Research Article

Mutually Permutable Products of Finite Groups

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Let G be a finite group and G_1, G_2 are two subgroups of G . We say that G_1 and G_2 are mutually permutable if G_1 is permutable with every subgroup of G_2 and G_2 is permutable with every subgroup of G_1 . We prove that if $G = G_1G_2 = G_1G_3 = G_2G_3$ is the product of three supersolvable subgroups G_1, G_2 , and G_3 , where G_i and G_j are mutually permutable for all i and j with $i \neq j$ and the Sylow subgroups of G are abelian, then G is supersolvable. As a corollary of this result, we also prove that if G possesses three supersolvable subgroups G_i ($i = 1, 2, 3$) whose indices are pairwise relatively prime, and G_i and G_j are mutually permutable for all i and j with $i \neq j$, then G is supersolvable.

1. Introduction

Throughout this paper, G will denote a finite group. We write $\sigma(G)$ for the set of prime divisors of the order of G and $|\sigma(G)|$ for their number. In [1], Doerk determined the structure of minimal non-supersolvable groups (nonsupersolvable groups and all of whose proper subgroups are supersolvable). He proved that if G is a minimal non-supersolvable group, then G is solvable and $2 \leq |\sigma(G)| \leq 3$. Therefore, if G is a minimal non-supersolvable group with $|\sigma(G)| = 3$, then G possesses three supersolvable subgroups G_i ($i = 1, 2, 3$) such that $G = G_1G_2 = G_1G_3 = G_2G_3$. In [2], Kegel proved that if $G = G_1G_2 = G_1G_3 = G_2G_3$, where G_1 and G_2 are nilpotent subgroups of G , and G_3 is a supersolvable subgroup of G , then G is supersolvable. In [3], Asaad and Shaalan proved the following result: assume that G_1 and G_2 are supersolvable subgroups of G , G' is nilpotent and $G = G_1G_2$. Assume further that G_1 is permutable with every subgroup of G_2 and G_2 is permutable with every subgroup of G_1 (G_1 and G_2 are mutually permutable). Then G is supersolvable. Further, they proved the following result: Assume that G_1 and G_2 are supersolvable subgroups of G and $G = G_1G_2$ and that every subgroup of G_1 is permutable with every subgroup of G_2 (G_1 and G_2 are totally permutable). Then, G is supersolvable.

In this paper, we are interested in the following question.

Assume that $G = G_1G_2 = G_1G_3 = G_2G_3$, where G_1 , G_2 , and G_3 are supersolvable subgroups and that G_i and G_j are mutually permutable for all i and j with $i \neq j$. Is G supersolvable? The answer is negative as the following example shows.

Example 1 (see [8, pages 8-9]). Let $H = \langle x \rangle \times \langle y \rangle$, where $|x| = |y| = 5$. The maps $\alpha : x \rightarrow x^2$, $y \rightarrow y^{-2}$, and $\beta : x \rightarrow y^{-1}$, $y \rightarrow x$ are automorphisms of H and generate a subgroup $A \leq \text{Aut}(H)$ of order 8 (A is isomorphic with a quaternion group). Take $G = [H]A$. Then $G_1 = H\langle\alpha\rangle$, $G_2 = H\langle\beta\rangle$, and $G_3 = H\langle\alpha\beta\rangle$ are normal supersolvable subgroups of G and $G = G_1G_2 = G_1G_3 = G_2G_3$, but G is not supersolvable.

We prove the following result.

Theorem 1.1. *If $G = G_1G_2 = G_1G_3 = G_2G_3$ is the product of three supersolvable subgroups G_1 , G_2 , and G_3 such that G_i and G_j are mutually permutable for all i and j with $i \neq j$, and the Sylow subgroups of G are abelian, then G is supersolvable.*

As a corollary of Theorem 1.1, we have the following.

