Research Article

Rational Solutions of Riccati Differential Equation with Coefficients Rational

Nadhém Echi

Department of Mathematics, Tunis El Manar University, Tunis 1002, Tunisia

Correspondence should be addressed to Nadhém Echi, echi_fst@yahoo.fr

Received 28 March 2011; Accepted 16 April 2011

Academic Editors: H.-T. Hu and X. Meng

Copyright © 2011 Nadhém Echi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper presents a simple and efficient method for determining the rational solution of Riccati differential equation with coefficients rational. In case the differential Galois group of the differential equation $(E_I) : y'' = ry, r \in \mathbb{C}(x)$ is reducible, we look for the rational solutions of Riccati differential equation $\theta' + \theta^2 = r$, by reducing the number of checks to be made and by accelerating the search for the partial fraction decomposition of the solution reserved for the poles of θ which are false poles of r . This partial fraction decomposition of solution can be used to code r . The examples demonstrate the effectiveness of the method.

1. Introduction

The quadratic Riccati differential equation:

$$(E_R) : \sigma' = p_2\sigma^2 + p_1\sigma + p_0, \quad (1.1)$$

where p_0, p_1 , and p_2 are in a differential field $\mathbb{K}$, $p_2 \neq 0$. The quadratic Riccati differential equation is first converted to a reduced Riccati differential equation:

$$(E_r) : \theta' + \theta^2 = r, \quad (1.2)$$

where $\theta = -p_2\sigma - (1/2)a$, with $a = (p_1'/p_2) + p_1$ and $r = (1/4)a^2 - (1/2)a' - p_2p_0$.

Furthermore, we put $y'/y = \theta$, reduced Riccati differential equation (1.2) is converted to a second-order linear ordinary differential equation

$$(E_l) : y'' = ry. \quad (1.3)$$

If we have a particular solution nonzero of (E_l) then general solution is $y = cu$ where $c' = \lambda/u^2$, λ constant (see [1–5]).

In the paper, we base ourselves mainly on the work of Kovacic [2] where the differential Galois group of the differential equation (E_l) is reducible and we take $\mathbb{K} = \mathbb{C}(X)$.

The case where every solution of (E_l) is Liouvillian corresponds to the case where reduced Riccati differential equation (E_r) has algebraic solution over $\mathbb{K}$. The case where differential Galois group is reducible corresponds to the case where the Riccati differential equation (E_r) has the rational solution (u'/u) u solution of (E_l) . The solution u of (E_l) is rational fraction if and only if u'/u is rational fraction with simples poles, integers residues and negative degree.

The field $\mathbb{C}(X)[u]$ is differential extension of $\mathbb{C}(X)$ by exponential of an integral and if $v' = 1/u^2$ then (u, v) two solutions of (E_l) are linearly independent over field of constants $\mathbb{C}$. The ordinary extension $\mathbb{C}(X)[u, v]$ is differential extension of $\mathbb{C}(X)[u]$, by an integral. $\mathbb{C}(X)[u, v]$ is Picard-Vessiot extension of $\mathbb{C}(X)[u]$ for the differential equation (E_l) (see [2, 3, 6–8]). The existence of rational solution u'/u of Riccati differential equation (E_r) and research primitive of $1/u^2$ give all solutions of (E_r) .

This paper presents a simple and efficient method for determining the solution of Riccati differential equation with coefficients rational. In case the differential Galois group of the differential equation $(E_l) : y'' = ry$, $r \in \mathbb{C}(x)$ is reducible, we look for the rational solutions of Riccati differential equation $\theta' + \theta^2 = r$, by reducing the number of checks to be made and by accelerating the search for the partial fraction decomposition of the solution reserved for the poles of θ which are false poles of r . This partial fraction decomposition of solution can be used to code r . The examples demonstrate the effectiveness of the method.

2. Form of Rational Solution of Equation: (E_r)

Let $r \in \mathbb{C}(x)$, $r \neq 0$ be rational fraction and let $\theta \in \mathbb{C}(x)$ be the rational solution of Riccati differential equation $(E_r) : \theta' + \theta^2 = r$.

2.1. Study in the Pole c of Multiplicity ν of θ

We put

$$\theta = \frac{\tau}{(x-c)^\nu}, \quad \text{where } \tau(c) \neq 0. \quad (2.1)$$

We have

$$r = \theta' + \theta^2 = (x-c)^{-2\nu} \left[\left(\tau - \frac{\nu}{2}(x-c)^{\nu-1} \right)^2 - \frac{\nu^2}{4}(x-c)^{2\nu-2} + \tau'(x-c)^\nu \right]. \quad (2.2)$$

Thus

$$\left(\tau - \frac{\nu}{2}(x-c)^{\nu-1}\right)^2 = (x-c)^{2\nu}r + (x-c)^\nu \left[\frac{\nu^2}{4}(x-c)^{\nu-2} - \tau' \right]. \quad (2.3)$$

Case 1 ($\nu \geq 2$). The function $(x-c)^{2\nu}r$ defines and is equal to $\tau(c)^2$ at c .
Thus c is pole of multiplicity 2ν of r where

$$\lim_{x \rightarrow c} (x-c)^{2\nu}r = \left(\lim_{x \rightarrow c} (x-c)^\nu \theta \right)^2. \quad (2.4)$$

Case 2 ($\nu = 1$). We have

$$\left(\tau - \frac{1}{2}\right)^2 = (x-c)^2r + \frac{1}{4} - (x-c)\tau'. \quad (2.5)$$

The function $(x-c)^2r$ defines and is equal to $\tau(c)(\tau(c) - 1)$ at c .

Situation 1 ($\lim_{x \rightarrow c} (x-c)^2r \neq 0, -1/4$)

c is double pole of r and the residue $\tau(c)$ of θ at c have two possibility values as follows:

$$\left(\tau(c) - \frac{1}{2}\right)^2 = \lim_{x \rightarrow c} (x-c)^2r + \frac{1}{4}. \quad (2.6)$$

Thus, c is double pole of r and the residue of θ at simple pole c equals

$$\tau(c) = \alpha_c + \frac{1}{2}, \quad (2.7)$$

where

$$\alpha_c^2 = \lim_{x \rightarrow c} (x-c)^2r + \frac{1}{4}. \quad (2.8)$$

Situation 2 ($\lim_{x \rightarrow c} (x-c)^2r = -1/4$)

c is double pole of r and the residue of θ at c is $1/2$.

Situation 3 ($\lim_{x \rightarrow c} (x-c)^2r = 0$)

c is a simple pole of r or is not pole of r and the residue of θ at simple pole c equals 1.

Proposition 2.1. Let $\theta \in \mathbb{C}(x)$ such as $\theta' + \theta^2 = r$.

(1) The fraction: $r = N/D$ with N and D being polynomials relatively prime

$$D = D_1 D_2^2 D_3^2 D_4^2, \quad (2.9)$$

where D_1, D_2, D_3 , and D_4 are polynomials relatively prime pairwise. D_1, D_2 , and D_3 are with simple roots, and D_4 is without simple root

$$\begin{aligned} \forall c \in \text{Root}(D_2) \quad \lim_{x \rightarrow c} (x - c)^2 r &= -\frac{1}{4}, \\ \forall c \in \text{Root}(D_3) \quad \lim_{x \rightarrow c} (x - c)^2 r &\neq -\frac{1}{4}. \end{aligned} \quad (2.10)$$

(2) (a) Let $\nu \geq 2$. c pole of multiplicity ν of $\theta \Leftrightarrow c \in \text{Root}(D_4)$.

(b) c simple pole of θ with residue $\neq 1, 1/2 \Leftrightarrow c \in \text{Root}(D_3)$, the residue of θ where c equals $\alpha_c + (1/2)$ where

$$\alpha_c^2 = \lim_{x \rightarrow c} (x - c)^2 r + \frac{1}{4}. \quad (2.11)$$

(c) c simple pole of θ with residue $= 1/2 \Leftrightarrow c \in \text{Roots}(D_2)$.

(d) c simple pole of θ with residue $= 1 \Leftrightarrow c \in \text{Roots}(D_1)$ or c pole of θ and false pole of r .

Corollary 2.2. One assumes that $r = N/D$ with N and D polynomials being relatively prime, $D = D_1 D_2^2 D_3^2 D_4^2$ where D_1, D_2, D_3 , and D_4 are polynomials relatively prime pairwise, D_1, D_2 , and D_3 are with simple roots, and D_4 is without simple root

$$\begin{aligned} \forall c \in \text{Root}(D_2) \quad \lim_{x \rightarrow c} (x - c)^2 r &= -\frac{1}{4}, \\ \forall c \in \text{Root}(D_3) \quad \lim_{x \rightarrow c} (x - c)^2 r &\neq -\frac{1}{4}. \end{aligned} \quad (2.12)$$

a rational fraction θ . Verify $\theta' + \theta^2 = r$ is the shape

$$\theta = E(\theta) + \sum_{c \in \text{Roots}(D_4)} \theta_c + \sum_{c \in \text{Roots}(D_3)} \frac{\alpha_c + 1/2}{x - c} + \frac{1}{2} \frac{D'_2}{D_2} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0} \quad (2.13)$$

with D_0 being monic polynomial with simple roots and the roots are false poles of r .

2.2. Study in the Infinity

Let $t \in \mathbb{C}(x)$, $t \neq 0$ rational fraction such as $(dt/dx) + t^2 = 0$.

Case 1. We assume that $d^o(\theta) < 0$.

We have $d^o(\theta') < 0$, and $d^o(\theta^2) < 0$ thus $d^o(r) < 0$. We put

$$\theta = t\sigma(t) \quad \text{with } \sigma \text{ being rational fraction defined at } 0. \quad (2.14)$$

We have

$$\begin{aligned} \sigma(0) &= \lim_{x \rightarrow \infty} x\theta = \text{sum the residues of } \theta, \\ r &= \theta' + \theta^2 = (\sigma^2 - \sigma)t^2 - \sigma't^3, \quad \lim_{x \rightarrow \infty} x^2r = \sigma(0)^2 - \sigma(0). \end{aligned} \quad (2.15)$$

Thus $d^o r \leq -2$ and $(\sigma(0) - (1/2))^2 = \lim_{x \rightarrow \infty} x^2r + (1/4)$.

If: $\lim_{x \rightarrow \infty} x^2r = -1/4$ then the sum of residues of θ : $\sigma(0) = 1/2$.

If: $\lim_{x \rightarrow \infty} x^2r \neq -1/4$ then the sum of residues of θ : $\sigma(0) = \alpha_\infty + (1/2)$ where

$$\alpha_\infty^2 = \lim_{x \rightarrow \infty} x^2r + \frac{1}{4}. \quad (2.16)$$

Case 2. We assume that $d^o(\theta) = 0$.

$E(\theta)$ constant $\neq 0$. We put

$$\theta = E(\theta) + t\sigma(t), \quad \text{with } \sigma \text{ being rational fraction defined at } 0. \quad (2.17)$$

We have

$$\begin{aligned} \sigma(0) &= \lim_{x \rightarrow \infty} x(\theta - E(\theta)) = \text{the sum of residues of } \theta, \\ r &= \theta' + \theta^2 = (E(\theta))^2 + 2E(\theta)t\sigma + (\sigma^2 - \sigma)t^2 - \sigma't^3. \end{aligned} \quad (2.18)$$

So $E(r)$ constant equals $E(\theta)^2$:

$$2E(\theta)[\text{sum the residues of } \theta] = \text{sum the residues of } r. \quad (2.19)$$

Case 3. We assume that $d^o(\theta) > 0$.

We put

$$v = d^o(\theta) \geq 1, \quad \theta = t^{-v}\sigma(t) \quad (2.20)$$

with σ being rational fraction defined at 0. The scalar $\sigma(0)$ is the dominant coefficient of $E(\theta)$.

We have

$$r = \theta' + \theta^2 = t^{-2v} [\sigma^2 + v t^{v+1} \sigma - t^{v+2} \sigma']. \quad (2.21)$$

Thus,

$$t^{2\nu}r = \sigma(0)^2 + o(t). \quad (2.22)$$

So $d^\circ r = 2\nu = 2d^\circ\theta$ and $\sigma^2(o)$ the dominant coefficient of $E(r)$.

Proposition 2.3. *Let $\theta \in \mathbb{C}(x)$ such as $\theta' + \theta^2 = r$.*

(1) $d^\circ(\theta) < 0 \Leftrightarrow d^\circ(r) < 0$. Thus $d^\circ(r) \leq -2$ and

$$\text{the sum of residues of } \theta = \begin{cases} \frac{1}{2} & \text{if } \lim_{x \rightarrow \infty} x^2 r = -\frac{1}{4}, \\ \alpha_\infty + \frac{1}{2} & \text{if } \lim_{x \rightarrow \infty} x^2 r \neq -\frac{1}{4}. \end{cases} \quad (2.23)$$

(2) $d^\circ(\theta) = 0 \Leftrightarrow d^\circ(r) = 0$. In the case $E(\theta)$ is square root of $E(r)$:

$$\begin{aligned} 2E(\theta)(\text{sum of residues of } \theta) &= \text{sum of residues of } r \\ &= \lim_{x \rightarrow \infty} x(r - E(r)). \end{aligned} \quad (2.24)$$

(3) We have $d^\circ(\theta) > 0 \Leftrightarrow d^\circ(r) > 0$. In the case

- (a) $d^\circ(r) = 2d^\circ(\theta)$;
- (b) the dominant coefficient of $E(\theta)$ is square root of $E(r)$.

