Research Article

Theories and Inequalities on the Satellite System

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We define the satellite system without any central, the satellite system with a central, and the satellite system of single point with a central. For the satellite system without any central $S\{\Gamma, \Gamma_N\}$, we establish the inequality: $\oint_{\Gamma} ds \oint_{\Gamma_N} |PQ| ds_Q \leq (N|\Gamma|^3/4\pi^2) \sin(2\pi/N)$. For the satellite system with a central $S\{O, \Gamma, \Gamma_N\}$, we establish the following inequality under the proper hypothesis: $\oint_{\Gamma} M_N^{[t]}(r) ds \leq (|\Gamma|^2/2\pi) \cos(\pi/N) (\forall t \in [-\infty, -2])$. As an application, we get the inequalities $\oint_{\Gamma} \rho_P ds \geq \oint_{\Gamma} r_P ds \geq 2|D(\Gamma)|$, for the satellite system of single point with a central $S\{O, \Gamma, P\}$. For these results, there are generalized backgrounds in the fields of differential geometry and space science.

1. Introduction and the Main Results

In this paper, we will use the following symbols: R and Z to denote the set of real numbers and the set of integers, respectively, $a = (a_1, a_2, \dots, a_n)$, and $R^n = \{a | a_i \in R, 1 \leq i \leq n\} [1, 2]$.

Definition 1.1. Let $\Gamma = \Gamma(AB) : r = r(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} (a \leq t \leq b)$ be a smooth curve in three-space R^3 . $A = r(a)$ and $B = r(b)$ are called the initial point and terminal point of Γ , respectively. If $a < t_1 < b$, $a \leq t_2 \leq b$, $t_1 \neq t_2$, and $r(t_1) = r(t_2)$, then $r(t_1)$ is called a coincident point of Γ ; smooth curve without any coincident point is called a simple curve; If a simple curve Γ satisfies $A = B$, then Γ is said to be a simple closed curve. Especially, we say Γ is a Jordan closed curve if Γ is a plane curve in R^3 . For a Jordan closed curve Γ , $D(\Gamma)$ denotes the closed region bounded by Γ and its area is written as $|D(\Gamma)|$, $|\Gamma|$ denotes the length of Γ , and we have the following Jordan curve theorem.

Theorem 1.2 (Jordan Curve [3]). *An arbitrary Jordan closed curve must divide a plane into two parts, where one part is bounded, and the other is unbounded. The bounded part is called interior and another is outside of the Jordan closed curve.*

Definition 1.3 (see [4, 5]). Assume that $\Gamma(AB)$ is a smooth space curve with end points A and B , P is a fixed point in R^3 , and $Q \in \Gamma(AB)$, where Q is a moving point, then the trail of the line segment PQ is called a bounded cone surface, written as $\Omega[P, \Gamma(AB)]$. We Also say PQ is the generating line, P is the vertex, and $\Gamma(AB)$ is the generating curve of the bounded cone surface. $|\Omega[P, \Gamma(AB)]|$ denotes the area of $\Omega[P, \Gamma(AB)]$.

If Γ is a given smooth curve in R^3 , then its length is written as $|\Gamma|$; $\Gamma(AB)$ denotes the smooth curve segment with end points A and B , and $\Gamma(AB) \subset \Gamma$.

Definition 1.4. Let Γ be a fixed smooth closed curve in R^3 , the N -polygon $\Gamma_N : A_1A_2 \cdots A_NA_1$ ($N \geq 3$) be inscribed in Γ , and, $A_i \in \Gamma(A_{i-1}A_{i+1})$ ($i = 1, 2, \dots, N$). If all vertices: $A_1, A_2, \dots, A_N$ of Γ_N move continuously along Γ and keep $l_i := |\Gamma(A_iA_{i+1})|$ invariable, where $0 < l_i < |\Gamma|/2$, $i = 1, 2, \dots, N$, we say that the set $S\{\Gamma, \Gamma_N\} := \{\Gamma, \Gamma_N\}$ is the satellite system without any central in three-space R^3 .

For $S\{\Gamma, \Gamma_N\}$, we write the set of the vertices of Γ_N as $V(\Gamma_N) := \{A_1, A_2, \dots, A_N\}$ and define $A_i = A_j \Leftrightarrow i \equiv j \pmod{N}$, for all $i, j \in Z$ [6].

