Research Article

L^1 -Algebra of a Locally Compact Groupoid

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For a locally compact groupoid G with a fixed Haar system λ and quasi-invariant measure μ , we introduce the notion of λ -measurability and construct the space $L^1(G, \lambda, \mu)$ of absolutely integrable functions on G and show that it is a Banach $*$ -algebra and a two-sided ideal in the algebra $M(G)$ of complex Radon measures on G . We find correspondences between representations of G on Hilbert bundles and certain class of nondegenerate representations of $L^1(G, \lambda, \mu)$.

1. Introduction and Preliminaries

For a locally compact group G with a Haar measure λ , the Banach algebra $L^1(G, \lambda)$ plays a central role in harmonic analysis on G [1]. This motivated us to define a similar notion in the case where G is a locally compact groupoid with a (fixed) Haar system λ and quasi-invariant measure μ . This paper is devoted to the study of such a groupoid L^1 -algebra $L^1(G, \lambda, \mu)$. One may expect that as the group case, there is a full interaction between the properties of G and that of $L^1(G, \lambda, \mu)$. This is not completely true. For instance, unlike the group case, not every nondegenerate representation of $L^1(G, \lambda, \mu)$ is integrated form a representation of G . In Section 2, we introduce the appropriate measurability notion used to define $L^1(G, \lambda, \mu)$. Sections 3 and 4 are devoted to the algebra structure of $L^1(G, \lambda, \mu)$ and its embedding into $M(G)$ as a closed ideal. In Section 5, we find the class of nondegenerate representations of $L^1(G, \lambda, \mu)$ which could be obtained by integrating a representation of G .

We start with some basic definitions. Our main reference for groupoids is the Renault's book [2]. In this paper, we frequently use the following version of Fubini's theorem for (not necessarily σ -finite) Radon measures [1, Theorem B.3.3].

Lemma 1.1. *Let ν and ρ be Radon measures on the Borel sets of the locally compact spaces X and Y , respectively. Then there exists a unique Radon measure $\nu \cdot \rho$ on $X \times Y$ such that*

(i) *if $f : X \times Y \rightarrow \mathbb{C}$ is $\nu \cdot \rho$ -integrable, then the partial integrals*

$$\int_X f(x, y) d\nu(x), \quad \int_Y f(x, y) d\rho(y) \quad (1.1)$$

define integrable functions such that Fubini's formula holds

$$\int_{X \times Y} f d\nu \cdot \rho = \int_X \int_Y f d\nu d\rho = \int_Y \int_X f d\rho d\nu, \quad (1.2)$$

(ii) *if f is measurable such that $A = \{(x, y) \in X \times Y : f(x, y) \neq 0\}$ is σ -finite and if one of the iterated integrals $\int_X \int_Y |f| d\nu d\rho$ or $\int_Y \int_X |f| d\rho d\nu$ is finite, then f is integrable and the Fubini formula holds.*

A groupoid is a set G endowed with a subset G^2 of $G \times G$, called the set of composable pairs, a product map: $G^2 \rightarrow G$; $(x, y) \mapsto xy$, and an inverse map: $G \rightarrow G$; $x \mapsto x^{-1}$, such that for each $x, y, z \in G$,

- (i) $(x^{-1})^{-1} = x$,
- (ii) if $(x, y), (y, z) \in G^2$ then $(xy, z), (x, yz) \in G^2$ and $(xy)z = x(yz)$,
- (iii) $(x^{-1}, x) \in G^2$ and if $(x, y) \in G^2$ then $x^{-1}(xy) = y$,
- (iv) $(x, x^{-1}) \in G^2$ and if $(z, x) \in G^2$ then $(zx)x^{-1} = z$.

If $x \in G$, $s(x) = x^{-1}x$ is called the source of x and $r(x) = xx^{-1}$ is called the range of x . The pair (x, y) is composable if and only if $s(x) = r(y)$. The set $G^0 = s(G) = r(G)$ is the unit space of G , and its elements are called units in the sense that $xs(x) = x$ and $r(x)x = x$.

For $u, v \in G^0$, $G^u = r^{-1}\{u\}$, $G_v = s^{-1}\{v\}$, $G_v^u = G^u \cap G_v$, and $G\{u\} = G_u^u$. The latter is called the isotropy group at u . We define $u \sim v$ if $G_v^u \neq \emptyset$. It is checked that $\sim$ is an equivalence relation on the unit space G^0 . The equivalence class of u is denoted by $[u]$ and is called the orbit of u .

A topological groupoid consists of a groupoid G and a topology compatible with the groupoid structure, such that the composition map is continuous on G^2 in the induced product topology, and the inversion map is continuous on G . A locally compact groupoid is a topological groupoid G which satisfies the following conditions:

- (i) G^0 is locally compact and Hausdorff in the relative topology.
- (ii) There is a countable family C of compact Hausdorff subsets of G whose interiors form a basis for the topology of G .
- (iii) Every G^u is locally compact Hausdorff in the relative topology.

A locally compact groupoid is r -discrete if its unit space is an open subset. Let G be a locally compact groupoid. The support of a function $f : G \rightarrow \mathbb{C}$ is the complement of the union of all open, Hausdorff subsets of G on which f vanishes. The space $C_c(G)$ consists of all continuous functions on G with compact support. A left Haar system for G consists of

measures $\{\lambda^u, u \in G^0\}$ on G such that

- (i) the support of λ^u is G^u ,
- (ii) (continuity) for each $f \in C_c(G)$, $u \mapsto \lambda(f)(u) = \int f d\lambda^u$ is continuous,
- (iii) (left invariance) for any $x \in G$ and any $f \in C_c(G)$,

$$\int f(xy) d\lambda^{s(x)}(y) = \int f(y) d\lambda^{r(x)}(y). \quad (1.3)$$

Let μ be a measure on G^0 . The measure on G induced by μ is $\nu = \int \lambda^u d\mu$, defined by $\int_G f d\nu = \int_{G^0} d\mu(u) \int_G f d\lambda^u$, for $f \in C_c(G)$. The measure on G^2 induced by μ is $\nu^2 = \int \lambda_u \times \lambda^u d\mu(u)$. The inversion of ν is $\nu^{-1} = \int \lambda_u d\mu(u)$. Note that the measures ν, ν^2, ν^{-1} are Radon. A measure μ on G^0 is said to be quasi invariant if its induced measure ν is equivalent to its inverse ν^{-1} . Let μ be a quasi-invariant measure on G^0 . The Radon-Nikodym derivative $D = d\nu/d\nu^{-1}$ is called the modular function of μ . An equivalent definition of modular function on G is given in [3, Definition 2.3], defining the modular function Δ as a strictly positive continuous homomorphism on G such that $\Delta|_{G_u^u}$ is modular function for G_u^u .