Corollary 1.2. *If G possesses three supersolvable subgroups G_i ($i=1,2,3$) whose indices in G are pairwise relatively prime, and G_i is permutable with every subgroup of G_j , for all i and j with $i \neq j$, then G is supersolvable.*

2. Preliminaries

We list here some basic results which are needed in this paper.

Lemma 2.1 (see [2]). *Let $G = G_1G_2 = G_1G_3 = G_2G_3$ be the product of three subgroups G_1 , G_2 , and G_3 . If G_1 , G_2 , and G_3 have normal Sylow p -subgroups for a certain prime p , then also G has a normal Sylow p -subgroup.*

Lemma 2.2 (see [4]). *Let $G = G_1G_2$ be a group such that G_1 and G_2 are mutually permutable.*

- (a) *If $G_1 \cap G_2 = 1$, then G_1 and G_2 are totally permutable.*
- (b) *$G_1 \cap G_2$ is a quasinormal subgroup of G_1 and of G_2 .*

Lemma 2.3 (see [3]). *Let $G = G_1G_2$ be a group such that G_1 and G_2 are totally permutable subgroups. If G_1 and G_2 are supersolvable subgroups of G , then G is supersolvable.*

Lemma 2.4 (see [5, page 213, Theorem 7.1.2]). *If H is a quasinormal subgroup of G , then H is subnormal in G .*

Lemma 2.5 (see [5, page 239, Theorem 7.7.1]). *Let $G = G_1G_2$ be a group such that G_1 and G_2 are subgroups of G . If H is a subnormal subgroup of G_1 and of G_2 , then H is subnormal in G .*

Lemma 2.6 (see [6]). *Let G be a group and H is a quasinormal subgroup of G . Then, $H/\text{Cor}_G(H)$ is nilpotent, where $\text{Cor}_G(H) = \bigcap_{g \in G} H^g$.*

Lemma 2.7 (see [7, page 29, Theorem 8.8(a)]). *$F(G) = \langle H : H \text{ is subnormal subgroup of } G \text{ and } H \text{ is nilpotent} \rangle$.*

Lemma 2.8 (see [2]). *Let the group $G = G_1G_2 = G_1G_3 = G_2G_3$ be the product of three subgroups G_1 , G_2 , and G_3 . If G_1 , G_2 , and G_3 are nilpotent subgroups of G , then G is nilpotent.*

Lemma 2.9 (see [8, page 196, Theorem 5.1(15)]). *If G is a supersolvable group, then $G/O_{p',p}(G)$ is abelian of exponent dividing $p - 1$ for all primes p .*

Lemma 2.10 (see [8, page 6, Theorem 1.9]). *Let P be a normal Sylow p -subgroup of G . If G/P is abelian of exponent dividing $p - 1$, then G is supersolvable.*

Lemma 2.11 (see [8, page 5, Theorem 1.6]). *The commutator subgroup of a supersolvable group is nilpotent.*

3. Proofs

Proof of Theorem 1.1. Assume that the result is not true, and let G be a counterexample of minimal order. Since G_i ($i = 1, 2, 3$) is supersolvable, it follows that G_i has a normal Sylow p -subgroup, where p is the largest prime dividing $|G|$. Then, by Lemma 2.1, G has a normal Sylow p -subgroup, say P . Certainly, every proper quotient group of G satisfies the hypothesis of the theorem. So every proper quotient group of G is supersolvable by the minimal choice of G . But the class of all supersolvable groups is a saturated formation, so $\Phi(G) = 1$, $P = F(G)$, and $C_G(P) = P$. We argue that $G_1 \cap G_2 \neq 1$. If not, $G_1 \cap G_2 = 1$. Then, $G = G_1G_2$ is a totally permutable product of G_1 and G_2 by Lemma 2.2(a). Then, by Lemma 2.3, G is supersolvable, a contradiction.