Definition 1.5. For $S\{\Gamma, \Gamma_N\}$, if the following statements are valid:

- (i) Γ is a fixed closed Jordan curve in R^3 ,
- (ii) O is a fixed point in the interior of Γ , and
- (iii) Γ_N is always a Jordan closed curve and $\Omega[O, \Gamma_N] = D(\Gamma_N)$ as all vertices of Γ_N move continuously along Γ ,

then we say the set $S\{O, \Gamma, \Gamma_N\} := \{O, \Gamma, \Gamma_N\}$ is the satellite system with a central and O is its central.

For $S\{O, \Gamma, \Gamma_N\}$, $r_i := \text{Dis}(O, A_iA_{i+1})$ denotes the distance from O to the line A_iA_{i+1} , $i = 1, 2, \dots, N$. And we write $r := (r_1, r_2, \dots, r_N)$.

Remark 1.6. The $S\{O, \Gamma, \Gamma_N\}$ may be explained as follows: the point O denotes the central of earth, Γ denotes the trajectory on which satellites move, and the vertexes of Γ_N are viewed as N satellites moving on the same trajectory Γ . In order to avoid hitting, they must move by same curve velocity, that is, $l_i := |\Gamma(A_iA_{i+1})|$ is invariable and $0 < l_i < |\Gamma|/2$, $i = 1, 2, \dots, N$.

Definition 1.7. The t th power mean of the positive real numbers sets $a = (a_1, a_2, \dots, a_N)$ ($N \geq 2$), written as $M_N^{[t]}(a)$, is defined by [7–15]:

$$M_N^{[t]}(a) := \begin{cases} \left(\frac{1}{N} \sum_{i=1}^N a_i^t \right)^{1/t}, & t \in R, \quad 0 < |t| < +\infty \\ \sqrt[t]{\prod_{i=1}^N a_i}, & t = 0. \end{cases} \quad (1.1)$$

Definition 1.8 (see [1, 7, 10, 11, 14]). Let Γ be a smooth curve in R^3 . For Riemann integrable function $f : \Gamma \rightarrow R$, $M(f, \Gamma) := (1/|\Gamma|) \int_{\Gamma} f ds$ is called function means of f on Γ .

For the function $f : \Gamma \rightarrow]0, \infty[$ and the real number t , if f^t ($t \in \mathbb{R}, t \neq 0$) and $\ln f$ are Riemann integrable on Γ , then

$$M^{[t]}(f, \Gamma) := \begin{cases} \left(\frac{1}{|\Gamma|} \int_{\Gamma} f^t ds \right)^{1/t}, & t \in \mathbb{R}, t \neq 0 \\ \exp \left(\frac{1}{|\Gamma|} \int_{\Gamma} \ln f ds \right), & t = 0 \end{cases} \quad (1.2)$$

is called the t th power means of f on Γ .

Now, we give our main results as follow.

Theorem 1.9. *Let $S\{\Gamma, \Gamma_N\}$ be a satellite system without any central. For any $P \in V(\Gamma_N) := \{A_1, A_2, \dots, A_N\}$, we have inequality*

$$\oint_{\Gamma} ds \oint_{\Gamma_N} |PQ| ds_Q \leq \frac{N|\Gamma|^3}{4\pi^2} \sin \frac{2\pi}{N}, \quad (1.3)$$

that is,

$$M(|\Omega[P, \Gamma_N]|, \Gamma) \leq \frac{N|\Gamma|^2}{8\pi^2} \sin \frac{2\pi}{N}, \quad (1.4)$$

where PQ is the generating line of the bounded cone surface $\Omega[P, \Gamma_N]$. A sufficient condition of equalities in (1.3) and (1.4) is that Γ is a circle and Γ_N is always a regular polygon with N sides.

Theorem 1.10. *Let $S\{O, \Gamma, \Gamma_N\}$ be a satellite system with a central. If $N = 3$ or $N \geq 4$, $0 < l_i \leq |\Gamma|/4$, $i = 1, 2, \dots, N$, we have inequality, for any the real number $t \leq -2$,*

$$\oint_{\Gamma} M_N^{[t]}(r) ds \leq \frac{|\Gamma|^2}{2\pi} \cos \frac{\pi}{N}, \quad (1.5)$$

that is,

$$M\left(M_N^{[t]}(r), \Gamma\right) \leq \frac{|\Gamma|}{2\pi} \cos \frac{\pi}{N}. \quad (1.6)$$

The equalities of (1.5) and (1.6) occur if and only if Γ is a circle, O is the central of the circle, and Γ_N is a regular polygon with sides N .

In Section 5, we will give some applications of these results and theories.