A subset A of G is called a G -set if the restrictions of r and s to it are one to one. Equivalently, A is a G -set if and only if AA^{-1} and $A^{-1}A$ are contained in G^0 .

We introduce some notations from [4, 5] which is related to the representations of $L^1(G, \lambda, \mu)$. Let $\{\lambda^u\}_u$ be a fixed Haar system on G . Let μ be a quasi-invariant measure, Δ its modular function, ν be the measure induced by μ on G , and $\nu_0 = \Delta^{-1/2}\nu$. Let

$$II_\mu(G, \nu, \mu) = \left\{ f \in L^1(G, \nu_0) : \|f\|_{II, \mu} < \infty \right\}, \quad (1.4)$$

where $\|f\|_{II, \mu}$ is defined by

$$\|f\|_{II, \mu} = \sup \left\{ \int |f(x)j(s(x))k(r(x))| d\nu_0(x); \int |j|^2 d\mu = \int |k|^2 d\mu = 1 \right\}. \quad (1.5)$$

If μ_1 and μ_2 are two equivalent quasi-invariant measures, then

$$\|f\|_{II, \mu_1} = \|f\|_{II, \mu_2}, \quad (1.6)$$

because $\|f\|_{II, \mu} = \|II_\mu(|f|)\|$ for each quasi-invariant measure μ , where II_μ is the one-dimensional trivial representation on μ . Now define

$$\|f\|_{II} = \sup_{\mu} \|f\|_{II, \mu}. \quad (1.7)$$

The supremum is taken over the class of quasi-invariant measures.

Let

$$I(G, \nu, \mu) = \left\{ f \in L^1(G, \nu_0) : \|f\|_{I, \mu} < \infty \right\}, \quad (1.8)$$

where

$$\|f\|_{I,\mu} = \max \left\{ \left\| u \rightarrow \int |f| d\lambda^u \right\|_\infty, \left\| u \rightarrow \int |f^*| d\lambda^u \right\|_\infty \right\}. \quad (1.9)$$

Under the canonical convolution and involution, $I(G, \nu, \mu)$ becomes a Banach $*$ -algebra [5, page 4]. Here $f^*(x) = \overline{f(x^{-1})}$. If we consider

$$\|f\|_{I,r} = \sup_u \left(\int |f| d\lambda^u \right), \quad \|f\|_{I,s} = \sup_u \left(\int |f| d\lambda_u \right), \quad (1.10)$$

and put $\|f\|_I = \max(\|f\|_{I,r}, \|f\|_{I,s})$, then $\|f\|_I = \|f\|_{I,\mu}$.

2. λ -Measurability

For the rest of the paper, G is a locally compact, Hausdorff, second countable groupoid which admits a left Haar system $\lambda = \{\lambda^u\}$.

Definition 2.1. A system of measures $\{\lambda^u\}_{u \in G^0}$ is said to be complete if for each $u \in G^0$, λ^u is complete on its σ -algebra $\mathfrak{M}_{\lambda^u}$. A Borel measurable set $E \subseteq G$ is called λ -measurable if for each $u \in G^0$, $E \cap G^u \in \mathfrak{M}_{\lambda^u}$. A function $f : G \rightarrow \mathbb{C}$ is λ -measurable if for every $u \in G^0$ and every open set $O \subseteq \mathbb{C}$, $f^{-1}(O) \cap G^u \in \mathfrak{M}_{\lambda^u}$.

Proposition 2.2. If $\bar{\lambda} = \{\bar{\lambda}^u\}$ is the completion of $\lambda = \{\lambda^u\}$ and $f : G \rightarrow [0, \infty]$ is $\bar{\lambda}$ -measurable, then there is a λ -measurable function g such that $f = g$ on G^u (λ^u -a.e.).

Proof. Since f is λ -measurable for each $u \in G^0$, f is $\bar{\lambda}^u$ -measurable on G^u , and since $\bar{\lambda}^u$ is the completion of λ^u , there exists a λ^u -measurable function g_u such that $f = g_u$ ($\bar{\lambda}^u$ -a.e.). Now define $g = g_u$ on G^u and zero, elsewhere. Since for every $u \in G^0$ and every open set $O \subseteq \mathbb{C}$, $g^{-1}(O) \cap G^u = g_u^{-1}(O) \in \mathfrak{M}_{\lambda^u}$, g is λ -measurable, and since $\lambda^u(G^u)^c = 0$, we have $f = g$ on G^u (λ^u -a.e). $\square$

From now on, we assume that the Haar system λ is complete.

Lemma 2.3. For each $f : G \rightarrow \mathbb{C}$, λ -measurability of f is equivalent to ν -measurability of f .

Proof. We have $\text{supp } \lambda^u = G^u$ and

$$f^{-1}(O) = (f^{-1}(O) \cap G^u) \cup (f^{-1}(O) \cap (G^u)^c). \quad (2.1)$$

Since λ^u is complete and $(G^u)^c$ is λ^u -null, $(f^{-1}(O) \cap (G^u)^c) \subseteq (G^u)^c$ is in $\mathfrak{M}_{\lambda^u}$. Thus, for each $u \in G^0$ and open set $O \subseteq \mathbb{C}$

$$f^{-1}(O) \in \mathfrak{M}_{\lambda^u} \iff f^{-1}(O) \cap G^u \in \mathfrak{M}_{\lambda^u}. \quad (2.2)$$

Now for $\nu = \int \lambda^u d\mu(u)$, we have $\mathfrak{M}_\nu = \bigcap_{u \in G^0} \mathfrak{M}_{\lambda^u}$, hence f is ν -measurable if and only if f is λ -measurable. $\square$

If $f : G \rightarrow \mathbb{C}$ is λ -measurable and $g : \mathbb{C} \rightarrow \mathbb{C}$ is continuous, then $g \circ f : G \rightarrow \mathbb{C}$ is λ -measurable. Also, if $f, g : G \rightarrow \mathbb{R}$ is λ -measurable, $\Phi : \mathbb{C} \rightarrow Y$ is continuous, and $h(x) = \Phi(f(x), g(x))$, then h is λ -measurable. If $f = u + iv$ then f is λ -measurable if and only if u, v are λ -measurable. If $f, g : G \rightarrow \mathbb{R}$ are λ -measurable, so are $f + g$ and fg . Also, if $\{f_j\}_1^\infty$ is a sequence of $\overline{\mathbb{R}}$ -valued λ -measurable functions, then the functions $g_1(x) = \sup_j f_j(x)$, $g_2(x) = \inf_j f_j(x)$, $g_3(x) = \limsup_j f_j(x)$, and $g_4(x) = \liminf_j f_j(x)$ are all λ -measurable. If $f(x) = \lim_j f_j(x)$ exists for every $x \in G$, then f is λ -measurable. Thus if $\{f_j\}_1^\infty$ is a sequence of complex-valued λ -measurable functions and $f_j \rightarrow f$ λ^u -a.e, then f is λ -measurable.