2.3. Determination of $E(\theta)$; $d^\circ(r) = 2\nu > 0$

We assume that r is a rational fraction of degree $2\nu > 0$ and $\theta \in \mathbb{C}(x)$ such as $\theta' + \theta^2 = r$.

Let a be the dominant coefficient of $E(r)$. Thus $r \sim ax^{2\nu}$ if x tends to ∞ :

$$t^{2\nu} \frac{E(r)}{a} = 1 + a_1 t + \cdots + a_{2\nu} t^{2\nu}. \quad (2.25)$$

The Taylor's expansion of order $\nu + 1$ at 0

$$\left(t^{2\nu} \frac{E(r)}{a} \right)^{1/2} = 1 + s_1 t + \cdots + s_{\nu+1} t^{\nu+1} + o(t^{\nu+1}). \quad (2.26)$$

We have

$$\begin{aligned}
 t^{2\nu}r &= t^{2\nu}E(r) + o(t^{2\nu}) \\
 &= t^{2\nu}E(r) + o(t^{\nu+1}) \\
 &= a \left[\left(\frac{t^{2\nu}E(r)}{a} \right)^{1/2} \right]^2 + o(t^{\nu+1}).
 \end{aligned} \tag{2.27}$$

We have $\theta = t^{-\nu}\sigma(t)$ with σ rational fraction defined at 0, $\sigma(0)^2 = a$ and

$$\begin{aligned}
 \left(\sigma + \frac{\nu}{2}t^{\nu+1} \right)^2 &= t^{2\nu}r + \frac{\nu^2}{4}t^{2\nu+2} + t^{\nu+2}\sigma' \\
 &= a \left[1 + s_1t + \cdots + s_{\nu+1}t^{\nu+1} \right]^2 + o(t^{\nu+1}) \\
 \sigma + \frac{\nu}{2}t^{\nu+1} &= \sigma(0) \left[1 + s_1t + \cdots + s_{\nu+1}t^{\nu+1} \right] + o(t^{\nu+1}).
 \end{aligned} \tag{2.28}$$

Thus

$$\theta = t^{-\nu}\sigma = \sigma(0) \left[t^{-\nu} + s_1t^{-(\nu-1)} + \cdots + s_{\nu+1}t \right] - \frac{\nu}{2}t + o(t). \tag{2.29}$$

ImPLY

$$\begin{aligned}
 E(\theta) &= \sigma(0) \left[t^{-\nu} + s_1t^{-(\nu-1)} + \cdots + s_{\nu} \right], \\
 \sigma(0)s_{\nu+1} - \frac{\nu}{2} &= \text{sum of residues of } \theta.
 \end{aligned} \tag{2.30}$$

Proposition 2.4. Let r be a rational fraction of degree $2\nu > 0$, $\theta \in \mathbb{C}(x)$ such as $\theta' + \theta^2 = r$ and a the dominant coefficient of $E(r)$. If

$$\left(t^{2\nu} \frac{E(r)}{a} \right)^{1/2} = 1 + s_1t + \cdots + s_{\nu+1}t^{\nu+1} + o(t^{\nu+1}). \tag{2.31}$$

Then

$$\begin{aligned}
 E(\theta) &= \alpha \left[t^{-\nu} + s_1t^{-(\nu-1)} + \cdots + s_{\nu} \right], \\
 \alpha s_{\nu+1} - \frac{\nu}{2} &= \text{sum of residues of } \theta,
 \end{aligned} \tag{2.32}$$

where

$$\alpha^2 = a. \tag{2.33}$$

3. Determination of Partial Fraction Decomposition

Let $r = N/D$ be rational fraction with N and D polynomials being relatively prime, $D = D_1 D_2^2 D_3^2 D_4^2$ where D_1, D_2, D_3 , and D_4 polynomials being relatively prime pair-wise, D_1, D_2 , and D_3 with simple roots and D_4 without simple root

$$\begin{aligned} \forall c \in \text{Root}(D_2) \quad \lim_{x \rightarrow c} (x - c)^2 r &= -\frac{1}{4}, \\ \forall c \in \text{Root}(D_3) \quad \lim_{x \rightarrow c} (x - c)^2 r &\neq -\frac{1}{4}. \end{aligned} \quad (3.1)$$

Let $\theta \in \mathbb{C}(x)$ be rational fraction. Verify $\theta' + \theta^2 = r$.

3.1. Case $d^0 D_3 = 0$ and $d^0 D_4 = 0$

We have $r = N/D_1 D_2^2$.

This case corresponds to the fact that one pole c of r is simple or double with

$$\lim_{x \rightarrow c} (x - c)^2 r = -\frac{1}{4}. \quad (3.2)$$

Proposition 3.1. *One assumes $d^0 r < 0$ and $d^0 D_3 = d^0 D_4 = 0$. One has:*

$$\lim_{x \rightarrow \infty} x^2 r + \frac{1}{4} = \left(\frac{q}{2}\right)^2 \quad (3.3)$$

with q being positive integer of parity against that of $d^0 D_2$

$$\theta = \frac{1}{2} \frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \quad (3.4)$$

with D_0 polynomial of degree

$$d^0 D_0 = \frac{1}{2}(q + 1 - d^0 D_2) - d^0 D_1. \quad (3.5)$$

Proof. We have

$$\theta = \frac{1}{2} \frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}. \quad (3.6)$$

The sum of residues of θ equal $(1/2)d^0 D_2 + d^0 D_1 + d^0 D_0 = \alpha_\infty + (1/2)$ with $\alpha_\infty^2 = \lim_{x \rightarrow \infty} x^2 r + (1/4)$. In particular $\alpha_\infty = q/2$ with $q \in \mathbb{N}$. $\square$

Remark 3.2. If $\lim_{x \rightarrow \infty} x^2 r = -1/4$ then $d^0 D_2 = 1, d^0 D_1 = d^0 D_0 = 0, \theta = (1/2)(1/(x - c))$, and $r = -1/4(x - c)^2$.

Proposition 3.3. *One assumes $d^o r = 0$ and $d^o D_3 = d^o D_4 = 0$.*

- (1) $E(\theta)$ square root of $E(r)$ such as $p = (1/E(\theta)) \lim_{x \rightarrow \infty} x[r - E(r)]$ is positive integer of same parity as $d^o D_2$.
- (2) One has

$$\theta = E(\theta) + \frac{1}{2} \frac{D'_2}{D_2} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0} \quad (3.7)$$

with D_0 being polynomial of degree

$$d^o D_0 = \frac{1}{2} p - d^o D_1 - \frac{1}{2} d^o D_2. \quad (3.8)$$

Proof. We have

$$\theta = E(\theta) + \frac{1}{2} \frac{D'_2}{D_2} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0}. \quad (3.9)$$

We have

$$\begin{aligned} 2E(\theta) \text{ sum of residues } (\theta) &= \text{sum of residues } (r) = \lim_{x \rightarrow \infty} x[r - E(r)], \\ 2E(\theta) \left[\frac{1}{2} d^o D_2 + d^o D_1 + d^o D_0 \right] &= \lim_{x \rightarrow \infty} x[r - E(r)]. \end{aligned} \quad (3.10)$$

Thus

$$d^o D_2 + 2d^o D_1 + 2d^o D_0 = \frac{1}{E(\theta)} \lim_{x \rightarrow \infty} x[r - E(r)]. \quad (3.11)$$

If r is constant then $D_1 = D_2 = D_0 = 1$ and θ are constant.

If r is nonconstant then r is not polynomial thus

$$p = \frac{1}{E(\theta)} \lim_{x \rightarrow \infty} x[r - E(r)] \quad (3.12)$$

is a positive integer of same parity as $d^o D_2$. □

Proposition 3.4. *We assume $d^o r = 2\nu > 0$ and $d^o D_3 = d^o D_4 = 0$.*

Let a be the dominant coefficient of $E(r)$ and one considers the Taylor's expansion at infinity:

$$\left(t^{2\nu} \frac{E(r)}{a}\right)^{1/2} = 1 + s_1 t + \cdots + s_{\nu+1} t^{\nu+1} + o(t^{\nu+1}). \quad (3.13)$$

(1)

$$4as_{\nu+1}^2 = p^2, \quad (3.14)$$

where p is a positive integer, $p \geq d^0 D_2 + 2d^0 D_1 + \nu$, and same parity of $d^0 D_2 + 2d^0 D_1 + \nu$.

(2) One has

$$\theta = E(\theta) + \frac{1}{2} \frac{D'_2}{D_2} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0} \quad (3.15)$$

with D_0 being polynomial of degree

$$d^0 D_0 = \frac{p - \nu - d^0 D_2}{2} - d^0 D_1 \quad (3.16)$$

and α the dominant coefficient of $E(\theta)$ where

$$p = 2\alpha s_{\nu+1}. \quad (3.17)$$

Proof. We have

$$\begin{aligned} E(\theta) &= \alpha \left[t^{-\nu} + s_1 t^{-(\nu-1)} + \cdots + s_\nu \right], \\ \alpha s_{\nu+1} - \frac{\nu}{2} &= \text{sum of residues of } \theta, \\ \theta &= E(\theta) + \frac{1}{2} \frac{D'_2}{D_2} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0}. \end{aligned} \quad (3.18)$$

Thus

$$\frac{1}{2} d^0 D_2 + d^0 D_1 + d^0 D_0 = \alpha s_{\nu+1} - \frac{\nu}{2} \quad (3.19)$$

$\alpha s_{\nu+1} = p/2$ with p positive integer. Thus $p \geq d^0 D_2 + 2d^0 D_1 + \nu$, p and $d^0 D_2 + 2d^0 D_1 + \nu$ have the same parity. $\square$

3.2. Case: $D_3 = X - c$ **and** $d^o D_4 = 0$

This case corresponds to the fact that a pole of r is simple or double with only double pole c such as:

$$\lim_{x \rightarrow c} (x - c)^2 r \neq -\frac{1}{4}, \quad (3.20)$$

$$r = \frac{N}{D_1 D_2^2 (x - c)^2}. \quad (3.21)$$

Proposition 3.5. Consider (3.21) and let $\theta \in \mathbb{C}(x)$ be a rational fraction. Verify $\theta' + \theta^2 = r$. One assumes $d^o r < 0$. Accordingly, in view of (2.11) and (2.16) one has

$$\theta = \frac{\alpha_c + (1/2)}{x - c} + \frac{1}{2} \frac{D'_2}{D_2} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0} \quad (3.22)$$

with D_0 being polynomial of degree

$$d^o D_0 = \lambda - \frac{1}{2} d^o D_2 - d^o D_1, \quad (3.23)$$

where

$$\lambda = \alpha_\infty - \alpha_c \quad (3.24)$$

one half positive integer of same parity as $d^o D_2$.

Proof. We have

$$\begin{aligned} \lim_{x \rightarrow \infty} x\theta &= \alpha_c + \frac{1}{2} + \frac{1}{2} d^o D_2 + d^o D_1 + d^o D_0, \\ \left(\alpha_c + \frac{1}{2} d^o D_2 + d^o D_1 + d^o D_0 \right)^2 &= \lim_{x \rightarrow \infty} x^2 r + \frac{1}{4} = \alpha_\infty^2. \end{aligned} \quad (3.25)$$

Thus

$$\frac{1}{2} d^o D_2 + d^o D_1 + d^o D_0 = \alpha_\infty - \alpha_c. \quad (3.26)$$

□

Proposition 3.6. Consider (3.21) and let $\theta \in \mathbb{C}(x)$ be a rational fraction. Verify: $\theta' + \theta^2 = r$. One assumes $d^o r = 0$. One has:

$$\theta = E(\theta) + \frac{\alpha_c + (1/2)}{x - c} + \frac{1}{2} \frac{D'_2}{D_2} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0} \quad (3.27)$$

with $E^2(\theta) = E(r)$ and D_0 being polynomial of degree

$$d^\circ D_0 = \frac{1}{2}\lambda - \frac{1}{2}d^\circ D_2 - d^\circ D_1 - \frac{1}{2}, \quad (3.28)$$

where

$$\lambda = \frac{1}{E(\theta)} \lim_{x \rightarrow \infty} x[r - E(r)] - 2\alpha_c \quad (3.29)$$

λ is positive integer of parity against that of $d^\circ D_2$.

