Finite Groups Whose Certain Subgroups of Prime Power Order Are $S$-Semipermutable

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Let $G$ be a finite group. A subgroup $H$ of $G$ is said to be $S$-semipermutable in $G$ if $H$ permutes with every Sylow $p$-subgroup of $G$ with $(p, |H|) = 1$. In this paper, we study the influence of $S$-permutability property of certain abelian subgroups of prime power order of a finite group on its structure.

1. Introduction

All groups considered in this paper will be finite. Two subgroups $H$ and $K$ of a group $G$ are said to permute if $HK = KH$. It is easily seen that $H$ and $K$ permute if and only if $HK$ is a subgroup of $G$. We say, following Kegel [1], that a subgroup of $G$ is $S$-quasinormal in $G$ if it permutes with every Sylow subgroup of $G$. Chen [2] introduced the following concept: a subgroup $H$ of $G$ is said to be $S$-semipermutable in $G$ if $H$ permutes with every Sylow $p$-subgroup of $G$ with $(p, |H|) = 1$. Obviously, every $S$-quasinormal subgroup of $G$ is an $S$-semipermutable subgroup of $G$. In contrast to the fact that every $S$-quasinormal subgroup of $G$ is a subnormal subgroup of $G$ (see [1]), it does not hold in general that every $S$-semipermutable subgroup of $G$ is a subnormal subgroup of $G$. It suffices to consider the alternating group of degree 4.

Several authors have investigated the structure of a finite group when some information is known about some subgroups of prime power order in the group. Huppert [3] proved that a finite group $G$ is solvable provided that all subgroups of prime order are normal in $G$. Buckley [4], proved that a group $G$ of odd order is supersolvable provided that all subgroups of prime order are normal in $G$. Srinivasan [5], and proved that a finite group $G$ is supersolvable if the maximal subgroups of every Sylow subgroup of $G$ are normal in $G$. 
Developing the result of Srinivasan, Ramadan [6] proved that if \( G \) is a solvable group and the maximal subgroups of every Sylow subgroup of the Fitting subgroup \( F(G) \) of \( G \) are normal in \( G \), then \( G \) is supersolvable.

For a finite \( p \)-group \( P \), we denote

\[
\Omega(P) = \Omega_2(P) \quad \text{if } p > 2, \quad \Omega(P) = \langle \Omega_1(P), \Omega_2(P) \rangle \quad \text{if } p = 2, \tag{1.1}
\]

where \( \Omega_i(P) = \langle x \in P : |x| = p^i \rangle \).

Of late there has been a considerable interest to investigate the influence of the abelian subgroups of largest possible exponent of prime power order (we call such subgroups ALPE-subgroups) on the structure of the group. Asaad et al. [7] proved that if \( G \) is a group such that for every prime \( p \) and every Sylow \( p \)-subgroup \( G_p \) of \( G \), the ALPE-subgroups of \( G_p \) (resp., \( \Omega(G_p) \)) are normal in \( G \), then \( G \) is supersolvable. Ramadan [8] proved the following two results. (1) Let \( G \) be a group such that for every prime \( p \) and every Sylow \( p \)-subgroup \( G_p \) of \( G \), the ALPE-subgroups of \( G_p \) (resp., \( \Omega(G_p) \)) are \( S \)-quasinormal in \( G \), then \( G \) is supersolvable. (2) Let \( K \) be a normal subgroup of \( G \) such that \( G/K \) is supersolvable. If for every prime \( p \) and every Sylow \( p \)-subgroup \( K_p \) of \( K \), the ALPE-subgroups of \( K_p \) (resp., \( \Omega(K_p) \)) are \( S \)-quasinormal in \( G \), then \( G \) is supersolvable.

In this paper, we study the structure of a finite group under the assumption that certain subgroups of prime power order are \( S \)-semipermutable in the group. We focus our attention on \( S \)-semipermutability property of the ALPE-subgroups of a fixed ALPE-subgroup having maximal order of the Sylow subgroups of a finite group. Furthermore, we improve and extend the above-mentioned results by using the concept of \( S \)-semipermutability.

\section{2. Preliminaries}

In this section, we give some results which will be useful in the sequel.

\textbf{Lemma 2.1} (see [2, Lemmas 1 and 2]). Let \( G \) be a group.