3. The Algebra $L^1(G, \lambda, \mu)$

In this section, we define the space of integrable functions on G with respect to a fixed Haar system λ and quasi-invariant measure μ and show that it is a Banach $*$ -algebra under canonical convolution and involution.

Definition 3.1. Suppose μ is a quasi-invariant probability measure on G^0 , and ν is Radon measure induced by μ , then we define

$$L^1(G, \nu) = L^1(G, \lambda, \mu) = \left\{ f : G \rightarrow \mathbb{C} : f \text{ is } \lambda\text{-measurable, } \|f\|_1 = \int_G |f(x)| d\nu(x) < \infty \right\}. \quad (3.1)$$

It is clear that $L^1(G, \lambda, \mu)$ is a Banach space. We show that the Banach space $L^1(G, \lambda, \mu)$ is a Banach $*$ -algebra under the following convolution product.

$$(f * g)(x) = \int_{G^r(x)} f(y) g(y^{-1}x) d\nu(y). \quad (3.2)$$

Lemma 3.2. If $f, g \in L^1(G, \lambda, \mu)$, then $f * g \in L^1(G, \lambda, \mu)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Proof. For each f, g ,

$$\begin{aligned} \|f * g\|_1 &= \int_G |f * g(x)| d\nu(x) \\ &\leq \int_G \int_{G^r(x)} |f(y)| |g(y^{-1}x)| d\nu(y) d\nu(x) \\ &\leq \iint_G |f(y)| |g(y^{-1}x)| d\nu(y) d\nu(x) \\ &= \iint_G |f(y)| |g(x)| d\nu(y) d\nu(yx) \\ &= \int_{G^0} \int_G |g(x)| \int_{G^0} \int_G |f(y)| d\lambda^u(y) d\mu(u) d\lambda^u(yx) d\mu(u) \end{aligned}$$

$$\begin{aligned}
&= \int_G |g(x)| \left(\int_G |f(y)| d\nu(y) \right) d\nu(x) \\
&= \|f\|_1 \|g\|_1.
\end{aligned} \tag{3.3}$$

The measurability of $f * g$ follows from λ -measurability of f, g . $\square$

Next, we define an involution on $L^1(G, \lambda, \mu)$. We say that the assertion $P(x)$ holds for λ -a.e. x if for $E = \{x : \neg P(x)\}$, $\mu\{u : \lambda_u(E) > 0\} = 0$. Clearly an assertion holds λ -almost everywhere if and only if it holds ν -almost everywhere.

Lemma 3.3. *Suppose $D_u : G^u \rightarrow \mathbb{R}^+$ with $D_u(x) = d\lambda^u(x)/d\lambda_u(x)$ ($x \in G$). Then $D = D_u$ on G^u (a.e.).*

Proof. Suppose $E \subseteq G^u$. We have

$$\nu(E) = \int_{G^0} \lambda^u(E) d\mu(u) = \int_{G^0} \int_E d\lambda^u(x) d\mu(u) = \int_{G^0} \int_E D_u(x) d\lambda_u(x) d\mu(u). \tag{3.4}$$

Also from $d\nu = D d\nu^{-1}$ we have

$$\nu(E) = \int_E D(x) d\nu^{-1}(x) = \int_E D(x) \int_{G^0} d\lambda_u(x) d\mu(u) = \int_{G^0} \int_E D(x) d\lambda_u(x) d\mu(u). \tag{3.5}$$

Thus

$$\int_{G^0} \int_E (D_u(x) - D(x)) d\lambda_u(x) d\mu(u) = 0. \tag{3.6}$$

Now, let $E^u = \{x \in G^u : D_u(x) > D(x)\}$. If $\lambda_u(E^u) > 0$ then

$$\int_{E^u} (D_u(x) - D(x)) d\lambda_u(x) > 0. \tag{3.7}$$

But $\int_{E^u} (D_u(x) - D(x)) d\lambda_u(x) = 0$ (μ -a.e.), hence $\lambda_u(E^u) = 0$ (μ -a.e.). Thus $\mu\{u : \lambda_u(E^u) > 0\} = 0$. Therefore, $D_u(x) \leq D(x)$ (λ -a.e.). A similar argument leads to $D_u(x) \geq D(x)$ (λ -a.e.). $\square$

Proposition 3.4. *The map $*$: $L^1(G, \lambda, \mu) \rightarrow L^1(G, \lambda, \mu)$; $f \mapsto f^*$, where $f^*(x) = \overline{f}(x^{-1})D(x^{-1})$, is an isometric involution on $L^1(G, \lambda, \mu)$.*

Note that from [5, page 9], we have

$$\|f\|_{L^1(G, \lambda, \mu)} = \|f^*\|_{L^1(G, \lambda, \mu)} = \|f\|_{L^1(G, \nu_0)} = \|f^*\|_{L^1(G, \nu_0)} \leq \|f\|_{II, \mu} = \|f^*\|_{II, \mu} \leq \|f\|_{I, \mu} = \|f^*\|_{I, \mu}. \tag{3.8}$$

Hence the space of $L^1(G, \lambda, \mu)$ is in general bigger than $I(G, \nu, \mu)$ and $II_\mu(G, \nu, \mu)$ with respect to I -norm and II -norm, indeed $I(G, \nu, \mu) \subseteq II_\mu(G) \subseteq L^1(G, \lambda, \mu)$.

According to [2, Lemma 1.4], I -norm topology is coarser than the inductive limit topology. Also [5, page 15] shows that $C_c(G)$ has a two-sided bounded approximate identity $(e_n)_{n=1}^\infty$ with respect to the inductive limit topology with the following properties:

- (i) $e_n(x) \geq 0$ for all $x \in G$,
- (ii) $|\int e_n(x) d\lambda^u(x) - 1| < 1/n$ for all $u \in K_n$, where $\cup_n K_n = G^0$ and K_n 's are compact,
- (iii) $e_n(x) = e_n(x^{-1})$ for all $x \in G$.

An argument in [5, page 15] shows that there is $M > 0$ such that $\|e_n\|_{II, \mu} \leq M$ for all n . Since $C_c(G)$ is dense in $L^1(G, \lambda, \mu)$, thus $L^1(G, \lambda, \mu)$ has a two-sided bounded approximate identity.

For each $f \in L^1(G, \lambda, \mu)$ define

$$L_x f(y) = f(x^{-1}y), \quad R_x f(y) = f(yx), \quad (3.9)$$

when the multiplications on the right hand sides are defined. It is easy to check that the maps L_x, R_x are homomorphisms.