Proof. We have

$$2E(\theta) \left[\alpha_c + \frac{1}{2} + \frac{1}{2}d^\circ D_2 + d^\circ D_1 + d^\circ D_0 \right] = \lim_{x \rightarrow \infty} x[r - E(r)]. \quad (3.30)$$

Thus

$$1 + d^\circ D_2 + 2d^\circ D_1 + 2d^\circ D_0 = \frac{1}{E(\theta)} \lim_{x \rightarrow \infty} x[r - E(r)] - 2\alpha_c. \quad (3.31)$$

□

Proposition 3.7. Consider (3.21) and let $\theta \in \mathbb{C}(x)$ be a rational fraction. Verify $\theta' + \theta^2 = r$. One assumes $d^\circ r = 2\nu > 0$. Let a be the dominant coefficient of $E(r)$, let α be the dominant coefficient of $E(\theta)$, $t = 1/x$ and Taylors expansion:

$$\left(t^{2\nu} \frac{E(r)}{a} \right)^{1/2} = 1 + s_1 t + \cdots + s_{\nu+1} t^{\nu+1} + o(t^{\nu+1}). \quad (3.32)$$

One has

$$\theta = E(\theta) + \frac{\alpha_c + 1/2}{x - c} + \frac{1}{2} \frac{D'_2}{D_2} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0}, \quad (3.33)$$

with D_0 being polynomial of degree:

$$d^\circ D_0 = \alpha s_{\nu+1} - \alpha_c - \frac{1 + \nu + d^\circ D_2}{2} - d^\circ D_1, \quad \alpha^2 = a. \quad (3.34)$$

Proof. We have

$$\begin{aligned} E(\theta) &= \alpha \left[t^{-\nu} + s_1 t^{-(\nu-1)} + \cdots + s_\nu \right], \\ \alpha s_{\nu+1} - \frac{\nu}{2} &= \text{sum of residues of } \theta. \end{aligned} \quad (3.35)$$

Thus

$$\begin{aligned}\theta &= E(\theta) + \frac{\alpha_c + (1/2)}{x - c} + \frac{1}{2} \frac{D'_2}{D_2} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0} \\ \alpha_{S_{v+1}} - \frac{v}{2} &= \alpha_c + \frac{1}{2} + \frac{1}{2} d^o D_2 + d^o D_1 + d^o D_0.\end{aligned}\tag{3.36}$$

Thus

$$d^o D_0 = \alpha_{S_{v+1}} - \alpha_c - \frac{1 + v + d^o D_2}{2} - d^o D_1.\tag{3.37}$$

□

3.3. Case: ($d^o D_3 \neq 0$ **and** $d^o D_4 \neq 0$) or ($d^o D_3 \geq 2$ **and** $d^o D_4 = 0$)

$D_1 D_2^2$ and $D_3 D_4$ are polynomials being relatively prime. Thus there are only two polynomials N_1 and N_2 such as

$$\begin{aligned}D_1 D_2^2 D_3^2 D_4^2 (r - E(r)) &= N_1 (D_1 D_2^2) + N_2 (D_3 D_4), \\ d^o N_1 &< d^o (D_3 D_4).\end{aligned}\tag{3.38}$$

Thus

$$r = E(r) + \frac{N_1}{D_3^2 D_4^2} + \frac{N_2}{D_1 D_2^2 D_3 D_4}.\tag{3.39}$$

We have

$$d^o \left(\frac{N_1}{D_3^2 D_4^2} \right) = d^o \left(\frac{N_1}{D_3 D_4} \frac{1}{D_3 D_4} \right) < d^o \left(\frac{1}{D_3 D_4} \right) \leq -2.\tag{3.40}$$

Because D_4 does not have simple roots verify $d^o D_4 = 0$ or $d^o D_4 \geq 2$. Thus $d^o (D_3 D_4) \geq 2$. Thus

$$\begin{aligned}\lim_{x \rightarrow \infty} x^2 \frac{N_1}{D_3^2 D_4^2} &= 0, \\ d^o (r - E(r)) &< 0, \\ d^o \left(\frac{N_2}{D_1 D_2^2 D_3 D_4} \right) &< 0.\end{aligned}\tag{3.41}$$

We consider the rational fraction

$$F = E(r) + \frac{N_1}{D_3^2 D_4^2} - \delta,\tag{3.42}$$

where

$$\delta = \frac{1}{4} \left[\frac{1}{k} \left(\frac{D'_3}{D_3} \right)^2 + \left(\frac{1}{k} + 1 \right) \left(\frac{D'_3}{D_3} \right)' \right], \quad (3.43)$$

$$k = d^0 D_3 \neq 0.$$

Proposition 3.8. One assumes $\left\{ \begin{smallmatrix} d^0 D_3 \neq 0 \\ d^0 D_4 \neq 0 \end{smallmatrix} \right\}$ or $\left\{ \begin{smallmatrix} d^0 D_3 \geq 2 \\ d^0 D_4 = 0 \end{smallmatrix} \right\}$.

- (1) If $d^0 r \geq 0$ then: $d^0 F = d^0 r$ and $E(F) = E(r)$.
- (2) If $d^0 r < 0$ then: $d^0 F = -2$ where $\lim_{x \rightarrow \infty} x^2 F = 1/4$.
- (3) For all c root of D_3 one has

$$\lim_{x \rightarrow c} (x - c)^2 F = \lim_{x \rightarrow c} (x - c)^2 r + \frac{1}{4}. \quad (3.44)$$

- (4) For all c root of D_4 of multiplicity v one has

$$(x - c)^{2v} r = (x - c)^{2v} F + o\left((x - c)^{v-1}\right). \quad (3.45)$$

Proof. (1) $r - F = N_2/D_1 D_2^2 D_3 D_4 + \delta$ is from negative degree.

(2) $\lim_{x \rightarrow \infty} x^2 F = \lim_{x \rightarrow \infty} (x^2 N_1/D_3^2 D_4^2) - \lim_{x \rightarrow \infty} x^2 \delta = -\lim_{x \rightarrow \infty} x^2 \delta = 1/4$.

(3) Let c be root of D_3

$$\begin{aligned} \lim_{x \rightarrow c} (x - c)^2 (r - F) &= \lim_{x \rightarrow c} \frac{N_2}{D_1 D_2^2 D_4} \frac{(x - c)^2}{D_3} + \lim_{x \rightarrow c} (x - c)^2 \delta \\ &= \lim_{x \rightarrow c} (x - c)^2 \delta = -\frac{1}{4}. \end{aligned} \quad (3.46)$$

(4) Let c be root of D_4

$$\begin{aligned} (x - c)^{2v} (r - F) &= \frac{N_2}{D_1 D_2^2 D_3} \frac{(x - c)^{2v}}{D_4} + (x - c)^{2v} \delta \\ &= o\left((x - c)^{v-1}\right). \end{aligned} \quad (3.47) \quad \square$$

Lemma 3.9. Let Z be nonzero rational fraction. Σ is a finished set such as

$$\Sigma \cap [\text{Roots}(Z) \cup \text{poles}(Z)] = \emptyset. \quad (3.48)$$

- (1) There exists Open connected set on which one has a square root holomorphic of Z , containing for every $c \in \Sigma$ a half-right closed by origin c .
- (2) If, besides, Z is from even degree then one can choose O of complementary compact.

Proof. We fix c_0 is not an element of Σ . Let Σ' be a finished set containing roots, poles of Z and c_0 . They consider all rights linked to the different pairs of points of $\Sigma \cup \Sigma'$. They choose a point w not being on these rights. For all $c \in \Sigma \cup \Sigma'$, straight line which joins right w and c does not contain any other point of $\Sigma \cup \Sigma'$.

Case 1 ($d^o Z$ even). Replacing Z by rational fraction $Z/(x - c_0)^{d^o Z}$.

Assume $d^o Z = 0$. We put $K = \bigcup_{c \in \Sigma'} [w, c]$. K is compact-connected.

We put $O = \mathbb{C} \setminus K$. O is open at infinity such as for all $c \in \Sigma$ reaching right w in c private of w is contained in O .

If γ is a shoelace of O then K is included in one connected component of $\mathbb{C} \setminus \gamma$. Thus:

$$\frac{1}{2i\pi} \int_{\gamma} \frac{Z'}{Z}(x) dx = \pm \sum_{c \in \Sigma'} \text{residue} \left(\frac{Z'}{Z}, c \right) = \pm d^o Z = 0. \quad (3.49)$$

Thus there exist the primitive of Z'/Z and the determination of logarithm of Z and the square root of Z in O .

Case 2 ($d^o Z$ odd). They use the previous case in replacing Z by rational fraction $Z/(x - c_0)$.

□

Notation. We choose a square root of the polynomial of even degree $N_1 + D_3^2 D_4^2 [E(r) - \delta]$ on a connected open at infinity, roots of D_3 and root of D_4 . We put

$$(F)^{1/2} = \frac{1}{D_3 D_4} \left(N_1 + D_3^2 D_4^2 [E(r) - \delta] \right)^{1/2}. \quad (3.50)$$

Proposition 3.10. *One assumes $\left\{ \begin{smallmatrix} d^o D_3 \neq 0 \\ d^o D_4 \neq 0 \end{smallmatrix} \right\}$ or $\left\{ \begin{smallmatrix} d^o D_3 \geq 2 \\ d^o D_4 = 0 \end{smallmatrix} \right\}$.*

One has

$$\theta = E(\theta) + \sum_{c \in \text{Root}(D_3) \cup \text{Root}(D_4)} \varepsilon_c \left(\text{partial fraction of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} \frac{(D_2 D_3 D_4)'}{D_2 D_3 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}, \quad (3.51)$$

where $\varepsilon_c = \pm 1$.

Proof. For c root of multiplicity $\nu \geq 2$ of D_4

$$\left((x - c)^\nu \theta - \frac{\nu}{2} (x - c)^{\nu-1} \right)^2 = \left((x - c)^\nu (F)^{1/2} \right)^2 + o\left((x - c)^{\nu-1} \right). \quad (3.52)$$

Thus

$$\begin{aligned} (x - c)^\nu \theta - \frac{\nu}{2} (x - c)^{\nu-1} &= \varepsilon_c (x - c)^\nu (F)^{1/2} + o\left((x - c)^{\nu-1} \right), \\ \theta - \frac{\nu}{2(x - c)} &= \varepsilon_c (F)^{1/2} + o\left(\frac{1}{x - c} \right). \end{aligned} \quad (3.53)$$

Partial fraction of θ , associated in root c of (D_4) , minus $\nu/2(x-c)$, is in sign meadows that of $(F)^{1/2}$.

For c root of D_3

$$\left[\left(\text{residue of } \theta \text{ at } c - \frac{1}{2} \right) \right]^2 = \lim_{x \rightarrow c} \left((x-c)(F)^{1/2} \right)^2. \quad (3.54)$$

Thus

$$\left[\left(\text{residue of } \theta \text{ at } c - \frac{1}{2} \right) \right] = \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right). \quad (3.55)$$

Partial fraction of θ , associated in root c of (D_3) , minus $1/2(x-c)$, is in sign meadows that of $(F)^{1/2}$. Thus

$$\theta = E(\theta) + \sum_{c \in \text{Root}(D_3) \cup \text{Root}(D_4)} \varepsilon_c \left(\text{partial fraction of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} \frac{(D_2 D_3 D_4)'}{D_2 D_3 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}. \quad (3.56)$$

□

Proposition 3.11. *One assumes $d^o r < 0$ and $\left\{ \begin{smallmatrix} d^o D_3 \neq 0 \\ d^o D_4 \neq 0 \end{smallmatrix} \right\}$ or $\left\{ \begin{smallmatrix} d^o D_3 \geq 2 \\ d^o D_4 = 0 \end{smallmatrix} \right\}$.*

One has

$$\theta = \sum_{c \in \text{Root}(D_3) \cup \text{Root}(D_4)} \varepsilon_c \left(\text{partial fraction of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} \frac{(D_2 D_3 D_4)'}{D_2 D_3 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}, \quad (3.57)$$

where D_0 being a polynomial of degree

$$\left[\sum_{c \in \text{Roots}(D_3) \cup \text{Roots}(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o (D_2 D_3 D_4) + d^o D_1 + d^o D_0 - \frac{1}{2} \right]^2 \quad (3.58)$$

$$= \lim_{x \rightarrow \infty} x^2 r + \frac{1}{4}.$$

Proof. We have

$$\lim_{x \rightarrow \infty} x\theta = \sum_{c \in \text{Roots}(D_3) \cup \text{Roots}(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o (D_2 D_3 D_4) + d^o D_1 + d^o D_0,$$

$$\lim_{x \rightarrow \infty} x^2 r + \frac{1}{4} = \left(\lim_{x \rightarrow \infty} x\theta - \frac{1}{2} \right)^2. \quad (3.59)$$

Thus

$$\left[\sum_{c \in \text{Roots}(D_3) \cup \text{Zéros}(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o(D_2 D_3 D_4) + d^o D_1 + d^o D_0 - \frac{1}{2} \right]^2$$

$$= \lim_{x \rightarrow \infty} x^2 r + \frac{1}{4}. \quad (3.60)$$

□

Proposition 3.12. *One assumes $d^o r = 0$. One has*

$$\theta = E(\theta) + \sum_{c \in \text{Root}(D_3) \cup \text{Root}(D_4)} \varepsilon_c \left(\text{partial fraction of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} \frac{(D_2 D_3 D_4)'}{D_2 D_3 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}, \quad (3.61)$$

where $E^2(\theta) = E(r)$ and D_0 being a polynomial of degree:

$$d^o D_0 = \frac{1}{2E(\theta)} \lim_{x \rightarrow \infty} x[r - E(r)]$$

$$- \left[\sum_{c \in R(D_3) \cup R(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o(D_2 D_3 D_4) + d^o D_1 \right]. \quad (3.62)$$