(i) If \( H \) is a \( S \)-semipermutable subgroup of \( G \) and \( K \) is a subgroup of \( G \) such that \( H \leq K \leq G \), then \( H \) is \( S \)-semipermutable in \( K \).

(ii) Let \( \pi \) be a set of primes, \( N \) a normal \( \pi' \)-subgroup of \( G \), and \( H \) a \( \pi \)-subgroup of \( G \). If \( H \) is \( S \)-semipermutable in \( G \), then \( HN/N \) is \( S \)-semipermutable in \( G/N \).

\textbf{Lemma 2.2} (see [9, Lemma A]). Let \( H \) be a \( p \)-subgroup of \( G \); for some prime \( p \). Then \( H \) is \( S \)-quasinormal in \( G \) if and only if \( O^p(G) \leq N_G(H) \), where \( O^p(G) \) is the normal subgroup of \( G \) generated by all \( p' \)-elements of \( G \).

\textbf{Lemma 2.3}. Let \( H \) be a \( p \)-subgroup of \( G \), \( p \) is a prime. Then the following statements are equivalent:

(i) \( H \) is \( S \)-quasinormal in \( G \);

(ii) \( H \leq O_p(G) \) and \( H \) is \( S \)-semipermutable in \( G \).

\textit{Proof.} (i) \( \Rightarrow \) (ii): Suppose that \( H \) is \( S \)-quasinormal in \( G \). So it follows by [1, Satz 1, page 209] that \( H \) is subnormal in \( G \) and then by [10, Lemma 8.6(a), page 28] that \( H \leq O_p(G) \). Since \( H \) is \( S \)-quasinormal in \( G \), obviously, it is \( S \)-semipermutable in \( G \). Thus (ii) holds.
is a subgroup of \(G\), and we have by Lemma 2.3, we have that 
\[\text{if } G \leq M < G, \text{ then } M \text{ is } p\text{-nilpotent}.\]

It is clear to see by Lemma 2.1 that the ALPE-subgroups of \(P\) are \(S\)-semipermutable in \(G\), so that \(M\) satisfies the hypothesis of the theorem. Thus, the minimality of \(G\) yields that \(M\) is \(p\)-nilpotent.

(2) \(N_G(P)\) is \(p\)-Nilpotent

Suppose that \(P\) is normal in \(G\). Let \(H\) be an ALPE-subgroup of \(P\) (in particular, we may take \(H = P\)). By hypothesis, \(H\) is \(S\)-semipermutable in \(G\) and so by Lemma 2.3, we have that \(H\) is \(S\)-quasinormal in \(G\). Hence \(H_G = G\) is a subgroup of \(G\), where \(G\) is a Sylow \(q\)-subgroup of \(G\) with \(q \neq p\). Clearly, \(H\) is a subnormal Hall subgroup of \(H_G\). Thus \(H\) is normal in \(H_G\) and hence \(H\) is normal in \(P\) as \(P\) is abelian. Thus \(P\) is supersolvably embedded in \(P\) by Lemma 2.4 and so \(P \leq Q_\infty(P)\). Since \(Q_\infty(P) \leq \text{gen}_{\infty}(P)\) by [14, page 34], it follows by Lemma 2.5 that \(P\) is \(p\)-nilpotent. Thus \(P = P \times P\), hence \(G\) is contained in \(C(P)\), so that \(Q_\infty(G) \leq C(P)\). If \(C(P) < G\), then \(C(P)\) is \(p\)-nilpotent by (1). Thus \(Q_\infty(G)\) is \(p\)-nilpotent and so \(G\) is \(p\)-nilpotent: a contradiction. Thus we may assume that \(C(P) = G\). Then \(P \leq Z(G)\), in particular, \(P \leq Z(G)\). So, \(P = G_p\) by the maximality of \(P\) and we have by [15, Theorem 4.3, page 252] that \(G\) is \(p\)-nilpotent: a contradiction. Thus we may assume that \(N_G(P) < G\). According to (1), we have that \(N_G(P)\) is \(p\)-nilpotent.