Proposition 3.5. *Let I be a closed subspace of $L^1(G, \lambda, \mu)$. Then I is a left ideal if and only if it is closed under left translation, and I is a right ideal if and only if it is closed under right translation.*

Proof. Note that since $f * g = \int_{G^r(y)} f(y) L_y g dv(y)$,

$$\begin{aligned} L_x(f * g) &= \int_{G^r(y)} f(y) L_x L_y g dv(y) = \int_{G^r(y)} f(y) L_{xy} g dv(y) \\ &= \int_{G^r(y)} f(x^{-1}y) L_y g dv(y) = \int_{G^r(y)} L_x f(y) L_y g dv(y) = (L_x f) * g. \end{aligned} \quad (3.10)$$

Now suppose $(e_n)_n$ is a bounded approximate identity for $L^1(G, \lambda, \mu)$. For the first assertion, if $f \in L^1(G, \lambda, \mu)$ and $g \in I$ and I is a left ideal, then we have

$$L_x(e_n) * f = L_x(e_n * f) \longrightarrow L_x f. \quad (3.11)$$

Conversely, if I is closed under left translation and $f \in L^1(G, \lambda, \mu)$ and $g \in I$,

$$f * g = \int_{G^r(y)} f(y) L_y g dv(y) \quad (3.12)$$

is in the closed linear span of the functions $L_y g$ and hence in I . The other assertion is proved similarly. $\square$

4. $M(G)$ as Banach $*$ -Algebra

Let $M(G)$ be the space of complex Radon measures on G . If $\eta, \theta \in M(G)$, then the map $\psi \mapsto I(\psi)$ on $C_0(G)$ defined by $I(\psi) = \int_G \int_{G^r(y)} \psi(xy) d\eta(x) d\theta(y)$ is a linear functional on

$C_0(G)$ satisfying $|I(\varphi)| \leq \|\varphi\|_{\sup} \|\eta\| \|\theta\|$, so by Riesz representation theorem, it is given by a measure shown as $\eta * \theta$ called the convolution of η, θ with $\|\eta * \theta\| \leq \|\eta\| \|\theta\|$. If we define $\eta^*(E) = \overline{\eta(E^{-1})}$, then $\eta \mapsto \eta^*$ is an involution on $M(G)$, and $M(G)$ is a Banach $*$ -algebra. In this section, we show that the space $L^1(G, \lambda, \mu)$ is a closed two-sided ideal of $M(G)$.

Proposition 4.1. *The map $L^1(G, \lambda, \mu) \hookrightarrow M(G)$ with $\nu_f(E) = \int_E f \chi_E d\nu$, for $(E \subseteq G)$; $f \mapsto \nu_f$ is an isometric embedding.*

Proof. If $f \in L^1(G, \lambda, \mu)$, then f is λ -measurable so the integral exists, and it is easy to check that ν_f is a measure on G . We show that ν_f is Radon. If $f = u + iv$, then $\nu_f = \nu_u + i\nu_v$, so ν_f is Radon if and only if ν_u and ν_v are Radon. Since G is LCH and second countable, we have $\nu_u(K) = \int_K u d\nu \leq \int_G |u| d\nu = \|u\|_1 < \infty$, for each compact set K , thus ν_u is Radon. Similarly ν_v is Radon, and so is ν_f .

By definition, $\|\nu_f\| = \sup\{\sum_1^n |\nu_f(E_i)| : n \in \mathbb{N}, G = \bigsqcup_1^n E_i\}$, so for each $\epsilon > 0$, there exists a partition $\{E_i\}_1^n$ of G such that

$$\|\nu_f\| - \epsilon < \sum_1^n |\nu_f(E_i)| = \sum_1^n \left| \int_G f \chi_{E_i} d\nu \right| \leq \int_G \left(|f| \sum_1^n \chi_{E_i} \right) d\nu = \|f\|_1. \quad (4.1)$$

Thus, $\|\nu_f\| \leq \|f\|_1$. Conversely, suppose $f \geq 0$, then $\nu_f \geq 0$ and for every partition $\{E_i\}_1^n$ of G we have,

$$\|\nu_f\| \geq \sum_1^n \nu_f(E_i) = \sum_1^n \int_G f \chi_{E_i} d\nu = \int_G f d\nu = \|f\|_1. \quad (4.2)$$

If $f = u + iv = (f_1 - f_2) + i(f_3 - f_4)$, where $f_i \geq 0$ then $\|\nu_f\| = \|\nu_{f_1}\| + \|\nu_{f_2}\| + \|\nu_{f_3}\| + \|\nu_{f_4}\| \geq \|f_1\|_1 + \|f_2\|_1 + \|f_3\|_1 + \|f_4\|_1 \geq \|f\|_1$. Hence $\|\nu_f\| \geq \|f\|_1$ and equality holds. $\square$

Corollary 4.2. $L^1(G, \lambda, \mu)$ is a closed subspace of $M(G)$.

Lemma 4.3. *If $f, g \in L^1(G, \lambda, \mu)$, then $\nu_{(f*g)} = \nu_f * \nu_g$.*

Proof. For each compact set K , we have

$$\begin{aligned} \nu_f * \nu_g(K) &= \int_G \chi_K(x) d(\nu_f * \nu_g)(x) = \int_G \int_{G^{r(x)}} \chi_K(yx) d\nu_f(y) d\nu_g(x) \\ &= \int_G f(y) \int_{G^{s(y)}} \chi_K(yx) d\nu_g(x) d\nu(y) = \int_G \int_{G^{s(y)}} f(y) g(y^{-1}x) \chi_K(x) d\nu(y) d\nu(x) \\ &= \int_G \chi_K(x) \int_{G^{r(x)}} f(y) g(y^{-1}x) d\nu(y) d\nu(x) = \int_G (f * g)(x) \chi_K(x) d\nu(x) = \nu_{f*g}(K). \end{aligned} \quad (4.3)$$

Since ν_{f*g} and $\nu_f * \nu_g$ are regular measures, the equality holds for each open set and then for each measurable set. $\square$

If $f \in L^1(G, \lambda, \mu)$ and $\eta \in M(G)$, we will define $\eta * f$ such that $\nu_{\eta * f} = \eta * \nu_f$. Suppose $\varphi \in C_0(G)$, we put

$$\eta * f(\varphi) = \int_G \varphi(x) d\nu_{\eta * f}(x) = \int_G \varphi(x) (\eta * f)(x) d\nu(x). \quad (4.4)$$

On the other hand,

$$\begin{aligned} \eta * \nu_f(\varphi) &= \int_G \int_{G^r(x)} \varphi(yx) d\eta(y) d\nu_f(x) = \int_G \int_{G^r(x)} \varphi(yx) d\eta(y) f(x) d\nu(x) \\ &= \int_G \int_{G^s(y)} f(y^{-1}x) \varphi(x) d\eta(y) d\nu(y^{-1}x) = \int_G \varphi(x) \int_{G^s(y)} f(y^{-1}x) d\eta(y) d\nu(x). \end{aligned} \quad (4.5)$$