2. Preliminaries

Lemma 2.1. For the quadrilateral $ABCD$ in R^3 , writing $a := |\overrightarrow{AB}|$, $b := |\overrightarrow{BC}|$, $c := |\overrightarrow{CD}|$, $d := |\overrightarrow{DA}|$, we obtain

$$|\Delta BAD| + |\Delta BCD| \leq \frac{1}{4} \sqrt{[(a+d)^2 - (b-c)^2][(b+c)^2 - (a-d)^2]}, \quad (2.1)$$

with equality if and only if $\angle BAD + \angle BCD = \pi$. If the quadrilateral $ABCD$ is convex and in plane, equality holds if and only if the quadrilateral $ABCD$ is inscribed in a circle.

Proof. Write $s := |\Delta BAD| + |\Delta BCD|$, $\angle BAD = \alpha$, $\angle BCD = \beta$. By the area formula of triangle and cosine theorem, we get

$$s = \frac{1}{2}(ad \sin \alpha + bc \sin \beta), \quad (2.2)$$

$$a^2 + d^2 - 2ad \cos \alpha = b^2 + c^2 - 2bc \cos \beta, \quad 0 \leq \alpha, \beta \leq \pi. \quad (2.3)$$

Consider $\beta = \beta(\alpha)$ as the implicit function with respect to α . Finding the derivatives of the two sides of (2.2) and (2.3), respectively, we have

$$\frac{ds}{d\alpha} = \frac{1}{2} \left(ad \cos \alpha + bc \cos \beta \frac{d\beta}{d\alpha} \right), \quad \frac{d\beta}{d\alpha} = \frac{ad \sin \alpha}{bc \sin \beta} \geq 0. \quad (2.4)$$

Therefore, $(ds/d\alpha) = (1/2)ad \sin \alpha (\cot \alpha + \cot \beta)$ and $\beta = \beta(\alpha)$ is increasing with respect to α . Since

$$\begin{aligned} |b-c| &\leq |BD| \leq a+d, & |a-d| &\leq |BD| \leq b+c, \\ \Rightarrow |b-c| &\leq a+d, & |a-d| &\leq b+c, \\ \Rightarrow (a+d)^2 - (b-c)^2 &\geq 0, & (a-d)^2 - (b+c)^2 &\leq 0, \\ \Rightarrow -2(ad+bc) &\leq a^2 + d^2 - b^2 - c^2 \leq 2(ad+bc), \\ \Rightarrow \left| \frac{a^2 + d^2 - b^2 - c^2}{2(ad+bc)} \right| &\leq 1, \end{aligned} \quad (2.5)$$

we have

$$\begin{aligned}
\frac{ds}{d\alpha} = 0 &\iff \alpha + \beta = \pi, & a^2 + d^2 - 2ad \cos \alpha &= b^2 + c^2 - 2bc \cos \beta, \\
&\iff \alpha = \alpha_0 := \arccos \frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)}, & \beta &= \beta(\alpha_0) = \pi - \alpha_0, \\
0 \leq \alpha < \alpha_0 &\implies 0 \leq \beta = \beta(\alpha) < \beta(\alpha_0) = \pi - \alpha_0 \implies 0 \leq \beta < \pi - \alpha \implies \frac{ds}{d\alpha} > 0, \\
\alpha_0 < \alpha \leq \pi &\implies \pi \geq \beta = \beta(\alpha) > \beta(\alpha_0) = \pi - \alpha_0 \implies \pi \geq \beta > \pi - \alpha \implies \frac{ds}{d\alpha} < 0.
\end{aligned} \tag{2.6}$$

Hence s is max if and only if $\alpha + \beta = \pi$. When s is max, $\beta = \pi - \alpha$, $\alpha = \alpha_0$. Thus,

$$\begin{aligned}
s &\leq \frac{1}{2}(ad \sin \alpha_0 + bc \sin \alpha_0) = \frac{1}{2}(ad + bc) \sqrt{1 - \left[\frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)} \right]^2} \\
&= \frac{1}{4} \sqrt{[(a + d)^2 - (b - c)^2][(b + c)^2 - (a - d)^2]}.
\end{aligned} \tag{2.7}$$

In other words, (2.1) holds. Equality holds if and only if $\alpha + \beta = \pi \Leftrightarrow \angle BAD + \angle BCD = \pi$. This completes the proof. $\square$

Lemma 2.2. *Let $S\{\Gamma, \Gamma_N\}$ be a satellite system without any central. For any $P \in V(\Gamma_N)$, we have inequality:*

$$\frac{1}{2} \oint_{\Gamma_N} |PQ| ds_Q = |\Omega[P, \Gamma_N]| \leq \frac{|\Gamma_N|^2}{4N} \cot \frac{\pi}{N}, \tag{2.8}$$

where PQ is the generating line of the bounded cone surface $\Omega[P, \Gamma_N]$. The second equality occurs if Γ_N is a regular polygon with sides N .