3. Main Results

Theorem 3.1. Let \(p\) be the smallest prime dividing the order of a group \(G\), and let \(G_p\) be a Sylow \(p\)-subgroup of \(G\). Fix an ALPE-subgroup \(P\) of \(G_p\) having maximal order. If the ALPE-subgroups of \(P\) are \(S\)-semipermutable in \(G\), then \(G\) is \(p\)-nilpotent.

Proof. Suppose that the theorem is false, and let \(G\) be a counterexample of minimal order. We prove the following steps.

(1) \(If \ P \leq M < G, Then \ M \text{ Is } p\text{-Nilpotent}\)

(2) \(N_G(P)\) is \(p\)-Nilpotent

Suppose that \(P\) is normal in \(G\). Let \(H\) be an ALPE-subgroup of \(P\) (in particular, we may take \(H = P\)). By hypothesis, \(H\) is \(S\)-semipermutable in \(G\) and so by Lemma 2.3, we have that \(H\) is \(S\)-quasinormal in \(G\). Hence \(H_G = G\) is a subgroup of \(G\), where \(G\) is a Sylow \(q\)-subgroup of \(G\) with \(q \neq p\). Clearly, \(H\) is a subnormal Hall subgroup of \(H_G\). Thus \(H\) is normal in \(H_G\) and hence \(H\) is normal in \(P\) as \(P\) is abelian. Thus \(P\) is supersolvably embedded in \(P\) by Lemma 2.4 and so \(P \leq Q_\infty(P)\). Since \(Q_\infty(P) \leq \text{gen}_{\infty}(P)\) by [14, page 34], it follows by Lemma 2.5 that \(P\) is \(p\)-nilpotent. Thus \(P = P \times P\), hence \(G\) is contained in \(C(P)\), so that \(Q_\infty(G) \leq C(P)\). If \(C(P) < G\), then \(C(P)\) is \(p\)-nilpotent by (1). Thus \(Q_\infty(G)\) is \(p\)-nilpotent and so \(G\) is \(p\)-nilpotent: a contradiction. Thus we may assume that \(C(P) = G\). Then \(P \leq Z(G)\), in particular, \(P \leq Z(G)\). So, \(P = G_p\) by the maximality of \(P\) and we have by [15, Theorem 4.3, page 252] that \(G\) is \(p\)-nilpotent: a contradiction. Thus we may assume that \(N_G(P) < G\). According to (1), we have that \(N_G(P)\) is \(p\)-nilpotent.
(3) $O_p'(G) = 1$

If $O_p(G) \neq 1$, we consider the quotient group $G/O_p'(G)$. Clearly, $G_pO_p'(G)/O_p'(G)$ is a Sylow $p$-subgroup of $G/O_p'(G)$ and $PQ_p(G)/O_p'(G)$ is an ALPE-Subgroup of $G_pO_p'(G)/O_p'(G)$ having maximal order. By Lemma 2.1, the hypotheses are inherited over $G/O_p'(G)$. Thus, the minimality of $G$ implies that $G/O_p'(G)$ is $p$-nilpotent, hence $G$ is $p$-nilpotent, which is a contradiction.

(4) $G = G_pG_q$. Where $G_q$ Is a Sylow $q$-Subgroup of $G$ with $q \neq p$

Since $G$ is not $p$-nilpotent by [15, Theorem 4.5, page 253], there exists a subgroup $H$ of $G_p$ such that $N_C(H)$ is not $p$-nilpotent. But $N_C(G_p)$ is $p$-nilpotent by a similar argument of the proof of the step (2). Thus we may choose a subgroup $H$ of $G_p$ such that $N_C(H)$ is not $p$-nilpotent but $N_C(K)$ is $p$-nilpotent for every subgroup $K$ of $G_p$ with $H < K \leq G_p$. It is easy to see that $N_C(G_p) \leq N_C(H) \leq G$. If $N_C(H) \neq G$, it follows by (1) that $N_C(H)$ is $p$-nilpotent: a contradiction. Thus $N_C(H) = G$. This leads to $O_p(G) \neq 1$ and $N_C(K)$ is $p$-nilpotent for every subgroup $K$ of $G_p$ with $O_p(G) < K \leq G_p$. Now, by [15, Theorem 4.5, page 253] again, we see that $G/O_p(G)$ is $p$-nilpotent and therefore that $G$ is $p$-solvable. Since $G$ is $p$-solvable, for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow $q$-subgroup $G_q$ of $G$ such that $G_pG_q \leq G$ by [15, Theorem 3.5, page 229]. If $G_pG_q < G$, then $G_pG_q$ is $p$-nilpotent by (1) and hence $O_p(G)G_q$ is $p$-nilpotent. Thus $O_p(G)G_q = O_p(G) \times G_q$. This leads to $G_q \leq C_G(O_p(G)) \leq O_p(G)$ by [15, Theorem 3.2, page 228] as $O_p(G) = 1$ by (3), which is a contradiction. Thus $G = G_pG_q$