Proof. We have

$$E^2(\theta) = E(r), \quad 2E(\theta) \sum_c \text{residue of } \theta \text{ at } c = \sum_c \text{residue of } r \text{ at } c. \quad (3.63)$$

Thus

$$2E(\theta) \left[\sum_{c \in R(D_3) \cup R(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o(D_2 D_3 D_4) + d^o D_1 + d^o D_0 \right]$$

$$= \lim_{x \rightarrow \infty} x(r - E(r)). \quad (3.64)$$

□

Proposition 3.13. *One assumes $d^o r = 2\nu > 0$. One has*

$$\theta = E(\theta) + \sum_{c \in \text{Root}(D_3) \cup \text{Root}(D_4)} \varepsilon_c \left(\text{partial fraction of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} \frac{(D_2 D_3 D_4)'}{D_2 D_3 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}, \quad (3.65)$$

where D_0 being a polynomial of degree:

$$d^o D_0 = \alpha s_{v+1} - \frac{v}{2} - \left[\sum_{c \in R(D_3) \cup R(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o (D_2 D_3 D_4) + d^o D_1 \right]. \quad (3.66)$$

Proof. Let a be dominant coefficient of $E(r)$, and we have $\alpha s_{v+1} - (v/2) = \text{sum of residues of } \theta$,

$$\theta = E(\theta) + \sum_{c \in \text{Zéros}(D_3) \cup \text{Zéros}(D_4)} \varepsilon_c \left(\text{partial fraction of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} \frac{(D_2 D_3 D_4)'}{D_2 D_3 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}. \quad (3.67)$$

Thus

$$\alpha s_{v+1} - \frac{v}{2} = \left[\sum_{c \in R(D_3) \cup R(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o (D_2 D_3 D_4) + d^o D_1 + d^o D_0 \right]. \quad (3.68)$$

Thus

$$d^o D_0 = \alpha s_{v+1} - \frac{v}{2} - \left[\sum_{c \in R(D_3) \cup R(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o (D_2 D_3 D_4) + d^o D_1 \right]. \quad (3.69)$$

□

3.4. Case $d^o D_3 = 0$ and $d^o D_4 \neq 0$

$D_1 D_2^2$ and D_4 are polynomials being relatively prime. Thus there are only two polynomials N_1 and N_2 such as

$$D_1 D_2^2 D_3^2 D_4^2 (r - E(r)) = N_1 (D_1 D_2^2) + N_2 D_4, \quad (3.70)$$

$$d^o N_1 < d^o D_4.$$

We have

$$r = E(r) + \frac{N_1}{D_4^2} + \frac{N_2}{D_1 D_2^2 D_4}, \quad (3.71)$$

$$\lim_{x \rightarrow \infty} x^2 \frac{N_1}{D_4^2} = 0.$$

We consider the rational fraction:

$$F = E(r) + \frac{N_1}{D_4^2} - \frac{1}{4d^0 D_4} \left(\frac{D'_4}{D_4} \right)' . \quad (3.72)$$

Proposition 3.14. *One assumes $\begin{cases} d^0 D_3 = 0 \\ d^0 D_4 \neq 0 \end{cases}$.*

- (1) *If $d^0 r \geq 0$ then: $d^0 F = d^0 r$ and $E(F) = E(r)$.*
- (2) *If $d^0 r < 0$ then: $d^0 F = -2$ where $\lim_{x \rightarrow \infty} x^2 F = 1/4$.*
- (3) *For all c root of D_4 of multiplicity v one has*

$$(x - c)^{2v} r = (x - c)^{2v} F + o\left((x - c)^{v-1}\right). \quad (3.73)$$

Proof. (1) $r - F = (N_2/D_1 D_2^2 D_4) + (1/4d^0 D_4)(D'_4/D_4)'$ is from negative degree.

(2)

$$\begin{aligned} \lim_{x \rightarrow \infty} x^2 F &= \lim_{x \rightarrow \infty} \frac{N_1}{D_4^2} - \lim_{x \rightarrow \infty} x^2 \frac{1}{4d^0 D_4} \left(\frac{D'_4}{D_4} \right)' \\ &= - \lim_{x \rightarrow \infty} x^2 \frac{1}{4d^0 D_4} \left(\frac{D'_4}{D_4} \right)' = \frac{1}{4}. \end{aligned} \quad (3.74)$$

(3) Let c be root of D_4 :

$$\begin{aligned} (x - c)^{2v} (r - F) &= \frac{N_2}{D_1 D_2^2} \frac{(x - c)^{2v}}{D_4} + (x - c)^{2v} \frac{1}{4d^0 D_4} \left(\frac{D'_4}{D_4} \right)' \\ &= o\left((x - c)^{v-1}\right). \end{aligned} \quad (3.75) \quad \square$$

Notation. We choose a square root of the polynomial of even degree: $N_1 + D_4^2[E(r) - (1/4d^0 D_4)(D'_4/D_4)']$ on a connected open at infinity, root of D_4 . We put

$$(F)^{1/2} = \frac{1}{D_4} \left(N_1 + D_4^2 \left[E(r) - \frac{1}{4d^0 D_4} \left(\frac{D'_4}{D_4} \right)' \right] \right)^{1/2}. \quad (3.76)$$

Proposition 3.15. *One assumes $d^0 r < 0$. One has*

$$\theta = \sum_{c \in \text{Roots}(D_4)} \varepsilon_c \left(\text{partial fraction of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} \frac{(D_2 D_4)'}{D_2 D_4} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0}, \quad (3.77)$$

where $\varepsilon_c = \pm 1$ and D_0 being a polynomial of degree:

$$\left[\sum_{c \in \text{Root}(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o(D_2 D_4) + d^o D_1 + d^o D_0 - \frac{1}{2} \right]^2 = \lim_{x \rightarrow \infty} x^2 r + \frac{1}{4}. \quad (3.78)$$

Proof.

$$\begin{aligned} \lim_{x \rightarrow \infty} x\theta &= \sum_{c \in \text{Roots}(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o(D_2 D_4) + d^o D_1 + d^o D_0, \\ \lim_{x \rightarrow \infty} x^2 r + \frac{1}{4} &= \left(\lim_{x \rightarrow \infty} x\theta - \frac{1}{2} \right)^2. \end{aligned} \quad (3.79)$$

Thus

$$\left[\sum_{c \in \text{Roots}(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o(D_2 D_4) + d^o D_1 + d^o D_0 - \frac{1}{2} \right]^2 = \lim_{x \rightarrow \infty} x^2 r + \frac{1}{4}. \quad (3.80)$$

□

Proposition 3.16. *One assumes $d^o r = 0$. One has*

$$\theta = E(\theta) + \sum_{c \in \text{Roots}(D_4)} \varepsilon_c \left(\text{partial fraction of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} \frac{(D_2 D_4)'}{D_2 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}, \quad (3.81)$$

where $\varepsilon_c = \pm 1$ and D_0 being a polynomial of degree

$$d^o D_0 = \frac{1}{2E(\theta)} \lim_{x \rightarrow \infty} x[r - E(r)] - \left[\sum_{c \in \text{Roots}(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o(D_2 D_4) + d^o D_1 \right]. \quad (3.82)$$

Proof. We have

$$E^2(\theta) = E(r), \quad 2E(\theta) \sum_c \text{residue of } \theta \text{ at } c = \sum_c \text{residue of } r \text{ at } c. \quad (3.83)$$

Thus

$$2E(\theta) \left[\sum_{c \in \text{Roots}(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o(D_2 D_4) + d^o D_1 + d^o D_0 \right] = \lim_{x \rightarrow \infty} x(r - E(r)). \quad (3.84)$$

□

Proposition 3.17. *One assumes $d^o r = 2\nu > 0$. One has*

$$\theta = E(\theta) + \sum_{c \in \text{Roots}(D_4)} \varepsilon_c \left(\text{partial fraction of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} \frac{(D_2 D_4)'}{D_2 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}, \quad (3.85)$$

where $\varepsilon_c = \pm 1$ and D_0 being a polynomial of degree

$$d^o D_0 = \alpha s_{\nu+1} - \frac{\nu}{2} - \left[\sum_{c \in \text{Roots}(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o (D_2 D_4) + d^o D_1 \right]. \quad (3.86)$$

Proof. Let a dominant coefficient of $E(r)$. $E(\theta) = \alpha[t^{-\nu} + s_1 t^{-(\nu-1)} + \dots + s_\nu]$. $\alpha s_{\nu+1} - (\nu/2) =$ sum residue of θ , where $\alpha^2 = a$. Thus

$$\theta = E(\theta) + \sum_{c \in \text{Roots}(D_4)} \varepsilon_c \left(\text{partial fraction of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} \frac{(D_2 D_4)'}{D_2 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}. \quad (3.87)$$

Thus

$$\alpha s_{\nu+1} - \frac{\nu}{2} = \left[\sum_{c \in \text{Roots}(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o (D_2 D_4) + d^o D_1 + d^o D_0 \right]. \quad (3.88)$$

Thus

$$d^o D_0 = \alpha s_{\nu+1} - \frac{\nu}{2} - \left[\sum_{c \in \text{Roots}(D_4)} \varepsilon_c \left(\text{residue of } (F)^{1/2} \text{ at } c \right) + \frac{1}{2} d^o (D_2 D_4) + d^o D_1 \right]. \quad (3.89)$$

□

4. Recurrent Method at Infinity

4.1. Presentation of D_0 as Determinant

Proposition 4.1. *Let $c_1, \dots, c_m$ be complex constants. We put*

$$\begin{aligned} P(x) &= (x - c_1) \cdots (x - c_m) \\ &= x^m - p_1 x^{m-1} + p_2 x^{m-2} - \dots + (-1)^m p_m. \end{aligned} \quad (4.1)$$

For $j \in \mathbb{N}$, we put $\sigma_j = c_1^j + \cdots + c_m^j$ and

$$\Delta = \begin{vmatrix} 1 & c_1 & \cdot & \cdot & c_1^{m-1} \\ 1 & c_2 & \cdot & \cdot & c_2^{m-1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 1 & c_m & \cdot & \cdot & c_m^{m-1} \end{vmatrix}. \quad (4.2)$$

(1) One has

$$\Delta^2 = \begin{vmatrix} \sigma_0 & \sigma_1 & \cdot & \cdot & \sigma_{m-1} \\ \sigma_1 & \sigma_2 & \cdot & \cdot & \sigma_m \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \sigma_{m-1} & \sigma_m & \cdot & \cdot & \sigma_{2m-2} \end{vmatrix}. \quad (4.3)$$

(2) One has

$$\Delta^2 P(x) = \begin{vmatrix} \sigma_0 & \cdot & \cdot & \sigma_{m-1} & 1 \\ \cdot & & & \cdot & x \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \sigma_m & \cdot & \cdot & \sigma_{2m-1} & x^m \end{vmatrix}. \quad (4.4)$$

Proof. Δ , $\begin{vmatrix} \sigma_0 & \sigma_1 & \cdot & \cdot & \sigma_{m-1} \\ \sigma_1 & \sigma_2 & \cdot & \cdot & \sigma_m \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \sigma_{m-1} & \sigma_m & \cdot & \cdot & \sigma_{2m-2} \end{vmatrix}$ are polynomials at $c_1, \dots, c_m$ with real coefficients. Thus, $\Delta^2 P(x)$ and $\begin{vmatrix} \sigma_0 & \cdot & \cdot & \sigma_{m-1} & 1 \\ \cdot & & & \cdot & x \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \sigma_m & \cdot & \cdot & \sigma_{2m-1} & x^m \end{vmatrix}$ are polynomials at $c_1, \dots, c_m, x$ with real coefficients. Thus, to have both identities we can assume $c_1, \dots, c_m, x$ are reals. Furthermore, we can restrict to the open of Zariski $c_1, \dots, c_m, x$ distinct real nonzero.

We have for all $j = 1, \dots, m$, $c_j^m = p_1 c_j^{m-1} - p_2 c_j^{m-2} - \cdots - (-1)^m p_m$.

