_{\alpha(x_k)_{y_n}} : k, n \in \mathbb{N}\}$ is a countable subcover of $\mathcal{U}$. Thus $X \times Y$ is Lindelöf, which completes the proof. $\square$

Remark 2.2. (a) Recall that a space X is a weak P -space [18] if, for each countable family $\{U_n : n \in \mathbb{N}\}$ of open sets in X , we have $\text{Cl}(\bigcup_{n \in \mathbb{N}} U_n) = \bigcup_{n \in \mathbb{N}} \text{Cl}(U_n)$. Clearly, (X, τ) is a weak P -space if and only if the countable union of regularly closed sets is regularly closed if and only if the countable intersection of regularly open sets is regularly open. Moreover, a space (X, τ) is a weak P -space if and only if (X, τ^*) is a P -space (see [19]). Note also that every P -space is a weak P -space but the converse is not necessarily true, since the finite complement topology on $\mathbb{R}$ is a weak P -space but it is not a P -space.

(b) It is a known fact that the semiregularization of a product space is the product of the semiregularizations of the factor spaces.

Lemma 2.3. *Finite product of weak P -spaces is a weak P -space.*

Proof. We prove for only two spaces using Remark 2.2. So let X and Y be two weak P -spaces. Then X^* and Y^* are P -spaces. Thus $X^* \times Y^* = (X \times Y)^*$ is a P -space. Therefore, $X \times Y$ is a weak P -space. $\square$

Note that infinite product of weak P -spaces is not necessarily a weak P -space, since if X is any discrete space containing more than one point and A is infinite; then the product space X^A is not a P -space, and, since it is semiregular, it cannot be a weak P -space either.

3. On Generalized Lindelöf Spaces

Definition 3.1 (see [1, 2, 4]). A topological space X is called nearly Lindelöf, almost Lindelöf, and weakly Lindelöf if, for every open cover $\{U_\alpha : \alpha \in \Delta\}$ of X , there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\} \subseteq \Delta$ such that

$$X = \bigcup_{n \in \mathbb{N}} \text{Int}(\text{Cl}(U_{\alpha_n})), \quad X = \bigcup_{n \in \mathbb{N}} \text{Cl}(U_{\alpha_n}), \quad X = \text{Cl}\left(\bigcup_{n \in \mathbb{N}} U_{\alpha_n}\right), \quad (3.1)$$

respectively.

One can easily show that if a space is semiregular and nearly Lindelöf (or regular and almost Lindelöf), then it is Lindelöf. And it is well known that the Sorgenfrey line E is regular and Lindelöf, but the Sorgenfrey plane $E \times E$ is not Lindelöf. Thus, neither of the properties almost Lindelöf and nearly Lindelöf is finitely productive.

The following proposition shows that if the product of topological spaces has any property of Definition 3.1, then each factor space has the same property.

Proposition 3.2. *Suppose that X_α is a nonempty topological space. If $\prod_{\alpha \in \Delta} X_\alpha$ is nearly Lindelöf (resp., almost Lindelöf or weakly Lindelöf), then X_α is nearly Lindelöf (resp., almost Lindelöf or weakly Lindelöf).*

Proof. Since the projection map $\pi_\alpha : \prod X_\alpha \rightarrow X_\alpha$ is a continuous and open function from $\prod X_\alpha$ onto X_α , it is almost continuous and almost open. Thus π_α is θ -continuous and R -map (see [20, 21]). Therefore, X_α is nearly Lindelöf, almost Lindelöf, and weakly Lindelöf (see [7, Corollary 3.1], [8, Corollary 3.3], and [22, Theorem 3.2], resp.). $\square$

In [5], it was shown that the product of a nearly Lindelöf space with a nearly compact space is nearly Lindelöf. Next we prove analogous results concerning almost Lindelöf and weakly Lindelöf spaces.

Proposition 3.3. *Let X be an almost Lindelöf (weakly Lindelöf) space and Y a nearly compact space. Then $X \times Y$ is almost Lindelöf (weakly Lindelöf).*

Proof. The proof of Proposition 3.3 is similar to the proof of an analogous result for nearly Lindelöf spaces (see [5, Proposition 1.9]). $\square$

Note that a space (X, τ) is nearly Lindelöf if and only if (X, τ^*) is Lindelöf (see [3, Theorem 1]). Thus, using this fact, the proof of the following theorem becomes easy.

Theorem 3.4. *The product of a nearly Lindelöf weak P -space with a nearly Lindelöf space is nearly Lindelöf.*

Proof. Let X be a nearly Lindelöf weak P -space and Y nearly Lindelöf. Thus, by Remark 2.2(a), X^* is a Lindelöf P -space and Y^* is Lindelöf. So, by Proposition 2.1, $X^* \times Y^* = (X \times Y)^*$ is Lindelöf. Therefore, $X \times Y$ is nearly Lindelöf. $\square$

Now on using Theorem 3.4 and Lemma 2.3, we conclude the following corollary.

Corollary 3.5. *The product of finitely many nearly Lindelöf spaces, all but one of which are weak P -spaces, is nearly Lindelöf.*

Next we prove that the result in Theorem 3.4 above is correct for almost Lindelöf and weakly Lindelöf spaces.