(5) The Final Contradiction

Let $N$ be a minimal normal subgroup of $G$ such that $N \leq O_p(G)$. Clearly, $N \cap Z(G_p) \neq 1$ and so $Z(G_p) \leq P$ by the maximality of $P$. Hence $1 \neq N \cap Z(G_p) \leq N \cap P$. By hypothesis, $PG_q \leq G$ for any Sylow $q$-subgroup $G_q$ of $G$ with $(q, |P|) = 1$. It is easy to see that $N \cap P = N \cap PG_q < PG_q$. Thus $O_p'(G) \leq N_C(N \cap P)$. If $N_C(N \cap P) < G$, then by (1), $N_C(N \cap P)$ is $p$-nilpotent. Hence $O_p'(G)$ is $p$-nilpotent and so also does $G$: a contradiction. Thus we may assume that $N_C(N \cap P) = G$. By the minimality of $N$ and since $N \cap P \neq 1$, we have that $N \cap P = N$ and so $N \leq P$. If $PG_q < G$, then $Pq$ is $p$-nilpotent by (1) and hence $NG_q$ is $p$-nilpotent. Thus $NG_q = N \times G_q$ and so $G_q \leq C_G(N)$. Thus by (4), $G/C_G(N)$ is a $p$-group and so by [14, Theorem 6.3, page 221], $N \leq Z_\infty(G)$. Since $Z_\infty(G) \leq Q_\infty(G)$, we have that $N \leq Q_\infty(G)$ which implies that $N$ is supersolvably embedded in $G$ and so clearly that $|N| = p$. Thus, it is easy to see that the quotient group $G/N$ satisfies the hypothesis of the theorem by Lemma 2.1. Now, by the minimality of $G$, we see that $G/N$ is $p$-nilpotent. Since the class of all $p$-nilpotent groups is a saturated formation, we have that $N$ is the unique minimal normal subgroup of $G$ and $N \neq \Phi(G)$. Thus $\Phi(G) = 1$ and hence $N = O_p(G)$ by Lemma 2.6 and so $F(G) = O_p(G) = N$ by (3). Hence $G_q \leq C_G(F(G))$. Since $G$ is solvable, it follows by [15, Theorem 2.6, page 216] that $C_G(F(G)) \leq F(G) = O_p(G)$: a contradiction. Thus we must have $G = PG_q$. Let $G_q$ be a Sylow $q$-subgroup of $N_C(P)$. By (2), we have that $G_q < N_C(P)$. Hence $N_C(P) = PG_q = P \times G_q$. Thus $P \leq Z(N_C(P))$, and therefore, $G$ is $p$-nilpotent by [15, Theorem 4.3, page 252]: a final contradiction.

We need the following result.
Theorem 3.2. Let $\mathcal{F}$ be a saturated formation containing the class of supersolvable groups $\mathcal{U}$. Let $G_p$ be a normal Sylow $p$-subgroup of a group $G$ such that $G/G_p \in \mathcal{F}$. Fix an ALPE-subgroup $P$ of $G_p$ having maximal order. If the ALPE-subgroups of $P$ are $S$-semipermutable in $G$, then $G \in \mathcal{F}$.

Proof. We treat the following two cases.

Case 1. $O_p(G) \neq 1$.