We put

$$\Delta(x) = \begin{vmatrix} 1 & c_1 & \cdot & \cdot & c_1^{m-1} & c_1^m \\ 1 & c_2 & \cdot & \cdot & c_2^{m-1} & c_2^m \\ \cdot & & & & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 1 & c_m & \cdot & \cdot & c_m^{m-1} & c_m^m \\ 1 & x & \cdot & \cdot & x^{m-1} & x^m \end{vmatrix}, \quad (4.5)$$

$$\Delta(x) = \begin{vmatrix} 1 & c_1 & \cdot & \cdot & c_1^{m-1} & 0 \\ 1 & c_2 & \cdot & \cdot & c_2^{m-1} & 0 \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ 1 & c_m & \cdot & \cdot & c_m^{m-1} & 0 \\ 1 & x & \cdot & \cdot & x^{m-1} & P(x) \end{vmatrix} = \Delta P(x).$$

We put

$$v_j = \begin{pmatrix} c_1^{j-1} \\ \cdot \\ \cdot \\ c_m^{j-1} \end{pmatrix}, \quad \text{for } j = 1, \dots, m. \quad (4.6)$$

The scalar product

$$\langle v_j, v_k \rangle = c_1^{j+k-2} + \dots + c_m^{j+k-2} = \sigma_{j+k-2}. \quad (4.7)$$

The Gram matrix of $v_1, \dots, v_m$ is

$$G = \begin{pmatrix} \sigma_0 & \sigma_1 & \cdot & \cdot & \sigma_{m-1} \\ \sigma_1 & \sigma_2 & \cdot & \cdot & \sigma_m \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \sigma_{m-1} & \sigma_m & \cdot & \cdot & \sigma_{2m-2} \end{pmatrix}. \quad (4.8)$$

Thus

$$\Delta^2 = \begin{vmatrix} \sigma_0 & \sigma_1 & \cdot & \cdot & \sigma_{m-1} \\ \sigma_1 & \sigma_2 & \cdot & \cdot & \sigma_m \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \sigma_{m-1} & \sigma_m & \cdot & \cdot & \sigma_{2m-2} \end{vmatrix}. \quad (4.9)$$

We put

$$v_j(x) = \begin{pmatrix} c_1^{j-1} \\ \cdot \\ \cdot \\ c_m^{j-1} \\ x^{j-1} \end{pmatrix}, \quad j = 1, \dots, m+1. \quad (4.10)$$

We notice

$$\langle v_j(x), v_k(x) \rangle = \langle v_j, v_k \rangle + x^{j+k-2} = \sigma_{j+k-2} + x^{j+k-2}. \quad (4.11)$$

The j th column of Gram of matrix which defines $\Delta(x)$ is

$$\begin{pmatrix} \sigma_{j-1} \\ \sigma_j \\ \cdot \\ \cdot \\ \sigma_{j+m-1} \end{pmatrix} + x^{j-1} \begin{pmatrix} 1 \\ x \\ \cdot \\ \cdot \\ x^m \end{pmatrix}. \quad (4.12)$$

Thus

$$\begin{aligned} \Delta^2(x) &= \begin{vmatrix} \sigma_0 & \sigma_1 & \cdot & \cdot & \sigma_m \\ \sigma_1 & \sigma_2 & \cdot & \cdot & \sigma_{m+1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \sigma_m & \sigma_{m+1} & \cdot & \cdot & \sigma_{2m} \end{vmatrix} + \begin{vmatrix} 1 & \sigma_1 & \cdot & \cdot & \sigma_m \\ x & \sigma_2 & \cdot & \cdot & \sigma_{m+1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ x^m & \sigma_{m+1} & \cdot & \cdot & \sigma_{2m} \end{vmatrix} + \dots + \begin{vmatrix} \sigma_0 & \cdot & \sigma_{m-1} & x^m \\ \sigma_1 & \cdot & \sigma_m & x^{m+1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \sigma_m & \cdot & \sigma_{2m-1} & x^{2m} \end{vmatrix} \\ &= \sum_{i,j} x^{i+j-2} \text{cofactor}_{i,j}(M), \end{aligned} \quad (4.13)$$

where

$$M = \begin{pmatrix} \sigma_0 & \sigma_1 & \cdot & \cdot & \sigma_m \\ \sigma_1 & \sigma_2 & \cdot & \cdot & \sigma_{m+1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \sigma_m & \sigma_{m+1} & \cdot & \cdot & \sigma_{2m} \end{pmatrix}. \quad (4.14)$$

We obtain

$$\Delta^2(x) = \Delta^2 P^2(x) = (1 \ x \cdots x^m) \text{Com}(M) \begin{pmatrix} 1 \\ x \\ \cdot \\ \cdot \\ x^m \end{pmatrix} \quad (4.15)$$

The cofactor $(m+1, m+1)$ of M is Δ^2 . The adjoint of M is nonzero. We prove that adjoint of M of rank 1. $\det(M) = 0$. Because: $\sigma_m = p_1\sigma_{m-1} - p_2\sigma_{m-2} - \cdots - (-1)^m p_m\sigma_0$ and for all $k \geq m$; $\sigma_k = p_1\sigma_{k-1} - p_2\sigma_{k-2} - \cdots - (-1)^m p_m\sigma_{k-m}$. Thus, the $(m+1)$ th column of M is a linear combination of other column.

We consider the matrix

$$E_{1,1} = \begin{pmatrix} 1 & 0 & \cdot & 0 \\ 0 & & \cdot & \\ \cdot & & \cdot & \\ 0 & \cdot & \cdot & 0 \end{pmatrix}, \quad (4.16)$$

$$M + E_{1,1} = \begin{pmatrix} \sigma_0 + 1 & \sigma_1 & \cdot & \cdot & \sigma_m \\ \sigma_1 & \sigma_2 & \cdot & \cdot & \sigma_{m+1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \sigma_m & \sigma_{m+1} & \cdot & \cdot & \sigma_{2m} \end{pmatrix}, \quad (4.17)$$

$$\det(M + E_{1,1}) = \det(M) + \begin{vmatrix} 1 & \sigma_1 & \cdot & \cdot & \sigma_m \\ 0 & \sigma_2 & \cdot & \cdot & \sigma_{m+1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & \sigma_{m+1} & \cdot & \cdot & \sigma_{2m} \end{vmatrix} = \begin{vmatrix} \sigma_2 & \sigma_3 & \cdot & \cdot & \sigma_{m+1} \\ \sigma_3 & \sigma_4 & \cdot & \cdot & \sigma_{m+2} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \sigma_{m+1} & \sigma_{m+2} & \cdot & \cdot & \sigma_{2m} \end{vmatrix}.$$

It is the determinant of Gram of

$$\begin{pmatrix} c_1 \\ \cdot \\ \cdot \\ c_m \end{pmatrix}, \dots, \begin{pmatrix} c_1^m \\ \cdot \\ \cdot \\ c_m^m \end{pmatrix}. \quad (4.18)$$

Thus

$$\det(M + E_{1,1}) = \begin{vmatrix} c_1 & \cdot & \cdot & \cdot & c_1^m \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ c_m & \cdot & \cdot & \cdot & c_m^m \end{vmatrix}^2. \quad (4.19)$$

But

$$\begin{vmatrix} c_1 & \cdot & \cdot & \cdot & c_1^m \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ c_m & \cdot & \cdot & \cdot & c_m^m \end{vmatrix} = (-1)^{m-1} p_m \begin{vmatrix} c_1 & \cdot & \cdot & \cdot & c_1^{m-1} & 1 \\ \cdot & & & & \cdot & 1 \\ \cdot & & & & \cdot & 1 \\ \cdot & & & & \cdot & 1 \\ c_m & \cdot & \cdot & \cdot & c_m^{m-1} & 1 \end{vmatrix} = p_m \Delta. \quad (4.20)$$

Thus $\det(M + E_{1,1}) = p_m^2 \Delta^2$. Or $p_m = c_1 \cdots c_m$ is scalar nonzero, $M + E_{1,1}$ is invertible matrix.

With M symmetric matrix, $\text{Com}(M)$ symmetric matrix, we have

$$M \text{Com}(M) = \text{Com}(M)M = \det(M)I = 0, \quad I \text{ being matrix identity.} \quad (4.21)$$

Thus

$$(M + E_{1,1})\text{Com}(M) = E_{1,1}\text{Com}(M). \quad (4.22)$$

Or $M + E_{1,1}$ is invertible and $\text{Com}(M) \neq 0$, $\text{Com}(M)$ of rank 1.

Thus

$$\text{cofactor}_{i,j}(M) = \frac{\text{cofactor}_{i,m+1}(M)}{\text{cofactor}_{m+1,m+1}(M)} \text{cofactor}_{m+1,j}(M). \quad (4.23)$$

For $i = 1, \dots, m+1$, $j = 1, \dots, m+1$, we put: $C_{i,j} = \text{cofactor}_{i,j}(M)$,

$$\frac{C_{i,j}}{\Delta^2} = \frac{C_{i,m+1}}{\Delta^2} \frac{C_{m+1,j}}{\Delta^2}. \quad (4.24)$$

Thus:

$$\begin{aligned}
 P(x)^2 &= \sum_{1 \leq i, j \leq m+1} \frac{C_{i,j}}{\Delta^2} x^{i+j-2} \\
 &= \sum_{1 \leq i, j \leq m+1} \frac{C_{i,m+1}}{\Delta^2} \frac{C_{j,m+1}}{\Delta^2} x^{i+j-2} \\
 &= \left(\sum_{1 \leq i \leq m+1} \frac{C_{i,m+1}}{\Delta^2} x^{i-1} \right)^2.
 \end{aligned} \tag{4.25}$$

Or P is monic polynomial and $C_{m+1,m+1} / \Delta^2 = 1$

$$P(x) = \sum_i \frac{C_{i,m+1}}{\Delta^2} x^{i-1} = \frac{1}{\Delta^2} \begin{vmatrix} \sigma_0 & \cdot & \cdot & \cdot & \sigma_{m-1} & 1 \\ \sigma_1 & \cdot & \cdot & \cdot & \sigma_m & x \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \sigma_m & \cdot & \cdot & \cdot & \sigma_{2m-1} & x^m \end{vmatrix}. \tag{4.26}$$

We obtain

$$\Delta^2 P(x) = \begin{vmatrix} \sigma_0 & \cdot & \cdot & \cdot & \sigma_{m-1} & 1 \\ \sigma_1 & \cdot & \cdot & \cdot & \sigma_m & x \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \sigma_m & \cdot & \cdot & \cdot & \sigma_{2m-1} & x^m \end{vmatrix}. \tag{4.27}$$

□

4.2. Taylor Expansion of P' / P

Proposition 4.2. Let $c_1, \dots, c_m$ be pairwise distinct constants.

(1) We put, for $j = 0, \dots, m$: $\sigma_j = c_1^j + \dots + c_m^j$ and $P(x) = (x - c_1) \cdots (x - c_m)$

$$\left(\frac{P'}{P} \right)' + \left(\frac{P'}{P} \right)^2 = 2 \sum_{i \neq j} \frac{1}{c_i - c_j} \frac{1}{x - c_i}. \tag{4.28}$$

(2) We put $t = 1/x$. Taylor expansion on the neighborhood of infinity:

(a)

$$\left(\frac{P'}{P} \right)' + \left(\frac{P'}{P} \right)^2 = \sum_{l=2}^{\infty} \left[\left(\sum_{\mu=0}^{l-2} \sigma_{\mu} \sigma_{l-2-\mu} \right) - (l-1) \sigma_{l-2} \right] t^l, \tag{4.29}$$

(b)

$$x \frac{P'}{P} = \sum_{l=0}^{\infty} \sigma_l t^l. \quad (4.30)$$

Proof. We have $P'/P = \sum_{i=1}^m 1/(x - c_i)$.

(1)

$$\begin{aligned} \left(\frac{P'}{P}\right)' + \left(\frac{P'}{P}\right)^2 &= \sum_{i \neq j} \frac{1}{x - c_i} \frac{1}{x - c_j} \\ &= \sum_{i \neq j} \left[\frac{1/(c_i - c_j)}{x - c_i} + \frac{1/(c_j - c_i)}{x - c_j} \right] \\ &= 2 \sum_{i \neq j} \frac{1}{c_i - c_j} \frac{1}{x - c_i}. \end{aligned} \quad (4.31)$$

(2) We put $t = 1/x$. We have

$$\begin{aligned} \left(\frac{P'}{P}\right)' + \left(\frac{P'}{P}\right)^2 &= 2t \sum_{i \neq j} \frac{1}{c_i - c_j} \frac{1}{1 - c_i t} \\ &= 2t \sum_{i \neq j} \frac{1}{c_i - c_j} \sum_{k=0}^{\infty} (c_i t)^k \\ &= \sum_{k=0}^{\infty} \left(2 \sum_{i \neq j} \frac{c_i^k}{c_i - c_j} \right) t^{k+1} \\ &= \sum_{k=1}^{\infty} \sum_{i \neq j} \frac{c_i^k - c_j^k}{c_i - c_j} t^{k+1}. \end{aligned} \quad (4.32)$$

as

$$\begin{aligned} \sum_{i \neq j} \frac{c_i^k - c_j^k}{c_i - c_j} &= \sum_{i \neq j} \sum_{\mu+\nu=k-1, 0 \leq \mu \leq k-1} c_i^{\mu} c_j^{\nu} \\ &= \sum_{\mu+\nu=k-1, 0 \leq \mu \leq k-1} \left[\sigma_{\mu} \sigma_{\nu} - \sum_{i=1}^m c_i^{\mu+\nu} \right] \\ &= \left[\sum_{\mu+\nu=k-1, 0 \leq \mu \leq k-1} \sigma_{\mu} \sigma_{\nu} \right] - k \sigma_{k-1}. \end{aligned} \quad (4.33)$$

Thus

$$\begin{aligned} \left(\frac{P'}{P}\right)' + \left(\frac{P'}{P}\right)^2 &= \sum_{k=1}^{\infty} \left[\left(\sum_{\mu=0}^{k-1} \sigma_{\mu} \sigma_{k-1-\mu} \right) - k \sigma_{k-1} \right] t^l \\ &= \sum_{l=2}^{\infty} \left[\left(\sum_{\mu=0}^{l-2} \sigma_{\mu} \sigma_{l-2-\mu} \right) - (l-1) \sigma_{l-2} \right] t^l. \end{aligned} \quad (4.34)$$

□

4.3. Research of D_0

Let $r = N/D_1 D_2^2 D_3^2 D_4^2$ be a rational fraction and $\theta \in \mathbb{C}(x)$ where $\theta' + \theta^2 = r$.