Comparing these equalities,

$$(\eta * f)(x) = \int_{G^r(x)} f(y^{-1}x) d\eta(y). \quad (4.6)$$

If $f \in L^1(G, \lambda, \mu)$, then it is easy to check that

$$\int_{G^u} R_y f(x) d\lambda^u(x) = \int_{G^u} f(x) d\lambda^u(xy^{-1}) = D(y^{-1}) \int_{G^u} f(x) d\lambda^u(x). \quad (4.7)$$

Thus

$$\int_G R_y f(x) d\nu(x) = D(y^{-1}) \int_G f(x) d\nu(x). \quad (4.8)$$

Similarly, we want to define $f * \eta$ in such a way that the equality $\nu_{(f * \eta)} = \nu_f * \eta$ holds. Again suppose $\varphi \in C_0(G)$. We have

$$\begin{aligned} \nu_{(f * \eta)}(\varphi) &= \int_G \varphi(x) d\nu_{f * \eta}(x) = \int_G \varphi(x) (f * \eta)(x) d\nu(x) = \int_{G^0} \int_G \varphi(x) (f * \eta)(x) d\lambda^u(x) d\mu(u) \\ &= \int_{G^0} \int_{G^u} \varphi(x) (f * \eta)(x) d\lambda^u(x) d\mu(u) = \int_{G^u} \varphi(x) (f * \eta)(x) d\nu(x). \end{aligned} \quad (4.9)$$

On the other hand,

$$\begin{aligned} (\nu_f * \eta)(\varphi) &= \int_G \int_{G^r(y)} \varphi(xy) d\nu_f(x) d\eta(y) \\ &= \int_G \int_{G^r(y)} \varphi(xy) f(x) d\nu(x) d\eta(y) \end{aligned}$$

$$\begin{aligned}
&= \int_G \int_{G^r(y)} \varphi(x) f(xy^{-1}) d\nu(xy^{-1}) d\eta(y) \\
&= \int_G \int_{G^r(y)} \varphi(x) f(xy^{-1}) D(y^{-1}) d\nu(x) d\eta(y) \\
&= \int_{G^s(x)} \varphi(x) \int_G f(xy^{-1}) D(y^{-1}) d\eta(y) d\nu(x).
\end{aligned} \tag{4.10}$$

Comparing the above equalities, we have

$$(f * \eta)(x) = \int_G f(xy^{-1}) D(y^{-1}) d\eta(y). \tag{4.11}$$

Lemma 4.4. $L^1(G, \lambda, \mu)$ is a two-sided ideal of $M(G)$.

Proof. Suppose $f \in L^1(G, \lambda, \mu)$ and $\eta \in M(G)$. Then we have

$$\begin{aligned}
\|\eta * f\|_1 &= \int_G |\eta * f(x)| d\nu(x) \leq \int_G \int_{G^r(x)} |f(y^{-1}x)| |d\eta|(y) d\nu(x) \\
&= \int_G \int_{G^r(x)} |f(x)| |d\eta|(y) d\nu(yx) = \int_G |f(x)| \int_{G^r(x)} |d\eta|(y) d\nu(x) \leq \|\eta\| \|f\|_1 < \infty.
\end{aligned} \tag{4.12}$$

Thus $\eta * f \in L^1(G, \lambda, \mu)$. Also

$$\begin{aligned}
\|f * \eta\|_1 &= \int_G |(f * \eta)(x)| d\nu(x) \leq \iint_G |f(xy^{-1}) D(y^{-1})| |d\eta|(y) d\nu(x) \\
&= \iint_G |f(x)| |D(y^{-1}) D(y)| d\nu(x) |d\eta|(y) \\
&= \iint_G |f| d\nu |d\eta| = \|f\|_1 \|\eta\| < \infty.
\end{aligned} \tag{4.13}$$

Hence, $f * \eta \in L^1(G, \lambda, \mu)$. □

5. Representation Theory of Locally Compact Groupoids

Let us briefly review some basic facts of representation theory on locally compact and Hausdorff groupoids. Recall that the measure on G defined by $\nu = \int \lambda^u d\mu(u)$ is a Radon measure. Let ν_0 be the measure on G given by

$$d\nu_0 = D^{-1/2} d\nu, \tag{5.1}$$

where D is modular function of μ . Then for each Borel subset E of G , we have

$$\begin{aligned} \nu_0(E) &= \int \chi_E(x) D^{-1/2}(x) d\nu(x) = \int \chi_E(x^{-1}) D^{-1/2}(x^{-1}) d\nu^{-1}(x) \\ &= \int \chi_{E^{-1}}(x) D^{1/2}(x) D^{-1}(x) d\nu(x) = \nu_0(E^{-1}). \end{aligned} \quad (5.2)$$

Hence, ν_0 is symmetric under inversion.

Definition 5.1. A representation of the locally compact groupoid G is a triple $(\mu, \{H_u\}_u, \pi)$ consisting of a Hilbert bundle $(G^0, \{H_u\}_{u \in G^0}, \mu)$, where μ is a quasi-invariant measure on G^0 (with associated Radon measures $\nu, \nu^{-1}, \nu^2, \nu_0$) and for each $x \in G$, a unitary element $\pi(x) \in \mathcal{B}(H_{s(x)}, H_{r(x)})$ such that

- (i) $\pi(u)$ is the identity map on H_u for all u ,
- (ii) $\pi(xy) = \pi(x)\pi(y)$ for ν^2 -a.e. $(x, y) \in G^2$,
- (iii) $\pi(x^{-1}) = \pi^{-1}(x)$ for ν -a.e. $x \in G$,
- (iv) for any $\xi, \eta \in L^2(G^0, \{H_u\}_u, \mu)$, the map

$$x \mapsto \langle \pi(x)\xi(s(x)), \eta(r(x)) \rangle \quad (5.3)$$

is ν -measurable on G .

Definition 5.2. A representation Π of $L^1(G, \lambda, \mu)$ on a Hilbert space, H is a $*$ -homomorphism

$$\Pi : L^1(G, \lambda, \mu) \longrightarrow \mathcal{B}(H). \quad (5.4)$$

It is called nondegenerate if $\langle \Pi(f)\xi : f \in L^1, \xi \in H \rangle$ is dense in H .

Continuity of Π automatically holds, because each $*$ -homomorphism from a Banach $*$ -algebra to a C^* -algebra is norm decreasing, namely, $\|\Pi(f)\| \leq \|f\|_1$, for each $f \in L^1$.

Our main aim here is to find a correspondence between unitary representations of G and nondegenerate representations of $L^1(G, \lambda, \mu)$. Unfortunately, this is impossible in general. Such a correspondence exists between representations of G and those of $C_c(G)$, when H is separable, G is second countable and admits sufficiently many nonsingular G -sets [2, Theorem 1.21].