Clearly, $G_pO_p(G)/O_p(G)$ is a normal Sylow $p$-subgroup of $G/O_p(G)$ and $PO_p(G)/O_p(G)$ is an ALPE-subgroup of $G_pO_p(G)/O_p(G)$ having maximal order. By hypothesis and Lemma 2.1, the ALPE-subgroups of $PO_p(G)/O_p(G)$ are $S$-semipermutable in $G/O_p(G)$. Clearly,

$$
\frac{(G/O_p(G))}{(G_pO_p(G)/G_p)} \cong \frac{G}{G_pO_p(G)} \equiv \frac{(G/O_p(G))}{(G_pO_p(G)/O_p(G))} \in \mathcal{F}.
$$

Thus, our hypothesis carries over to $G/O_p(G)$ and so $G/O_p(G) \in \mathcal{F}$ by induction on the order of $G$. Therefore, $G/(O_p(G) \cap G_p) \equiv G \in \mathcal{F}$.

Case 2. $O_p(G) = 1$.

Let $H$ be an ALPE-subgroup of $P$. Then $H$ is $S$-quasinormal in $G$ by Lemma 2.3 and hence $O_p(G) \leq N_G(H)$ by Lemma 2.2. Let $T = PO_p(G)$. Then $H$ is normal in $T$. Thus Lemma 2.4 implies that $P$ is supersolvably embedded in $T$. Then, $T/C_T(P)$ is supersolvable by [14, Lemma 7.15, page 35]. Clearly, $T_p = G_p \cap T \triangleleft T$, where $T_p$ is a Sylow $p$-subgroup of $T$. Let $Q$ be a $p'$-subgroup of $C_T(P)$. Then $QP = Q \times P$ is a group of automorphisms of $T_p = O_p(T)$. But $C_{T_p}(P) = P$, and consequently, $Q$ acts trivially on $C_{T_p}(P)$. Then $Q$ acts trivially on $T_p$ by [15, Theorem 3.4, page 179], that is, $Q \leq C_T(T_p)$. It is easy to see that $T$ is subnormal in $G$ and so $O_p(T) \leq O_p(G) = 1$. Hence $F(T) = T_p$. Since $T$ is solvable, it follows by [15, Theorem 2.6, page 216] that $Q \leq C_T(F(T)) \leq F(T) = T_p$: a contradiction. Hence $C_T(P)$ must be a $p'$-group and so $C_T(P) = P$. Thus, $T/C_T(P) = T/P$ is supersolvable which implies that $T$ is supersolvable by [16, Theorem 4]. Thus $O_p(G)$ is supersolvable and therefore, $G = G_pO_p(G)$ is supersolvable by [17, Exercise 7.2.23, page 159]. Hence, $G \in \mathcal{U} \subseteq \mathcal{F}$.

As an immediate consequence of Theorem 3.2, we have the following theorem.

Corollary 3.3. Let $G_p$ be a normal Sylow $p$-subgroup of a group $G$ such that $G/G_p$ is supersolvable. Fix an ALPE-subgroup $P$ of $G_p$ having maximal order. If the ALPE-subgroups of $P$ are $S$-semipermutable in $G$, then $G$ is supersolvable.

We now prove the following theorem.

Theorem 3.4. Let $G$ be a group. For every prime $p$ and every Sylow $p$-subgroup $G_p$ of $G$, fix an ALPE-subgroup $P$ of $G_p$ having maximal order. If the ALPE-subgroups of $P$ are $S$-semipermutable in $G$, then $G$ is supersolvable.

Proof. By repeated applications of Theorem 3.1, the group $G$ has a Sylow tower of supersolvable type. Hence $G$ has a normal Sylow $p$-subgroup $G_{p}$, where $p$ is the largest prime dividing the order of $G$. By Lemma 2.1, our hypothesis carries over to $G/G_p$. Thus $G/G_p$ is
supersolvable by induction on the order of $G$. Now, it follows from Corollary 3.3 that $G$ is supersolvable.

As an immediate consequence of Theorem 3.4, we have the following corollary.

**Corollary 3.5** (Asaad et al. [7]). If $G$ is a group such that the ALPE-subgroups of every Sylow subgroup of $G$ are normal in $G$, then $G$ is supersolvable.

**Corollary 3.6** (Ramadan [8]). If $G$ is a group such that the ALPE-subgroups of every Sylow subgroup of $G$ are $S$-quasinormal in $G$, then $G$ is supersolvable.

We need the following Lemma.