Thus, $G_1 \cap G_2 \neq 1$. Analogously, $G_1 \cap G_3 \neq 1$ and $G_2 \cap G_3 \neq 1$. Since $G = G_1G_2$ is a mutually permutable product of G_1 and G_2 , it follows that $G_1 \cap G_2$ is a quasinormal subgroup of G_1 and of G_2 by Lemma 2.2(b). Then, by Lemma 2.4, $G_1 \cap G_2$ is a subnormal subgroup of G_1 and of G_2 . Hence $G_1 \cap G_2$ is a subnormal subgroup of G by Lemma 2.5.

If $\text{Cor}_{G_1}(G_1 \cap G_2) = 1$, then, by Lemma 2.6, $G_1 \cap G_2$ is nilpotent. Hence, $G_1 \cap G_2$ is a subnormal nilpotent subgroup of G . So $G_1 \cap G_2 \leq P$ by Lemma 2.7. Taken into consideration that $G_1 \cap G_2$ is quasinormal in G_1 , $G_1 \cap G_2 \leq P$ and P is abelian, it follows that $G_1 \cap G_2$ is normal in G_1 and so $G_1 \cap G_2 = \text{Cor}_{G_1}(G_1 \cap G_2) = 1$, a contradiction. Thus $\text{Cor}_{G_1}(G_1 \cap G_2) \neq 1$. Analogously, $\text{Cor}_{G_2}(G_2 \cap G_3) \neq 1$ and $\text{Cor}_{G_3}(G_3 \cap G_1) \neq 1$. Set $L_1 = \text{Cor}_{G_1}(G_1 \cap G_2)$, $L_2 = \text{Cor}_{G_2}(G_2 \cap G_3)$ and $L_3 = \text{Cor}_{G_3}(G_3 \cap G_1)$. Then $L_1^G = L_1^{G_1G_2} \leq G_2$, $L_2^G = L_2^{G_2G_3} \leq G_3$, and $L_3^G = L_3^{G_3G_1} \leq G_1$. Now, $P \leq L_i^G$ ($i = 1, 2, 3$) as P is a unique minimal normal subgroup of G . Hence, $P \leq G_i$ ($i = 1, 2, 3$).

Now, we finish the proof of the theorem. Since G_i is supersolvable, $C_G(P) = P = F(G)$ and $P \leq G_i$ ($i = 1, 2, 3$), it follows that $C_{G_i}(P) = C_{G_i}(F(G_i)) = P$ ($i = 1, 2, 3$) and G_i/P is abelian. Hence, by Lemma 2.8, G/P is nilpotent, and, since the Sylow subgroups of G are abelian, it follows that G/P is abelian. On the other hand, by Lemma 2.9, G_i/P ($i = 1, 2$) is of exponent dividing $p - 1$. Hence, G/P is abelian of exponent dividing $p - 1$ and so G is supersolvable, by Lemma 2.10, a final contradiction completing the proof of the theorem. $\square$

Proof of Corollary 1.2. Assume that the result is not true and let G be a counterexample of minimal order. Let P be a Sylow p -subgroup of G , where p is the largest prime dividing $|G|$. Then, P is normal in G by Lemma 2.1. Certainly, every proper quotient group of G satisfies the hypothesis of the corollary. So every proper quotient group of G is supersolvable by the minimal choice of G . But the class of all supersolvable groups is a saturated formation, so $\Phi(G) = 1$, $P = F(G)$, and $C_G(P) = P$. Since G_1 , G_2 , and G_3 have coprime indices, we can

assume that p does not divide $|G : G_1|$ and p does not divide $|G : G_2|$. Then, $P \leq G_1$ and $P \leq G_2$ and so $P = C_{G_1}(P) = C_{G_2}(P) = F(G_2) = F(G_1)$ as $P = F(G) = C_G(P)$. Then, G_1/P and G_2/P are abelian subgroups of G/P by Lemma 2.11. This together with $(|G : G_1|, |G : G_2|) = 1$ imply that the Sylow subgroups of G/P are abelian. Now Theorem 1.1 implies that G is supersolvable, a contradiction completing the proof of the corollary. $\square$

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