We put

$$\theta = S + \frac{D'_0}{D_0}, \quad (4.35)$$

where $S = E(\theta) + \sum_{c \text{ poles of } r} \theta_c$ and D_0 monic polynomial of degree m . We have

$$\begin{aligned} \theta' + \theta^2 &= S' + S^2 + \left(\frac{D'_0}{D_0}\right)' + \left(\frac{D'_0}{D_0}\right)^2 + 2S \frac{D'_0}{D_0}, \\ \theta' + \theta^2 = r &\iff \left(\frac{D'_0}{D_0}\right)' + \left(\frac{D'_0}{D_0}\right)^2 + 2S \frac{D'_0}{D_0} = \frac{R}{B} \end{aligned} \quad (4.36)$$

with $B = D_1 D_2 D_3 D_4$ and R polynomials as $r - S' - S^2 = R/B$.

Case 1 ($d^o r = 2\nu \geq 0$). We have $d^o S = \nu$ and $d^o R/B = \nu - 1$. Taylors expansion of $(1/x^\nu)S$, $(1/x^{\nu-1})(R/B)$ and $x(D'_0/D_0)$ at order μ .

We obtain the Taylors expansion of $(1/x^{\nu-1})[(R/B) - 2S(D'_0/D_0)]$ at order μ .

The Taylors expansion equals the Taylors expansion of $(1/x^{\nu-1})[(D'_0/D_0)' + (D'_0/D_0)^2]$. Let $t = 1/x$. In Taylors expansion of $(1/x^{\nu-1})[(P'/P)' + (P'/P)^2]$ (see Proposition 4.2) the coefficient of $t^{\nu+1}$ is: $\sigma_0^2 - \sigma_0 = m^2 - m$. For $l \geq 2$, coefficient of $t^{l+\nu-1}$ use: $\sigma_0, \dots, \sigma_{l-2}$.

Besides, we have

$$x \frac{D'_0}{D_0} = \sum_{i=1}^m \frac{1}{1 - c_i t} = \sum_{i=1}^m \sum_{k=0}^{\infty} c_i^k t^k = \sum_{k=0}^{\infty} \sigma_k t^k. \quad (4.37)$$

Thus, Taylors expansion of

$$\Phi = \frac{1}{x^{\nu-1}} \left[\left(\frac{D'_0}{D_0}\right)' + \left(\frac{D'_0}{D_0}\right)^2 \right] - \frac{1}{x^{\nu-1}} \left[\frac{R}{B} - 2S \frac{D'_0}{D_0} \right] \quad (4.38)$$

and the coefficient of t^k is

$$-2\alpha\sigma_k + \text{polynomial at } \sigma_0, \dots, \sigma_{k-1}, \quad (4.39)$$

$\Phi = 0$ determine σ_k by recurrence at $k = \nu$.

Case 2 ($d^0 r < 0$). The coefficient of t^k in Taylors expansion of

$$\Psi = x^2 \left[\left(\frac{D'_0}{D_0} \right)' + \left(\frac{D'_0}{D_0} \right)^2 \right] - x^2 \frac{R}{B} + 2xS \left(x \frac{D'_0}{D_0} \right) \quad (4.40)$$

is

$$2m\sigma_k - (k+1)\sigma_k + \left(2 \lim_{x \rightarrow \infty} xS \right) \sigma_k + \text{polynomial at } \sigma_0, \dots, \sigma_{k-1}. \quad (4.41)$$

By recurrence σ_k at condition

$$k \neq 2 \lim_{x \rightarrow \infty} xS + 2m - 1. \quad (4.42)$$

We have $\lim_{x \rightarrow \infty} xS + m = \alpha_\infty^2 + (1/2)$ with $\alpha_\infty^2 = \lim_{x \rightarrow \infty} x^2 r + (1/4)$.

For $k \neq 2\alpha_\infty$, to be σ_k at function of $\sigma_0, \dots, \sigma_{k-1}$.

If $2\alpha_\infty \in \mathbb{N}$ then, the coefficient of $t^{2\alpha_\infty}$ is nonzero implied that there is no solution, or the coefficient of $t^{2\alpha_\infty}$ is zero implied that $\sigma_{2\alpha_\infty}$ is arbitrary and $\sigma_k; k > 2\alpha_\infty$ depend in a unique way.

D_0 is polynomial determined by $\sigma_0, \dots, \sigma_{2m-1}$ by determinant formulae; the problem of nonunique suite $(\sigma_k)_{k \in \mathbb{N}}$ is put if $2\alpha_\infty$ is positive integer equal to or less than $2m - 1$, then $4 \lim_{x \rightarrow \infty} x^2 r + 1$ is square of integer and

$$\lim_{x \rightarrow \infty} x^2 r \leq m^2 - m. \quad (4.43)$$

Example 4.3. In this example we consider the Riccati differential equation (1.2) where

$$r = -1 + \frac{z^2 - (1/4)}{x^2}; \quad z \in \mathbb{R}^*. \quad (4.44)$$

We have $D_1 = 1, D_2 = 1, D_3 = x$ and $E^2(\theta) = -1$.

We can assume $E(\theta) = i$,

$$\begin{aligned} \lim_{x \rightarrow 0} x^2 r + \frac{1}{4} &= z^2 = \alpha_0^2, \\ \theta &= i + \frac{\alpha_0 + (1/2)}{x} + \frac{D'_0}{D_0}, \end{aligned} \quad (4.45)$$

$2i(\alpha_0 + (1/2) + d^0 D_0) = 0$. Thus $m = d^0 D_0 = -\alpha_0 - (1/2) \in \mathbb{N}$.

Thus $\alpha_0 = -m - (1/2)$, $r = -1 + ((m + (1/2))^2 - (1/4))/x^2 = -1 + (m^2 + m)/x^2$ and $\theta = i - (m/x) + (D'_0/D_0)$. θ is a solution of Riccati equation if and only if D_0 verifies

$$\left(\frac{D'_0}{D_0}\right)' + \left(\frac{D'_0}{D_0}\right)^2 + 2\left(i - \frac{m}{x}\right)\frac{D'_0}{D_0} = 2i\frac{m}{x}. \quad (4.46)$$

Thus $\Phi = 0$ where

$$\Phi = x \left[\left(\frac{D'_0}{D_0}\right)' + \left(\frac{D'_0}{D_0}\right)^2 \right] + 2\left(i - \frac{m}{x}\right) \left(x \frac{D'_0}{D_0}\right) - 2im. \quad (4.47)$$

Taylor's expansion of $\Phi = 0$ at $t = 1/x$

$$\Phi = \sum_{k=1}^{\infty} \left[\left(\sum_{\mu=1}^{k-1} \sigma_{\mu} \sigma_{k-1-\mu} \right) - k\sigma_{k-1} - 2m\sigma_{k-1} + 2i\sigma_k \right] t^k. \quad (4.48)$$

As $\sigma_0 = m$. Thus

$$\Phi = 0 \iff \forall k \geq 1, \quad \left(\sum_{\mu=0}^{k-1} \sigma_{\mu} \sigma_{k-1-\mu} \right) - k\sigma_{k-1} - 2m\sigma_{k-1} + 2i\sigma_k = 0. \quad (4.49)$$

For $k = 1$, $\sigma_0^2 - \sigma_0 - 2m\sigma_0 + 2i\sigma_1 = 0$ equivalent to $2i\sigma_1 = m^2 + m$.

For all, $k \geq 2$, $2i\sigma_k = k\sigma_{k-1} - \sum_{\mu=1}^{k-2} \sigma_{\mu} \sigma_{k-1-\mu}$. For example

$$m = 3 : \begin{cases} 2i\sigma_1 = 12, \\ 2i\sigma_2 = 2\sigma_1, \\ 2i\sigma_3 = 3\sigma_2 - \sigma_1^2, \\ 2i\sigma_4 = 4\sigma_3 - 2\sigma_1\sigma_2, \\ 2i\sigma_5 = 5\sigma_4 - 2\sigma_1\sigma_3 - \sigma_2^2. \end{cases} \quad (4.50)$$

Thus, the polynomial D_0 is partner to

$$\begin{vmatrix} \sigma_0 & \sigma_1 & \sigma_2 & 1 \\ \sigma_1 & \sigma_2 & \sigma_3 & x \\ \sigma_2 & \sigma_3 & \sigma_4 & x^2 \\ \sigma_3 & \sigma_4 & \sigma_5 & x^3 \end{vmatrix} = \begin{vmatrix} 3 & -6i & -6 & 1 \\ -6i & -6 & -9i & x \\ -6 & -9i & -54 & x^2 \\ -9i & -54 & 99i & x^3 \end{vmatrix} = 135x^3 + 810ix^2 - 2025x - 2025i. \quad (4.51)$$

Thus

$$D_0 = x^3 + 6ix^2 - 15x - 15i. \quad (4.52)$$

5. Method of Last Minor

Let $r = N/D_1 D_2^2 D_3^2 D_4^2$ be a rational fraction and $\theta \in \mathbb{C}(x)$ as: $\theta' + \theta^2 = r$.
 θ solution of (1.2) if and only if D_0 verifies

$$BD_0'' + 2SBD_0' = RD_0. \quad (5.1)$$

We choose a complex number c not pole of r and we use the expression of polynomials following the powers of $x - c$.

For the sake of simplicity, in the following we assume that $c = 0$. Constant coefficient constant of B is nonzero.

Denote by a_k , b_k , and r_k coefficients of x^k , in A , B , and R , respectively, $k \in \mathbb{N}$. If $k \in \mathbb{Z}$ then $a_k = b_k = r_k = 0$.

5.1. Case $d^0 r < 0$

We put $S = A/B$.

If $d^0 B = 1$ then r are pole unique, it is simple or double. If c simple pole of r then r of degree -1 and (1.2) has no rational solution.

If c double pole of r then $A = \alpha_c + (1/2)$, $r = (\alpha_c^2 - (1/4))/(x - c)^2$ and $R = 0$.

Thus, (5.1)

$$\frac{D_0''}{D_0'} = \frac{-2(\alpha_c + (1/2))}{x - c}. \quad (5.2)$$

The solutions

$$\begin{aligned} D_0' &= (-2\alpha_c)(x - c)^{-2\alpha_c - 1}, \\ D_0 &= (x - c)^{-2\alpha_c} + \beta, \end{aligned} \quad (5.3)$$

where β nonzero constant.

We have, an infinity of the other rational solution when $-2\alpha_c \in \mathbb{N}^*$, with the same parity of S .

We assume $d^0 B \geq 2$ and we consider D_0 a polynomial of degree $m \geq 1$ that verifies (5.1).

Proposition 5.1. One assumes $d^o r < 0$, $\left\{ \begin{smallmatrix} d^o B \geq 2 \\ d^o = m \geq 1 \end{smallmatrix} \right.$ and D_0 verifies (5.1).

(1) One has

$$\begin{aligned} d^o(BD_0'' + 2AD_0') &\leq d^o B + m - 2, \\ d^o R &\leq d^o B - 2. \end{aligned} \quad (5.4)$$

(2) One has $r_{d^o B-2} = 2a_{d^o B-1}m + m(m-1)$.

(3) One has

$$\begin{aligned} a_{d^o B-1} &= \frac{1}{2} - m + \alpha_\infty, \\ r_{d^o B-2} &= m(2\alpha_\infty - m). \end{aligned} \quad (5.5)$$

Proof. (1) $d^o(BD_0'' + 2AD_0') \leq \sup(d^o(BD_0''), d^o(2AD_0'))$. If $d^o D_0 = 1$ then

$$\sup(d^o(BD_0''), d^o(2AD_0')) = d^o(2AD_0') = d^o A \leq d^o B - 1 = d^o B + d^o D_0 - 2. \quad (5.6)$$

If $d^o D_0 \geq 2$ then

$$\begin{aligned} d^o(BD_0'') &= d^o B + m - 2, \\ d^o(2AD_0') &= d^o A + m - 1 \leq d^o B + m - 2. \end{aligned} \quad (5.7)$$

(2) We have $d^o(RD_0) \leq d^o B + m - 2$ and coefficients of $x^{d^o B+m-2}$ in (5.1).

(3) Accordingly, in view of $\lim_{x \rightarrow \infty} x\theta = \alpha_\infty + (1/2)$. □

We put

$$v(x) = -R \begin{pmatrix} 1 \\ x \\ x^2 \\ \cdot \\ \cdot \\ x^m \end{pmatrix} + 2A \begin{pmatrix} 0 & 1 \\ 2x \\ \cdot \\ \cdot \\ mx^{m-1} \end{pmatrix} + B \begin{pmatrix} 0 \\ 0 \\ 2 \\ \cdot \\ \cdot \\ m(m-1)x^{m-2} \end{pmatrix} = \begin{pmatrix} v_1(x) \\ v_2(x) \\ v_3(x) \\ \cdot \\ \cdot \\ v_{m+1}(x) \end{pmatrix}. \quad (5.8)$$

The k th element of $v(x)$ is

$$v_k(x) = B(k-1)(k-2)x^{k-3} + 2A(k-1)x^{k-2} - Rx^{k-1}, \quad (5.9)$$

$v_k(x)$ of degree equal to or less than $d^o B + k - 3$ and $x^{d^o B + k - 3}$ of coefficient

$$\begin{aligned} (k-1)(k-2) + 2a_{d^o B-1}(k-1) - r_{d^o B-2} &= (k-m-1) \left(\frac{r_{d^o B-2}}{m} + k-1 \right) \\ &= (k-m-1)(2a_{d^o B-1} + m + k - 2). \end{aligned} \quad (5.10)$$

For $k \leq m$, the coefficient of $x^{d^o B + k - 3}$ is zero if and only if

$$2a_{d^o B-1} = -(m+k-2) \in \{-(m-1), -m, \dots, -2(m-1)\}, \quad (5.11)$$

v_{m+1} of degree equal to or less than $d^o B + m - 3$.