**Lemma 3.7.** Let $K$ be a normal $p$-subgroup of a group $G$ such that $G/K$ is supersolvable. Fix an ALPE-subgroup $P$ of $K$ having maximal order. If the ALPE-subgroups of $P$ are $S$-semipermutable in $G$, then $G$ is supersolvable.

**Proof.** Let $G_p$ be a Sylow $p$-subgroup of $G$. We treat the following two cases.

**Case 1.** $K = G_p$.

Then by Corollary 3.3, $G$ is supersolvable.

**Case 2.** $K < G_p$.

Put $\pi(G) = \{p_1, p_2, \ldots, p_n\}$, where $p_1 > p_2 > \cdots > p_n$. Since $G/K$ is supersolvable, it follows by [18, Theorem 5, page 5] that $G/K$ possesses supersolvable subgroups $M/K$ and $L/K$ such that $|G/K : M/K| = p_1$ and $|G/K : L/K| = p_n$. Since $M/K$ and $L/K$ are supersolvable, it follows that $M$ and $L$ are supersolvable by induction on the order of $G$. Since $|G : M| = |G/K : M/K| = p_1$ and $|G : L| = |G/K : L/K| = p_n$, it follows again by [18, Theorem 5, page 5] that $G$ is supersolvable.

Now, we can prove the following theorem.

**Theorem 3.8.** Let $K$ be a normal subgroup of $G$ such that $G/K$ is supersolvable. For every prime $p$ dividing the order of $K$ and every Sylow $p$-subgroup $K_p$ of $K$, fix an ALPE-subgroup $P$ of $K_p$ having maximal order. If the ALPE-subgroups of $P$ are $S$-semipermutable in $G$, then $G$ is supersolvable.

**Proof.** By Lemma 2.1, the ALPE-subgroups of $P$ are $S$-semipermutable in $K$. Hence $K$ is supersolvable by Theorem 3.4. Thus $K$ has a normal Sylow $p$-subgroup $K_p$, where $p$ is the largest prime dividing the order of $K$. Since $K_p$ is characteristic in $K$ and $K < G$, we have that $K_p < G$. Clearly, $(G/K_p)/(K/K_p) \cong G/K$ is supersolvable. By Lemma 2.1, our hypothesis carries over to $G/K_p$ and hence $G/K_p$ is supersolvable by induction on the order of $G$. Now, it follows from Lemma 3.7 that $G$ is supersolvable.

As an immediate consequence of Theorem 3.8, we have the following corollary.

**Corollary 3.9** (Ramadan [8]). Let $K$ be a normal subgroup of a group $G$ such that $G/K$ is supersolvable. If the ALPE-subgroups of every Sylow subgroup of $K$ are $S$-quasinormal in $G$, then $G$ is supersolvable.
4. Similar Results

Following similar arguments to those of Theorem 3.1, it is possible to prove the following result.

**Theorem 4.1.** Let $p$ be the smallest prime dividing the order of a group $G$ and let $G_p$ be a Sylow $p$-subgroup of $G$. Fix an ALPE-subgroup $P$ of $\Omega(G_p)$ having maximal order. If the ALPE-subgroups of $P$ are $S$-semipermutable in $G$, then $G$ is $p$-nilpotent.

We prove the following lemma.

**Lemma 4.2.** Let $K$ be a normal $p$-subgroup of a group $G$ such that $G/K$ is supersolvable. Fix an ALPE-subgroup $P$ of $\Omega(K)$ having maximal order. If the ALPE-subgroups of $P$ are $S$-semipermutable in $G$, then $G$ is supersolvable.

**Proof.** Let $G_p$ be a Sylow $p$-subgroup of $G$. We treat the following two cases.

**Case 1.** By [15, Theorem 2.1, page 221], there exists a $p'$-Hall subgroup $T$, which is a complement to $G_p$ in $G$. Hence $G/G_p \cong T$ is supersolvable. Since $\Omega(G_p)$ is characteristic in $G_p$ and $G_p \leq G$, we have that $\Omega(G_p) \leq G$. Clearly, $\Omega(G_p)T/\Omega(G_p) \cong T$ is supersolvable. Thus, our hypothesis and Corollary 3.3 imply that $\Omega(G_p)T$ is supersolvable. Therefore, $G$ is supersolvable by Lemma 2.7.