Proof. By the definition and geometric meaning of curve integral, we obtain

$$\frac{1}{2} \oint_{\Gamma_N} |PQ| ds_Q = |\Omega[P, \Gamma_N]|. \tag{2.9}$$

Now we prove the inequality in (2.8).

If $N = 3$, (2.8) is known; if $N = 4$, by Lemma 2.1, (2.8) holds. In the following, we suppose that $N \geq 5$. First, fix the value of $|\Gamma_N|$. Without loss of generality, we set $P = A_1$. By the theory of differential geometry, we know a bounded cone surface is a developable surface, which implies that $\Omega[P, \Gamma_N]$ can be developed into a bounded cone surface $\Omega[A_1^*, \Gamma_N^*]$ in

the plane, where $\Gamma_N^* : A_1^* A_2^* \cdots A_N^* A_1^* (P = A_1 = A_1^*)$ is the developed graph of Γ_N in the plane and it is a polygon with sides N (may not be a closed Jordan curve) and satisfies

$$\begin{aligned} \Delta A_{i-1}^* A_1^* A_i^* \cap \Delta A_i^* A_1^* A_{i+1}^* &= A_1^* A_i^* \quad (i = 3, 4, \dots, N-1), \quad |\Gamma_N^*| = |\Gamma_N|, \\ |\Omega[P, \Gamma_N]| &= |\Omega[A_1^*, \Gamma_N^*]| = \sum_{i=3}^N |\Delta A_{i-1}^* A_1^* A_i^*|. \end{aligned} \quad (2.10)$$

When $|\Omega[A_1^*, \Gamma_N^*]|$ is max, for any $i \in \{2, 3, \dots, N-1\}$, the quadrilateral $A_{i-1}^* A_i^* A_{i+1}^* A_{i+2}^* A_{i-1}^*$ must be convex and the polygon with 5 sides $A_1^* A_{i-1}^* A_i^* A_{i+1}^* A_{i+2}^* A_1^* (A_{N+1}^* = A_1^*)$ must be also convex by the plane geometry. Now, we prove the four points $A_{i-1}^*, A_i^*, A_{i+1}^*$ and A_{i+2}^* are on a common circle. Otherwise, there exists some $i \in \{2, 3, \dots, N-1\}$ such that $A_{i-1}^*, A_i^*, A_{i+1}^*, A_{i+2}^*$ are not on a common circle. Therefore, fix the point $A_j^* (j \in \{1, 2, \dots, N\} - \{i, i+1\})$ and modify A_i^*, A_{i+1}^* to A_i^{**}, A_{i+1}^{**} such that $A_{i-1}^*, A_i^{**}, A_{i+1}^{**}, A_{i+2}^*$ are on a common circle and

$$\left| \overrightarrow{A_{i-1}^* A_i^{**}} \right| = \left| \overrightarrow{A_{i-1}^* A_i^*} \right|, \left| \overrightarrow{A_i^{**} A_{i+1}^{**}} \right| = \left| \overrightarrow{A_i^* A_{i+1}^*} \right|, \left| \overrightarrow{A_{i+1}^{**} A_{i+2}^*} \right| = \left| \overrightarrow{A_{i+1}^* A_{i+2}^*} \right|. \quad (2.11)$$

Hence, we obtain a new polygon with N sides and write

$$\Gamma_N^{**} : A_1^* A_2^* \cdots A_{i-1}^* A_i^{**} A_{i+1}^{**} A_{i+2}^* \cdots A_N^* A_1^*. \quad (2.12)$$

It follows that

$$\begin{aligned} |\Omega[A_1^*, \Gamma_N^*]| &= |\Omega[A_1^*, A_1^* A_2^* \cdots A_{i-1}^* A_{i+2}^* \cdots A_N^* A_1^*]| + |D(A_{i-1}^* A_i^* A_{i+1}^* A_{i+2}^* A_{i-1}^*)|, \\ |\Omega[A_1^*, \Gamma_N^{**}]| &= |\Omega[A_1^*, A_1^* A_2^* \cdots A_{i-1}^* A_{i+2}^* \cdots A_N^* A_1^*]| + |D(A_{i-1}^* A_i^{**} A_{i+1}^{**} A_{i+2}^* A_{i-1}^*)|. \end{aligned} \quad (2.13)$$