For $k \geq 3$ the coefficient of more low degree of v_k is coefficient of x^{k-3} equal $b_0(k-1)(k-2)$.

Let V matrix of $m+1$ row and k th row is row l_k of coefficients of $v_k(x)$ in basis of $\mathbb{C}_{d^o B + m - 3}[X]$.

$l_3, \dots, l_{m+1}$ is linearly independent system.

Equation (5.1) gives

$$D_0 = d_0 + d_1 x + \dots + x^m, \quad (5.12)$$

obtained

$$\begin{aligned} d_0 v_1(x) + \dots + d_{m-1} v_m(x) + v_{m+1}(x) &= 0, \\ d_0 l_1 + \dots + d_{m-1} l_m + l_{m+1} &= 0. \end{aligned} \quad (5.13)$$

Thus, the matrix V is rank $m-1$, m or $m+1$. Accordingly, existence of D_0 correspondent is linearly dependent in row l_{m+1} of $l_1, \dots, l_m$.

Let $c_1, \dots, c_m$ roots of D_0 distinct.

We put, for $j \in \mathbb{N}$: $\sigma_j = c_1^j + \dots + c_m^j$ and

$$\Delta^2 = \begin{vmatrix} \sigma_0 & \sigma_1 & \cdot & \cdot & \sigma_{m-1} \\ \sigma_1 & \sigma_2 & \cdot & \cdot & \sigma_m \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \sigma_{m-1} & \sigma_m & \cdot & \cdot & \sigma_{2m-2} \end{vmatrix}. \quad (5.14)$$

The m column vector $\begin{pmatrix} \sigma_0 \\ \cdot \\ \sigma_m \end{pmatrix}, \dots, \begin{pmatrix} \sigma_{m-1} \\ \cdot \\ \sigma_{2m-1} \end{pmatrix}$ of $m+1$ column is linearly independent.

The equation of hyperplane H engendered by vectors $\begin{pmatrix} \sigma_0 \\ \vdots \\ \sigma_m \end{pmatrix}, \dots, \begin{pmatrix} \sigma_{m-1} \\ \vdots \\ \sigma_{2m-1} \end{pmatrix}$ is

$$\begin{vmatrix} \sigma_0 & \cdot & \cdot & \cdot & \sigma_{m-1} & x_1 \\ \sigma_1 & \cdot & \cdot & \cdot & \sigma_m & x_2 \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \sigma_m & \cdot & \cdot & \cdot & \sigma_{2m-1} & x_{m+1} \end{vmatrix} = 0. \quad (5.15)$$

The coefficient of x_{m+1} is $\Delta^2 \neq 0$ and H as equation

$$x_{m+1} = \lambda_1 x_1 + \dots + \lambda_m x_m, \quad (5.16)$$

where $\lambda_1 \dots \lambda_m$ scalar satisfying

$$(\sigma_m, \dots, \sigma_{2m-1}) = \lambda_1(\sigma_0, \dots, \sigma_{m-1}) + \dots + \lambda_m(\sigma_{m-1}, \dots, \sigma_{2m-2}). \quad (5.17)$$

The derivative of

$$\begin{vmatrix} \sigma_0 & \cdot & \cdot & \cdot & \sigma_{m-1} & 1 \\ \sigma_1 & \cdot & \cdot & \cdot & \sigma_m & x \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \sigma_m & \cdot & \cdot & \cdot & \sigma_{2m-1} & x^m \end{vmatrix} \quad (5.18)$$

is derived from last column. Equation (5.1) is equivalent to:

$$\begin{vmatrix} \sigma_0 & \cdot & \cdot & \cdot & \sigma_{m-1} & 1 \\ \sigma_1 & \cdot & \cdot & \cdot & \sigma_m & v_1(x) \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \sigma_m & \cdot & \cdot & \cdot & \sigma_{2m-1} & v_{m+1}(x) \end{vmatrix} = 0. \quad (5.19)$$

Thus, for all $x, v(x) \in H$. With application of Taylor, this is equivalent to columns of matrix V in hyperplane H ,

$$\begin{aligned}
 l_{m+1} &= \lambda_1 l_1 + \cdots + \lambda_m l_m, \\
 \begin{vmatrix} \sigma_0 & \cdot & \cdot & \cdot & \sigma_{m-1} & 1 \\ \sigma_1 & \cdot & \cdot & \cdot & \sigma_m & x \\ \cdot & & & & \cdot & \\ \cdot & & & & \cdot & \\ \sigma_m & \cdot & \cdot & \cdot & \sigma_{2m-1} & x^m \end{vmatrix} &= \begin{vmatrix} \sigma_0 & \cdot & \cdot & \cdot & \sigma_{m-1} & 1 \\ \sigma_1 & \cdot & \cdot & \cdot & \sigma_m & x \\ \cdot & & & & \cdot & \\ \cdot & & & & \cdot & \\ \sigma_{m-1} & \cdot & \cdot & \cdot & \sigma_{2m-2} & x^{m-1} \\ 0 & \cdot & \cdot & \cdot & 0 & x^m - \lambda_1 - \lambda_2 x - \cdots - \lambda_m x^{m-1} \end{vmatrix} \\
 &= \Delta^2 \left[x^m - \lambda_1 - \lambda_2 x - \cdots - \lambda_m x^{m-1} \right].
 \end{aligned} \tag{5.20}$$

The polynomial D_0 is partner to

$$\begin{vmatrix} \sigma_0 & \cdot & \cdot & \cdot & \sigma_{m-1} & 1 \\ \sigma_1 & \cdot & \cdot & \cdot & \sigma_m & x \\ \cdot & & & & \cdot & \\ \cdot & & & & \cdot & \\ \sigma_m & \cdot & \cdot & \cdot & \sigma_{2m-1} & x^m \end{vmatrix}. \tag{5.21}$$

Thus

$$D_0(x) = x^m - \lambda_1 - \lambda_2 x - \cdots - \lambda_m x^{m-1}. \tag{5.22}$$

For $k = 1, \dots, m+1$, we put

$$v_k(x) = \rho_k(x) + x^{d^0 B - 2} w_k, \tag{5.23}$$

where $d^0 \rho_k(x) \leq d^0 B - 3$.

The matrix W partner to $\begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$ is triangular where k th entry diagonal equal $(k-1-m)(2a_{d^0 B-1} + m + k - 2)$. The coefficient is nonzero except possibly for a single value of k . Thus, rank of $\begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$ is m or $m-1$.

For $\text{rank}(V) = m$, (5.1) as solution is equivalent to $l_1, \dots, l_m$ which is linearly independent.

Hence, we assume $\text{rank}(V) = m-1$. In that case l_1 and l_2 are linear combinations of $l_3, \dots, l_{m+1}$. One of both combinations has to express l_{m+1} . Thus we have two situations.

Situation 1

l_2 linear combination of $l_3, \dots, l_{m+1}$ where coefficient of l_{m+1} is nonzero. By replacement of l_{m+1} , we obtain l_1 a linear combination of $l_2, \dots, l_m$.

Thus, w_1 is linear combination of $w_2, \dots, w_m$. ($w_2, \dots, w_m$) libre system where $d^0 w_k = k - 1$ for $0 \leq k = 2, \dots, m$. Thus: $w_1 = 0$.

If

$$l_1 = \sum_{j=2}^m \alpha_j l_j, \quad (5.24)$$

where α_j is constant, then

$$w_1 = \sum_{j=2}^m \alpha_j w_j. \quad (5.25)$$

As $w_1 = 0$, thus for all $j = 2, \dots, m$, $\alpha_j = 0$ equivalent to $l_1 = 0$. Thus $R = 0$.

First Way of Having D_0 :

$$\frac{D_0''}{D_0'} = \frac{-2A}{B} \quad (5.26)$$

$-2A/B$ has to be the sum of simple elements of the shape $\mu_c/(x - c)$ where μ_c is greater or equal to 1.

Thus $D_4 = D_2 = D_1 = 1$ and $D_3 = B$.

For all c root of D_3 , $-2\alpha_c = \mu_c + 1$ where $\alpha_c^2 = \lim_{x \rightarrow c} (x - c)^2 r + (1/4)$. Thus

$$\alpha_c < 0, \quad 4 \left[\lim_{x \rightarrow c} (x - c)^2 r + \frac{1}{4} \right] = (\mu_c + 1)^2. \quad (5.27)$$

Thus

$$D_0' = m \prod_{c \in \text{Root}(D_3)} (x - c)^{\mu_c}. \quad (5.28)$$

D_0 is primitive nonzero in roots of D_3 .

Second Way of Having D_0 :

$$l_{m+1} = \sum_{j=2}^m \lambda_j l_j, \quad (5.29)$$

where λ_j is constant ($j = 2, \dots, m$). The relations which are linearly dependent between l_{m+1} and $l_1, \dots, l_m$ are

$$l_{m+1} = \lambda l_1 + \sum_{j=2}^m \lambda_j l_j. \quad (5.30)$$

Thus we have an infinity of solutions:

$$D_0 = x^m - \lambda - \sum_{j=2}^m \lambda_j x^{j-1}, \quad (5.31)$$

where λ is arbitrarily constant.

Situation 2

l_1 is linear combination of $l_3, \dots, l_{m+1}$ where coefficient of l_{m+1} is nonzero and l_2 is linear combination of $l_3, \dots, l_m$. Thus, w_2 is linear combination of $w_3, \dots, w_m$. Thus, $w_2 = 0$. $w_1, w_3, \dots, w_m$ is libre system.

If

$$l_2 = \sum_{j=3}^m \alpha_j l_j, \quad (5.32)$$

then

$$w_2 = \sum_{j=3}^m \alpha_j w_j. \quad (5.33)$$

As $w_2 = 0$. Thus, for all $j = 3, \dots, m$, $\alpha_j = 0$ equivalent to $l_2 = 0$ and $2A - xR = 0$.

First Way of Having D_0 :

$$\begin{aligned} \frac{2A}{x} D_0 &= 2AD'_0 + BD''_0, \\ 2A(D_0 - xD'_0) &= xBD''_0. \end{aligned} \quad (5.34)$$

We put $Q = D_0 - xD'_0$. $Q'/Q = -2A/B$,

$$D_4 = D_2 = D_1 = 1, \quad Q = (1 - m) \prod_{c \in \text{Root}(D_3)} (x - c)^{\mu_c}. \quad (5.35)$$

Thus $D_0 = Cx$ with $C' = -Q/x^2$; C rational function.

The coefficient of x in Q is zero. $Q'(0) = 0$ ($A(0) = 0$; $2A = xR$).

Second Way of Having D_0

$$l_{m+1} = \lambda_1 l_1 + \sum_{j=3}^m \lambda_j l_j, \quad (5.36)$$

thus

$$D_0 = x^m - \sum_{j=1}^m \lambda_j x^{j-1}. \quad (5.37)$$

λ_2 is arbitrarily constant.

5.2. Case $d^o r = 2\nu \geq 0$

We put $E = E(\theta)$ and $S = E + (A/B)$. We have

$$\begin{aligned} d^o(BD_0'') &\leq d^o B + m - 2, \\ d^o(EB + A)D_0' &= d^o B + m - 1 + \nu. \end{aligned} \quad (5.38)$$

Thus

$$d^o(RD_0) = d^o B + m - 1 + \nu. \quad (5.39)$$

Thus, $d^o R = d^o B + \nu - 1$ and $r_{d^o B + \nu - 1} = 2\alpha m$ where α is dominant coefficient of E .

We put

$$v(x) = -R \begin{pmatrix} 1 \\ x \\ x^2 \\ \cdot \\ \cdot \\ x^m \end{pmatrix} + 2(EB + A) \begin{pmatrix} 0 \\ 1 \\ 2x \\ \cdot \\ \cdot \\ mx^{m-1} \end{pmatrix} + B \begin{pmatrix} 0 \\ 0 \\ 2 \\ \cdot \\ \cdot \\ m(m-1)x^{m-2} \end{pmatrix} = \begin{pmatrix} v_1(x) \\ v_2(x) \\ v_3(x) \\ \cdot \\ \cdot \\ v_{m+1}(x) \end{pmatrix}. \quad (5.40)$$

The k th element of $v(x)$ is

$$v_k(x) = B(k-1)(k-2)x^{k-3} + 2(EB + A)(k-1)x^{k-2} - Rx^{k-1}. \quad (5.41)$$

$v_k(x)$ is of degree equal to or less than $d^o B + \nu + k - 2$ where coefficient of $x^{d^o B + \nu + k - 2}$ equal $2\alpha(k - 1 - m)$. For $k = 1, \dots, m$, the k th element of $v(x)$ is of degree: $d^o R + k - 1$. The $(m + 1)$ th element of $v(x)$ is of degree equal to or less than $d^o R + m - 1$.