**Case 2.** Put $\pi(G) = \{p_1, p_2, \ldots, p_n\}$, where $p_1 > p_2 > \cdots > p_n$. Since $G/K$ is supersolvable, it follows by [18, Theorem 5, page 5] that $G/K$ possesses supersolvable subgroups $M/K$ and $L/K$ such that $|G/K : M/K| = p_1$ and $|G/K : L/K| = p_n$. Since $M/K$ and $L/K$ are supersolvable, it follows that $M$ and $L$ are supersolvable by induction on the order of $G$. Since $|G : M| = |G/K : M/K| = p_1$ and $|G : L| = |G/K : L/K| = p_n$, it follows again by [18, Theorem 5, page 5] that $G$ is supersolvable.

By a similar proof to the proof of Theorem 3.4, we can prove the following theorem.

**Theorem 4.3.** Let $G$ be a group. For every prime $p$ and every Sylow $p$-subgroup $G_p$ of $G$, fix an ALPE-subgroup $P$ of $\Omega(G_p)$ having maximal order. If the ALPE-subgroups of $P$ are $S$-semipermutable in $G$, then $G$ is supersolvable.

As an immediate consequence of Theorem 4.3, we have the following corollary.

**Corollary 4.4** (Asaad et al. [7]). If $G$ is a group such that for every prime $p$ and every Sylow $p$-subgroup $G_p$, the ALPE-subgroups of $\Omega(G_p)$ are normal in $G$, then $G$ is supersolvable.

**Corollary 4.5** (Ramadan [8]). If $G$ is a group such that for every prime $p$ and every Sylow $p$-subgroup $G_p$, the ALPE-subgroups of $\Omega(G_p)$ are $S$-quasinormal in $G$, then $G$ is supersolvable.

We can now prove the following corollary.

**Corollary 4.6.** Let $K$ be a normal subgroup of $G$ such that $G/K$ is supersolvable. For every prime $p$ dividing the order of $K$ and every Sylow $p$-subgroup $K_p$ of $K$, fix an ALPE-subgroup $P$ of $\Omega(K_p)$ having maximal order. If the ALPE-subgroups of $P$ are $S$-semipermutable in $G$, then $G$ is supersolvable.
Proof. By Lemma 2.1, the ALPE-subgroups of \( P \) are \( S \)-semipermutable in \( K \). Hence \( K \) is supersolvable by Theorem 4.3. Thus \( K \) has a normal Sylow \( p \)-subgroup \( K_p \), where \( p \) is the largest prime dividing the order of \( K \). Since \( K_p \) is characteristic in \( K \) and \( K \triangleleft G \), we have that \( K_p \triangleleft G \). Clearly, \( (G/K_p)/(K/K_p) \cong G/K \) is supersolvable. By Lemma 2.1, the hypothesis of our theorem carries over to \( G/K_p \). Thus \( G/K_p \) is supersolvable by induction on the order of \( G \) and it follows that \( G \) is supersolvable by Lemma 4.2.

Remarks 4.7. (a) The converse of Theorem 3.4 is not true. For example, set \( G = S_3 \times Z_3 \), where \( S_3 = \langle x, y \mid x^3 = y^2 = 1, xy = x^2y \rangle \) and \( Z_3 = \langle z \mid z^3 = 1 \rangle \). Clearly, \( G \) is supersolvable and \( G \) has an abelian Sylow 3-subgroup of exponent 3. It is easy to check that \( G \) contains a subgroup \( \langle xz \rangle \) of order 3 which fails to be \( S \)-semipermutable in \( G \).

(b) Theorem 4.3 is not true when the smallest prime dividing the order of \( G \) is even and \( \Omega(G_p) = \Omega_1(G_p) \), where \( G_p \) is a Sylow \( p \)-subgroup of \( G \). For example, if \( Q \) is the quaternion group \( \langle a, b \mid a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle \), \( C_9 \) is a cyclic group of order 9 with generator \( c \), and the action of \( C_9 \) on \( Q \) is given by \( a^e = b, b^c = ab \), then the semidirect product of \( Q \) by \( C_9 \) is a group of even order in which every subgroup of prime order is \( S \)-semipermutable. Clearly, the semidirect product of \( Q \) by \( C_9 \) is not supersolvable (see Buckley [4, Examples (ii)]).

References

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