Theorem 3.6. *The product of an almost Lindelöf weak P -space X with an almost Lindelöf space Y is almost Lindelöf.*

Proof. Since almost Lindelöf property is a semiregular property, that is, a space (X, τ) is almost Lindelöf if and only if (X, τ^*) is almost Lindelöf (see [6, Theorem 2.1]), it is sufficient to prove that $X^* \times Y^*$ is almost Lindelöf. Thus let $\{U_\alpha : \alpha \in \Delta\}$ be an open cover of $X^* \times Y^*$, and, without loss of generality, suppose that $U_\alpha = V_\alpha \times W_\alpha$ for every $\alpha \in \Delta$ where V_α is regularly open in X and W_α is regularly open in Y . Fix $x \in X$, and, for each $y(x) \in Y$, there exists $\alpha_{y(x)} \in \Delta$ such that $(x, y(x)) \in V_{\alpha_{y(x)}} \times W_{\alpha_{y(x)}}$.

Now $\{W_{\alpha_{y(x)}} : y(x) \in Y\}$ is an open cover of the almost Lindelöf space Y , so it has a countable subset $\{W_{\alpha_{y_n(x)}} : n \in \mathbb{N}\}$ such that $Y = \bigcup_{n \in \mathbb{N}} \text{Cl}(W_{\alpha_{y_n(x)}})$. Put $H_x = \bigcap_{n \in \mathbb{N}} V_{\alpha_{y_n(x)}}$. Since X is a weak P -space, H_x is a regularly open set in X . Thus $\{H_x : x \in X\}$ is an open cover of the almost Lindelöf space X , so it has a countable subset $\{H_{x_m} : m \in \mathbb{N}\}$ such that $X = \bigcup_{m \in \mathbb{N}} \text{Cl}(H_{x_m})$. Therefore,

$$\begin{aligned}
 X \times Y &= \bigcup_{m \in \mathbb{N}} \text{Cl}(H_{x_m}) \times \bigcup_{n \in \mathbb{N}} \text{Cl}(W_{\alpha_{y_n(x)}}) \\
 &= \bigcup_{m, n \in \mathbb{N}} \left(\text{Cl}(H_{x_m}) \times \text{Cl}(W_{\alpha_{y_n(x)}}) \right) \\
 &\subseteq \bigcup_{m, n \in \mathbb{N}} \text{Cl}\left(V_{\alpha_{y_n(x_m)}} \times W_{\alpha_{y_n(x)}}\right) \\
 &\subseteq \bigcup_{m, n \in \mathbb{N}} \text{Cl}\left(V_{\alpha_{y_n(x_m)}} \times W_{\alpha_{y_n(x_m)}}\right) \\
 &= \bigcup_{m, n \in \mathbb{N}} \text{Cl}\left(U_{\alpha_{y_n(x_m)}}\right).
 \end{aligned} \tag{3.2}$$

Since the last term is countable, thus $X^* \times Y^* = (X \times Y)^*$ is also almost Lindelöf and therefore $X \times Y$ is almost Lindelöf. $\square$

Corollary 3.7. *The product of finitely many almost Lindelöf spaces, all but one of which are weak P -spaces, is almost Lindelöf.*

For weakly Lindelöf spaces we give the following results.

Theorem 3.8. *The product of a weakly Lindelöf weak P -space X with a weakly Lindelöf space Y is weakly Lindelöf.*

Proof. The proof of Theorem 3.8 is similar to the proof of Theorem 3.6, thus the details are omitted. $\square$

Corollary 3.9. *The product of finitely many weakly Lindelöf spaces, all but one of which are weak P -spaces, is weakly Lindelöf.*

Next we prove that the product of separable spaces is weakly Lindelöf. First we recall that a space X is called separable if it has a countable dense subset, and one says that X has caliber $\aleph_1$ if, whenever $\mathcal{U}$ is a family of open subsets of X with $|\mathcal{U}| = \aleph_1$, a subfamily $\mathcal{V}$ of $\mathcal{U}$ exists with $|\mathcal{V}| = \aleph_1$ and $\bigcap \{V : V \in \mathcal{V}\} \neq \emptyset$. One also says that X satisfies the countable chain condition if every family of disjoint open subsets of X is countable. Moreover, X is called almost rc-Lindelöf [23] if every regularly closed cover of X has a countable subfamily whose union is dense in X .

Theorem 3.10. *The product of separable spaces is weakly Lindelöf.*

Proof. Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of separable spaces. Then $\prod X_\alpha$ has caliber $\aleph_1$. Thus $\prod X_\alpha$ satisfies the countable chain condition. So, $\prod X_\alpha$ is an almost rc-Lindelöf space (see [23, Proposition 2.2]). Therefore, $\prod X_\alpha$ is weakly Lindelöf (see [23, Theorem 2.1]). $\square$

Note that the product of two separable Lindelöf spaces need not be almost Lindelöf, since the Sorgenfrey line E is separable, but $E \times E$ is not almost Lindelöf since it is a regular non-Lindelöf space.

Since in weak P -spaces, for any countable open subsets $\{U_n : n \in \mathbb{N}\}$ of X , we have $\text{Cl}(\bigcup_{n \in \mathbb{N}} U_n) = \bigcup_{n \in \mathbb{N}} \text{Cl}(U_n)$. Thus we conclude the following lemma.

Lemma 3.11. *In weak P -spaces, weakly Lindelöf property and almost Lindelöf property are equivalent.*

So depending on Theorem 3.6 we conclude the following corollaries.

Corollary 3.12. *If X is a weakly Lindelöf weak P -space and Y is almost Lindelöf, then $X \times Y$ is almost Lindelöf.*

Corollary 3.13. *In weak P -spaces, finite product of weakly Lindelöf spaces is almost Lindelöf.*

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