Proposition 5.2. One puts $v_k(x) = \rho_k(x) + x^{d^o R} w_k(x)$ where $d^o \rho_k(x) < d^o R$.

(1)

$$\begin{aligned} d^o w_k(x) &= k - 1 \quad \text{for } k = 1, \dots, m, \\ d^o w_{m+1}(x) &\leq m - 1, \end{aligned} \tag{5.42}$$

(2) $(w_1, \dots, w_m)$ libre system and

$$w_{m+1}(x) = \lambda_1 w_1(x) + \dots + \lambda_m w_m(x), \tag{5.43}$$

where $\lambda_1, \dots, \lambda_m$ constants.

(3)

$$D_0 = x^m - \lambda_1 - \lambda_2 x - \dots - \lambda_{m-1} x^{m-1} \tag{5.44}$$

is solution if and only if

$$l_{m+1} = \lambda_1 l_1 + \dots + \lambda_m l_m. \tag{5.45}$$

Example 5.3. In this example we consider the Riccati differential equation (1.2) where

$$r = \frac{1}{(x+1)^4} - \frac{5}{(x+1)^3} + \frac{7}{4(x+1)^2} + \frac{1}{x+1} + x^2 + 2, \tag{5.46}$$

$D_1 = D_2 = D_3 = 1$, $D_4 = (x+1)^2$, $d^o(r) = 2$, $\nu = 1$. We have

$$\begin{aligned} N_1 &= 1 - 5(x+1), \\ F &= x^2 + 2 + \frac{1 - 5(x+1)}{(x+1)^4} + \frac{1}{4(x+1)^2}. \end{aligned} \tag{5.47}$$

Study at (-1)

Laurent series development at -1 :

$$(F)^{1/2} = \varepsilon_{-1} \left(\frac{1}{(x+1)^2} - \frac{5}{2(x+1)} + \dots \right); \quad \varepsilon_{-1} = \pm 1. \tag{5.48}$$

Study at Infinity

We have $E(r)/x^2 = 1 + t^2$ where $t = 1/x$,

$$\left(1 + 2t^2\right)^{1/2} = 1 + t^2 + o(t^2). \quad (5.49)$$

Thus $s_{\nu+1} = 1$ and $E(\theta) = \alpha x$, where $\alpha^2 = 1$,

$$\begin{aligned} \theta &= \alpha x + \varepsilon_{-1} \left(\frac{1}{(x+1)^2} - \frac{5}{2(x+1)} \right) + \frac{1}{(x+1)} + \frac{D'_0}{D_0}, \\ d^o D_0 &= \alpha s_{\nu+1} - \frac{\nu}{2} + \frac{5\varepsilon_{-1}}{2} - 1 = \alpha s_{\nu+1} + \frac{5\varepsilon_{-1}}{2} - \frac{3}{2}. \end{aligned} \quad (5.50)$$

Case $\left\{ \begin{smallmatrix} \alpha=-1 \\ \varepsilon_{-1}=-1 \end{smallmatrix} \right\}$ and $\left\{ \begin{smallmatrix} \alpha=1 \\ \varepsilon_{-1}=-1 \end{smallmatrix} \right\}$ are to be rejected because we obtain negative values of D_0 .

(i) If $\alpha = 1$ and $\varepsilon_{-1} = 1$ then: $d^o D_0 = 2$.

(ii) If $\alpha = -1$ and $\varepsilon_{-1} = 1$ then: $d^o D_0 = 0$.

Case 1 ($\alpha = 1$ and $\varepsilon_{-1} = 1$). We have the following

$$\theta = x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)} + \frac{D'_0}{D_0}, \quad (5.51)$$

where $d^o D_0 = 2$. Research of coefficients of D_0 ,

$$S = x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)} = \frac{x^3 + 2x^2 - (1/2)x - (1/2)}{(x+1)^2} \quad (5.52)$$

$r - S' - S^2 = R/B$, where $R = 4x^2 + 4x$ and $B = (1+x)^2$

$$\begin{aligned} v(x) &= -R \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} + 2SB \begin{pmatrix} 0 \\ 1 \\ 2x \end{pmatrix} + B \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -4x - 4x^2 \\ -1 - x - 2x^3 \\ 2 + 2x + 4x^3 \end{pmatrix}, \\ V &= \begin{pmatrix} 0 & -4 & -4 & 0 \\ -1 & -1 & 0 & -2 \\ 2 & 2 & 0 & 4 \end{pmatrix}, \end{aligned} \quad (5.53)$$

$l_3 = -2l_2$. Thus $D_0 = x^2 + 2x$,

$$\theta = x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)} + \frac{2(x+1)}{x^2 + 2x} = x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)} + \frac{1}{x} + \frac{1}{x+2}. \quad (5.54)$$

Case 2 ($\alpha = -1$ and $\varepsilon_{-1} = 1$). We obtain the rational fraction

$$\theta = -x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)}. \quad (5.55)$$

It is not a solution because the sum of this fraction with the solution already found is not the logarithmic derivative of a rational fraction.

Example 5.4. In this example we consider the Riccati differential equation (1.2) where

$$\begin{aligned} r = & \frac{1}{16} + \frac{1}{(x-1)^8} - \frac{4}{(x-1)^5} - \frac{29}{6(x-1)^4} - \frac{8}{9(x-1)^3} \\ & - \frac{64}{27(x-1)^2} - \frac{152}{18(x-1)} + \frac{30}{(x+2)^2} - \frac{10}{81(x+2)}, \end{aligned} \quad (5.56)$$

$D_1 = D_2 = 1$, $D_3 = x+2$ and $D_4 = (x-1)^4$. We have

$$\begin{aligned} N_1 &= (2418x - 2454)(x-1)^3 + (x+2)^2, \quad \delta = -\frac{1}{4(x+2)^2}, \\ F &= \frac{1}{16} + \frac{(2418x - 2454)(x-1)^3 + (x+2)^2}{(x-1)^8(x+2)^2} + \frac{1}{4(x+2)^2}. \end{aligned} \quad (5.57)$$

Laurent series development at 1:

$$(F)^{1/2} = \varepsilon_1 \left(\frac{1}{(x-1)^4} - \frac{2}{x-1} + \cdots \right); \quad \varepsilon_1 = \pm 1. \quad (5.58)$$

Laurent series development at -2:

$$(F)^{1/2} = \varepsilon_{-2} \left(\frac{11}{2(x+2)} + \cdots \right); \quad \varepsilon_{-2} = \pm 1. \quad (5.59)$$

We have

$$\frac{1}{2} \frac{(D_2 D_3 D_4)'}{D_2 D_3 D_4} = \frac{1}{2(x+2)} + \frac{2}{x-1}. \quad (5.60)$$

Thus

$$\theta = E(\theta) + \varepsilon_1 \left[\frac{1}{(x-1)^4} - \frac{2}{x-1} \right] + \varepsilon_{-2} \left[\frac{11}{2(x+2)} \right] + \frac{1}{2(x+2)} + \frac{2}{x-1} + \frac{D'_0}{D_0}, \quad (5.61)$$

where

$$\begin{aligned} E^2(\theta) &= E(r) = \frac{1}{16}, \\ d^0 D_0 &= -\frac{1}{E(\theta)} + 2\varepsilon_1 - \frac{11\varepsilon_{-2}}{2} - \frac{5}{2}. \end{aligned} \quad (5.62)$$

Cases $\left\{ \begin{smallmatrix} \varepsilon_1=1 \\ \varepsilon_{-2}=1 \end{smallmatrix} \right\}$ and $\left\{ \begin{smallmatrix} \varepsilon_1=-1 \\ \varepsilon_{-2}=1 \end{smallmatrix} \right\}$ are to be rejected because we obtain negative values of D_0 .

Case 1. If $\varepsilon_1 = 1$, $\varepsilon_{-2} = -1$, $E(\theta) = 1/4$ then $d^0 D_0 = 1$,

$$\theta = \frac{1}{4} + \frac{1}{(x-1)^4} - \frac{5}{x+2} + \frac{D'_0}{D_0}. \quad (5.63)$$

Research of coefficients of D_0 ,

$$S = \frac{1}{4} + \frac{1}{(x-1)^4} - \frac{5}{x+2}, \quad (5.64)$$

$r - (S' + S^2) = R/B$, where $R = (1/2)x^4 - 10x^3 + 19x^2 - 18x + (5/2)$, $B = (x-1)^4(x+2)$,

$$\begin{aligned} v(x) &= -R \begin{pmatrix} 1 \\ x \end{pmatrix} + 2SB \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x^4 + 10x^3 - 19x^2 + 18x - \frac{5}{2} \\ -x^4 + 20x^3 - 38x^2 + 36x - 1 \end{pmatrix}, \\ V &= \begin{pmatrix} -\frac{5}{2} & 18 & -19 & 10 & -\frac{1}{2} \\ -5 & 36 & -38 & 20 & -5 \end{pmatrix}, \end{aligned} \quad (5.65)$$

$l_2 = 2l_1$. Thus, $D_0 = x - 2$,

$$\theta_1 = \frac{1}{4} + \frac{1}{(x-1)^4} - \frac{5}{x+2} + \frac{1}{x-2}. \quad (5.66)$$

Case 2. If $\varepsilon_1 = 1$, $\varepsilon_{-2} = -1$, $E(\theta) = -1/4$ then: $d^0 D_0 = 9$,

$$\theta = \frac{-1}{4} + \frac{1}{(x-1)^4} - \frac{5}{x+2} + \frac{D'_0}{D_0}. \quad (5.67)$$

It is not a solution because the sum of this fraction with the solution already found is not the logarithmic derivative of a rational fraction.

Case 3. If $\varepsilon_1 = -1$, $\varepsilon_{-2} = -1$, $E(\theta) = -1/4$ then $d^0 D_0 = 5$,

$$\theta_3 = \frac{-1}{4} - \frac{1}{(x-1)^4} - \frac{5}{x+2} + \frac{4}{x-1} + \frac{D'_0}{D_0}. \quad (5.68)$$

Research of coefficients of D_0 ,

$$S = \frac{-1}{4} - \frac{1}{(x-1)^4} - \frac{5}{x+2} + \frac{4}{x-1},$$

$$R = -x^4 + 36x^3 - 137x^2 + 168x - 88, \quad B = (x-1)^4(x+2),$$

$$V = \begin{pmatrix} -37 & -168 & 137 & -36 & \frac{5}{2} & 0 & 0 & 0 & 0 \\ -32 & \frac{-151}{2} & -266 & 180 & -37 & 2 & 0 & 0 & 0 \\ 4 & -78 & -162 & -368 & 219 & -36 & \frac{3}{2} & 0 & 0 \\ 0 & 12 & -138 & \frac{-209}{2} & -474 & 254 & -33 & 1 & 0 \\ 0 & 0 & 24 & -212 & -95 & -584 & 285 & -28 & \frac{1}{2} \\ 0 & 0 & 0 & 40 & -300 & \frac{-139}{2} & -698 & 312 & -21 \end{pmatrix}. \quad (5.69)$$

We consider V_p the minor 6×6 obtained by column vectors 1, 2, 6, 7, 8, and 9. In $\mathbb{Z}/5\mathbb{Z}$

$$V_p = \begin{pmatrix} -2 & 2 & 0 & 0 & 0 & 0 \\ -2 & 2 & 2 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 & 2 & -1 \end{pmatrix} \quad (5.70)$$

$\det V_p = 1$. Accordingly, row 6 is not linear combination of other row and So no solution D_0 .

References

- [1] F. Ulmer, "On liouvillian solutions of linear differential equations," *Applicable Algebra in Engineering, Communication and Computing*, vol. 2, no. 3, pp. 171–193, 1992.
- [2] J. J. Kovacic, "An algorithm for solving second order linear homogeneous differential equations," *Journal of Symbolic Computation*, vol. 2, no. 1, pp. 3–43, 1986.
- [3] M. Bronstein, "On solutions of linear ordinary difference equations in their coefficient field," *Journal of Symbolic Computation*, vol. 29, no. 6, pp. 841–877, 2000.
- [4] M. Van Hoeij and J.-A. Weil, "An algorithm for computing invariants of differential Galois groups," *Journal of Pure and Applied Algebra*, vol. 117–118, pp. 353–379, 1997.
- [5] M. Van der Put and F. Ulmer, "Differential equations and finite groups," *Journal of Algebra*, vol. 226, no. 2, pp. 920–966, 2000.
- [6] I. Kaplansky, *An Introduction to Differential Algebra*, Hermann, Paris, France, 1957.
- [7] M. F. Singer and F. Ulmer, "Galois groups of second and third order linear differential equations," *Journal of Symbolic Computation*, vol. 16, no. 1, pp. 9–36, 1993.
- [8] M. Van der Put and M. F. Singer, *Galois Theory of Linear Differential Equations*, Grundlehren der mathematischen Wissenschaften, Springer, Berlin, Germany, 2003.