Proposition 5.3. *Let G be a second countable locally compact groupoid with Haar system and with sufficiently many nonsingular Borel G -sets. Then, every representation of $C_c(G)$ on a separable Hilbert space is the integrated form of a representation of G .*

These assumptions satisfied in the case of r -discrete groupoids and transformation groups. The main problem is that a continuous representation of $(L^1(G, \lambda, \mu), \|\cdot\|_1)$ is not necessarily continuous in the L^1 -norm. To get a partial result, we use the following result [2, Proposition 1.7].

Proposition 5.4. Suppose $(\pi, \{H_u\}_u, \mu)$ is a representation of G . For $\xi, \eta \in L^2(G^0, \{H_u\}_u, \mu)$ and $f \in C_c(G)$,

$$\langle \Pi(f)\xi, \eta \rangle = \int_G f(x) \langle \pi(x)\xi(s(x)), \eta(r(x)) \rangle d\nu_0(x) \quad (5.5)$$

defines a bounded nondegenerate $*$ -representation of $C_c(G)$ on $L^2(G^0, \{H_u\}_u, \mu)$ such that two equivalent representations of G give two equivalent representations of $C_c(G)$.

The equation above is called the integrated form of a unitary representation. If Π is a representation of $C_c(G)$, the above proposition says that Π should be of the form

$$\langle \Pi(f)\xi, \eta \rangle = \int_G f(x) \langle \pi(x)\xi(s(x)), \eta(r(x)) \rangle d\nu_0(x). \quad (5.6)$$

For each $\xi, \eta \in L^2(G^0, \{H_u\}_u, \mu)$ and $f \in C_c(G)$.

Next, we define a representation of $I(G, \lambda, \mu)$, denoted by Π_I as

$$\langle \Pi_I(f)\xi, \eta \rangle = \int_G f(x) \langle \pi(x)\xi(s(x)), \eta(r(x)) \rangle d\nu_0(x) \quad (5.7)$$

for each $\xi, \eta \in L^2(G^0, \{H_u\}_u, \mu)$ and $f \in I(G, \lambda, \mu)$

Lemma 5.5. Π_I is a bounded representation of $I(G, \lambda, \mu)$ on $\mathcal{B}(L^2(G^0, \{H_u\}_u, \mu))$.

Proof. We have

$$\begin{aligned} \langle \Pi_I(f)\xi, \eta \rangle &= \int_G f(x) \langle \pi(x)\xi(s(x)), \eta(r(x)) \rangle d\nu_0(x) \\ &= \int_{G^0} \int_{G^u} f(x) \langle \pi(x)\xi(s(x)), \eta(r(x)) \rangle d^{-1/2}(x) d\lambda^u(x) d\mu(u) \\ &= \int_{G^0} \left\langle \int_{G^u} f(x) \pi(x)\xi(s(x)) D^{-1/2}(x) d\lambda^u(x), \eta(u) \right\rangle d\mu(u). \end{aligned} \quad (5.8)$$

Thus, we may define for μ -a.e. $u \in G^0$,

$$\Pi_I(f)\xi(u) = \int_{G^u} f(x) \pi(x)\xi(s(x)) D^{-1/2}(x) d\lambda^u(x). \quad (5.9)$$

Now, we have

$$\begin{aligned} |\langle \Pi_I(f)\xi, \eta \rangle| &= \left| \int_G f(x) \langle \pi(x)\xi(s(x)), \eta(r(x)) \rangle d\nu_0(x) \right| \\ &\leq \int_G |f(x)| \|\xi \circ s(x)\| \|\eta \circ r(x)\| d\nu_0(x) \end{aligned}$$

$$\begin{aligned}
&\leq \left[\int_G |f(x)| \|\xi \circ s(x)\|^2 d\nu^{-1}(x) \right]^{1/2} \left[\int_G |f(x)| \|\eta \circ r(x)\|^2 d\nu(x) \right]^{1/2} \\
&\leq \left[\int_{G^0} \int_{G_u} |f(x)| d\lambda_u(x) \|\xi(u)\|^2 d\mu(u) \right]^{1/2} \\
&\quad \times \left[\int_{G^0} \int_{G^u} |f(x)| d\lambda^u(x) \|\eta(u)\|^2 d\mu(u) \right]^{1/2} \\
&\leq \|f\|_{I,s}^{1/2} \|\xi\| \|f\|_{I,r}^{1/2} \|\eta\| \\
&\leq \|f\|_I \|\xi\| \|\eta\|.
\end{aligned} \tag{5.10}$$

Therefore, the map $x \mapsto f(x) \langle \pi(x) \xi(s(x)), \eta(r(x)) \rangle$ is ν_0 -integrable, and $\Pi_I(f)$ is a bounded operator of norm $\|\Pi_I(f)\| \leq \|f\|_I$. We have to check that Π_I is a $*$ -homomorphism. For this, we define

$$F(x, t) = f(xt) g(t^{-1}) \langle \pi(x) \xi(s(x)), \eta(r(x)) \rangle d^{1/2}(x). \tag{5.11}$$

Then

$$\begin{aligned}
\langle \Pi_I(f * g) \xi, \eta \rangle &= \int_G f * g(x) \langle \pi(x) \xi(s(x)), \eta(r(x)) \rangle d\nu_0(x) \\
&= \int_G \left(\int_G f(xt) g(t^{-1}) d\lambda^{s(x)}(t) \right) \langle \pi(x) \xi \circ s(x), \eta \circ r(x) \rangle d^{-1/2}(x) D(x) d\nu^{-1}(x) \\
&= \int_{G^2} F(x, t) d\nu^2(x, t) \\
&= \int_{G^2} F(xt, t^{-1}) \rho^{-1}(xt, t^{-1}) d\nu^2(x, t) \\
&= \int_{G^0} \int_{G_u} f(x) \left\langle \pi(x) \left(\int_{G^{s(x)}} g(t) \pi(t) \xi \circ s(t) D^{-1/2}(t) d\lambda^{s(x)}(t) \right), \eta(u) \right\rangle \\
&\quad \times D^{-1/2}(x) d\lambda^u(x) d\mu(u) \\
&= \int_G f(x) \langle \pi(x) \Pi(g) \xi(s(x)), \eta(r(x)) \rangle d\nu_0(x) \\
&= \langle \Pi_I(f) \Pi_I(g) \xi, \eta \rangle.
\end{aligned} \tag{5.12}$$

Hence, $\Pi_I(f * g) = \Pi_I(f) \Pi_I(g)$. Also

$$\begin{aligned}
\langle \Pi_I(f^*) \xi, \eta \rangle &= \int_G f^*(x) \langle \pi(x) \xi(s(x)), \eta(r(x)) \rangle d\nu_0(x) \\
&= \int_G \overline{f(x^{-1})} D(x^{-1}) \langle \pi(x) \xi(s(x)), \eta(r(x)) \rangle d^{-1/2}(x) d\nu(x)
\end{aligned}$$

$$\begin{aligned}
&= \int_G \overline{f(x)} D(x^{-1}) \langle \pi^*(x) \xi(r(x)), \eta(s(x)) \rangle d^{1/2}(x) D(x^{-1}) dv(x) \\
&= \overline{\int_G f(x) \langle \eta(s(x)), \pi^*(x) \xi(r(x)) \rangle dv_0(x)} \\
&= \overline{\int_G f(x) \langle \pi(x) \eta(s(x)), \xi(r(x)) \rangle dv_0(x)} \\
&= \overline{\langle \Pi_I(f) \xi, \eta \rangle} = \langle \xi, \Pi_I(f) \eta \rangle = \langle \Pi_I^*(f) \xi, \eta \rangle.
\end{aligned} \tag{5.13}$$

The nondegeneracy follows from

$$\langle \Pi_I(f) \xi \mid f \in C_c(G), \xi \in H \rangle \subseteq \langle \Pi_I(f) \xi \mid f \in L^1(G, \|\cdot\|_I), \xi \in H \rangle \subseteq H. \tag{5.14}$$

□

Now let us turn to the problem that a continuous representation of $I(G, \lambda, \mu)$ is not necessarily continuous in L^1 -norm. Let $H = L^2(G^0, \{H_u\}_u, \mu)$ and put

$$H_{\Pi_1} = \overline{\langle \xi \in H : \text{The map } f \mapsto \langle \Pi_I(f) \xi, \xi \rangle \text{ is continuous in } L^1\text{-norm} \rangle}. \tag{5.15}$$

Observe that H_{Π_1} is a nontrivial subspace, as μ is a probability measure and if $\|\xi(u)\| = 1$ in H_u for each $u \in G^0$, then $\|\xi\| = 1$ in H and the calculations in the proof of Lemma 5.5 shows that $|\langle \Pi_I(f) \xi, \xi \rangle| \leq \|f\|_1$, hence $\xi \in H_1$. On the other hand, $\Pi_I(f) H_{\Pi_1} \subseteq H_{\Pi_1}$ for each $f \in I(G, \lambda, \mu)$, because the map

$$f \mapsto \langle \Pi_I(f) \Pi_I(g) \xi, \Pi_I(g) \xi \rangle = \langle \Pi_I(g^* * f * g) \xi, \xi \rangle \tag{5.16}$$

is continuous. Therefore, $\Pi_I(f) H_{\Pi_1}^\perp \subseteq H_{\Pi_1}^\perp$. Hence, we have $\Pi_I(f) = \Pi_I(f)|_{H_{\Pi_1}} \oplus \Pi_I(f)|_{H_{\Pi_1}^\perp}$ for each $f \in I(G, \lambda, \mu)$.

Put $H_1 = H_{\Pi_1}$ and define $\Pi_1(f) := \Pi_I(f)|_{H_1}$, for $f \in I(G, \lambda, \mu)$. Then, it follows from continuity of $f \mapsto \langle \Pi_I(f) \xi, \xi \rangle$ and polarization identity that Π_1 extends to a continuous representation of $L^1(G, \lambda, \mu)$ on H_1 , still denoted by Π_1 .

Next we focus on the notion of irreducibility which plays an important role in the theory of representations. We show that if π is an irreducible representation of G , the integrated representation Π_1 of $L^1(G, \lambda, \mu)$ on H_1 is irreducible. Basic materials come from [6].

Definition 5.6. Let $(G^0, \{H_u\}_{u \in G^0}, \mu)$ be a Hilbert bundle. A family $M = \{M_u\}_{u \in G^0}$, where M_u is a closed subspace of H_u for each $u \in G^0$, is called a subbundle. A subbundle $\{M_u\}_{u \in G^0}$ is called nontrivial if $0 \neq M_u \neq H_u$ for some $u \in G^0$. For a representation π of a locally compact groupoid G associated with the Hilbert bundle $(G^0, \{H_u\}_{u \in G^0}, \mu)$, a subbundle $\{M_u\}_{u \in G^0}$ is called invariant if $\pi(x) M_{s(x)} \subseteq M_{r(x)}$ for each $x \in G$. Note that if M is an invariant subbundle, and $0 \neq M_u \neq H_u$ for some $u \in G^0$, then $0 \neq M_w \neq H_w$ for every $w \in [u]$.

The following lemma is proved in [6, Lemma 3.4].

Lemma 5.7. *Let π be a representation of a locally compact groupoid G associated with the Hilbert bundle $(G^0, \{H_u\}_{u \in G^0}, \mu)$. If $M = \{M_u\}_{u \in G^0}$ is an invariant subbundle, then so is $M^\perp = \{M_u^\perp\}_{u \in G^0}$.*

Definition 5.8. A representation π of a locally compact groupoid G is called reducible if π admits a nontrivial invariant subbundle $M = \{M_u\}_{u \in G^0}$, otherwise π is called irreducible. In this case, it is easy to check that π^M with $\pi^M(x) = \pi(x)|_{M_{s(x)}} : M_{s(x)} \rightarrow M_{r(x)}$ is called a subrepresentation of π . If π and π' are two representations of a locally compact groupoid G associated with two Hilbert bundles $(G^0, \{H_u\}_{u \in G^0}, \mu)$ and $(G^0, \{H'_u\}_{u \in G^0}, \mu')$, respectively. Then, we put

$$\mathcal{C}(\pi, \pi') = \left\{ (T_u)_u \in \prod_{u \in G^0} \mathcal{B}(H_u, H'_u) : \pi'(x)T_{s(x)} = T_{r(x)}\pi(x) \ (x \in G) \right\} \quad (5.17)$$

and write $\mathcal{C}(\pi, \pi) = \mathcal{C}(\pi)$.

Two representations π and π' are called (unitarily) equivalent if $\mu \sim \mu'$, and there is $(T_u)_{u \in G^0} \in \mathcal{C}(\pi, \pi')$ such that T_u is a unitary operator for every $u \in G^0$. Note that if $(T_u)_{u \in G^0} \in \mathcal{C}(\pi)$ and T_u^* denotes the adjoint operator to T_u , then

$$\pi(x)T_{s(x)}^* = \left[T_{s(x)}\pi(x^{-1}) \right]^* = \left[\pi(x^{-1})T_{r(x)} \right]^* = T_{r(x)}^*\pi(x), \quad (5.18)$$

hence, $(T_u^*)_{u \in G^0} \in \mathcal{C}(\pi)$. We observe that $\mathcal{C}(\pi)$ is a unital $*$ -algebra, where the operations are defined pointwise.

Following [6], for a representation π of a locally compact groupoid G associated with the Hilbert bundle $(G^0, \{H_u\}_{u \in G^0}, \mu)$, we set

$$\Lambda = \left\{ (\lambda_u \pi(u))_u \in \prod_{u \in G^0} \mathcal{B}(H_u) : \lambda_u \in \mathbb{C}, \lambda_u = \lambda_w \text{ whenever } u \sim w \right\}. \quad (5.19)$$

If $(\lambda_u \pi(u))_u \in \mathcal{C}(\pi)$, then $\lambda_u = \lambda_w$ whenever $u \sim w$ that is $(\lambda_u \pi(u))_u \in \Lambda$. Therefore, if $(T_u)_{u \in G^0} \in \mathcal{C}(\pi) \setminus \Lambda$, then there exists $u \in G^0$ with T_u not in $\mathbb{C}\pi(u)$. It is also obvious that $\Lambda \subseteq \mathcal{C}(\pi)$.

We need the following version of Schur's lemma [6, Lemma 3.11].

Lemma 5.9. *A representation π of a locally compact groupoid G is irreducible if and only if $\Lambda = \mathcal{C}(\pi)$. In particular, in the case where G is transitive, then π is irreducible if and only if $\mathcal{C}(\pi) = \mathbb{C}(\pi(u))_{u \in G^0}$.*

Lemma 5.10. *If $\mathcal{M} = (G^0, \{M_u\}_u, \mu)$ is closed subbundle of $\mathcal{H} = (G^0, \{H_u\}_u, \mu)$ and P_u is an orthogonal projection H_u onto M_u , then $\mathcal{M}$ is invariant under π if and only if $(P_u)_u \in \mathcal{C}(\pi)$.*

Proof. If $(P_u)_u \in \mathcal{C}(\pi)$ and $m_u \in M_u$, then

$$\pi(x)m_{s(x)} = \pi(x)P_{s(x)}m_{s(x)} = P_{r(x)}\pi(x)m_{s(x)} \in M_{r(x)}. \quad (5.20)$$

Thus, $\pi(x)M_{s(x)} \subseteq M_{r(x)}$ so $\mathcal{M}$ is invariant. Conversely, if $\mathcal{M}$ is invariant, then for $m \in M_{s(x)}$

$$\pi(x)P_{s(x)}m = \pi(x)m = P_{r(x)}\pi(x)m \quad (5.21)$$

and for $m \in M_{s(x)}^\perp$

$$\pi(x)P_{s(x)}m = 0 = P_{r(x)}\pi(x)m. \quad (5.22)$$

Hence, $\pi(x)P_{s(x)} = P_{r(x)}\pi(x)$. □

We show that if a representation π of a locally compact groupoid G associated with the Hilbert bundle $(G^0, \{H_u\}_{u \in G^0}, \mu)$ is irreducible and Π_1 is the corresponding integrated representation of $L^1(G, \lambda, \mu)$ on $H_1 \subseteq L^2(G^0, \{H_u\}_u, \mu)$, then Π_1 is irreducible. If $H = L^2(G^0, \{H_u\}_u, \mu)$, then for each subbundle $\{M_u\}_u$, $M = L^2(G^0, \{M_u\}_u, \mu)$ is closed subspace of H with orthogonal complement $M^\perp = L^2(G^0, \{M_u^\perp\}_u, \mu)$. A map $\Theta : L^2(G^0, \{H_u\}_u, \mu) \rightarrow L^2(G^0, \{M_u\}_u, \mu)$; $\xi \mapsto \Theta\xi$ with $\Theta\xi(s(x)) \in M_{r(x)}$ is called an orthogonal projection of H onto M .

Let π be a representation of G on the Hilbert bundle $(G^0, \{H_u\}_{u \in G^0}, \mu)$ and Π_I be the bounded representation of $I(G, \lambda, \mu)$ on $\mathcal{B}(L^2(G^0, \{H_u\}_u, \mu))$ constructed above.

Lemma 5.11. *M is invariant under Π_I if and only if for each $f \in I(G, \lambda, \mu)$, $\Pi_I(f)\Theta = \Theta\Pi_I(f)$.*

Proof. Suppose $\Pi_I(f)\Theta = \Theta\Pi_I(f)$ and $\xi \in M$, then

$$\Pi_I(f)\xi = \Pi_I(f)\Theta\xi = \Theta\Pi_I(f)\xi \in M. \quad (5.23)$$

Hence, M is invariant. Conversely, if M is invariant, then for $\xi \in M$,

$$\Pi_I(f)\Theta\xi = \Pi_I(f)\xi = \Theta\Pi_I(f)\xi \quad (5.24)$$

and for $\eta \in M^\perp$,

$$\Pi_I(f)\Theta\eta = 0 = \Theta\Pi_I(f)\eta. \quad (5.25)$$

Thus, $\Pi_I(f)\Theta = \Theta\Pi_I(f)$. □

Proposition 5.12. *If π is an irreducible representation of G , then Π_I is an irreducible representation of $I(G, \lambda, \mu)$.*

Proof. If Π_I is reducible, then there exists a nontrivial invariant closed subspace M of $L^2(G^0, \{H_u\}_u, \mu)$, and, hence, there is an orthogonal projection $\Theta \in \mathcal{C}(\Pi_I)$. It follows that

$$\begin{aligned} & \int_G f(x) \langle \pi(x) P_{s(x)} \xi(s(x)), \eta(r(x)) \rangle d\nu_0(x) \\ &= \langle \Pi_I(f) \Theta \xi, \eta \rangle = \langle \Theta \Pi_I(f) \xi, \eta \rangle \\ &= \int_G f(x) \langle P_{r(x)} \pi(x) \xi(s(x)), \eta(r(x)) \rangle d\nu_0(x), \end{aligned} \quad (5.26)$$

for each $f \in C_c(G)$ and $\xi, \eta \in L^2(G^0, \{H_u\}_u, \mu)$. Therefore, $\pi(x) P_{s(x)} = P_{r(x)} \pi(x)$, for each $x \in G$, hence $(P_u)_u \in \mathcal{C}(\pi)$. Now by the Schur's lemma, π is reducible. $\square$

Theorem 5.13. *If π is an irreducible representation of G , then Π_1 is an irreducible representation of $L^1(G, \lambda, \mu)$ on H_1 .*

Proof. If Π_1 is reducible, then there exists a nontrivial invariant closed subspace M_1 of $H_1 \subseteq H = L^2(G^0, \{H_u\}_u, \mu)$. By the calculations after Lemma 5.5, $\Pi_I = \tilde{\Pi}_1 \oplus \tilde{\Pi}_2$, where $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ are the corresponding representations of $I(G, \lambda, \mu)$ on H_1 and $H_1^\perp$. Therefore, $M = M_1 \oplus H_1^\perp$ is a nontrivial invariant closed subspace H , and Π_I is reducible. Hence, π is reducible by the above proposition. $\square$

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