By Lemma 2.1, we have

$$|\Omega[A_1^*, \Gamma_N^{**}]| - |\Omega[A_1^*, \Gamma_N^*]| = |D(A_{i-1}^* A_i^{**} A_{i+1}^{**} A_{i+2}^* A_{i-1}^*)| - |D(A_{i-1}^* A_i^* A_{i+1}^* A_{i+2}^* A_{i-1}^*)| > 0. \quad (2.14)$$

This contradicts the greatest $|\Omega[A_1^*, \Gamma_N^*]|$. Since three points confirm a unique circle and for any $i \in \{2, 3, \dots, N-1\}$, we know $A_1^*, A_2^*, \dots, A_{N-1}^*, A_N^*$ are on a common circle if $|\Omega[A_1^*, \Gamma_N^*]|$ is greatest, and since

$$\Delta A_{i-1}^* A_1^* A_i^* \cap \Delta A_i^* A_1^* A_{i+1}^* = A_1^* A_i^*, \quad i = 3, 4, \dots, N-1, \quad (2.15)$$

Γ_N^* is inscribed in a circle and $|\Omega[P, \Gamma_N]| = |\Omega[A_1^*, \Gamma_N^*]| = |D(\Gamma_N^*)|$. Otherwise, there exists $k \in \{2, 3, \dots, N-1\}$ such that $\Delta A_{k-1}^* A_1^* A_k^* \cap \Delta A_k^* A_1^* A_{k+1}^* \neq A_1^* A_k^*$. When the perimeter of a circle is a fixed value, the area of regular polygon with N sides is greatest in all N -polygons inscribed in the circle [16]. Therefore, Γ_N^* is a regular polygon with N sides if $|\Omega[A_1^*, \Gamma_N^*]|$ is greatest.

Now let Γ_N^* be a regular polygon with N sides. By the plane geometry, we know

$$|\Omega[A_1^*, \Gamma_N^*]| = |D(\Gamma_N^*)| = \frac{|\Gamma_N^*|^2}{4N} \cot \frac{\pi}{N} = \frac{|\Gamma_N|^2}{4N} \cot \frac{\pi}{N}. \quad (2.16)$$

Hence, the inequality holds in (2.8). Equality occurs if Γ_N^* is a regular N -polygon. This completes the proof. $\square$

Lemma 2.3. *For the quadrilateral $ABCD$ in R^3 , we have*

$$|\overrightarrow{AC}|^2 + |\overrightarrow{BD}|^2 \leq |\overrightarrow{BC}|^2 + |\overrightarrow{DA}|^2 + 2|\overrightarrow{AB}| \cdot |\overrightarrow{CD}|, \quad (2.17)$$

with equality if and only if $\angle(\overrightarrow{AB}, \overrightarrow{CD}) = \pi$.

Proof. Write $(\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DA}, \overrightarrow{AC}, \overrightarrow{BD}) = (a, b, c, d, e, f)$, $a \cdot b := \langle a, b \rangle$ denotes the inner product of vectors a and b , especially, $a^2 = a \cdot a = |a|^2$. Thus, (2.17) is expressed as

$$e^2 + f^2 \leq b^2 + d^2 + 2|a| \cdot |c|. \quad (2.18)$$

Since

$$\begin{cases} a + b = e \\ c + d = -e \\ b + c = f \\ d + a = -f \end{cases} \implies \begin{cases} 2a \cdot b = e^2 - a^2 - b^2 \\ 2c \cdot d = e^2 - c^2 - d^2 \\ 2b \cdot c = f^2 - b^2 - c^2 \\ 2d \cdot a = f^2 - d^2 - a^2 \end{cases} \quad (2.19)$$

and $\sum a := a + b + c + d = 0$, which implies that

$$\begin{aligned} 0 &= \left(\sum a \right)^2 = a^2 + b^2 + c^2 + d^2 + 2(a \cdot b + c \cdot d + b \cdot c + d \cdot a) + 2(a \cdot c + b \cdot d) \\ &= -(a^2 + b^2 + c^2 + d^2) + 2(e^2 + f^2) + 2(a \cdot c + b \cdot d) \\ &= -2(a^2 + b^2 + c^2 + d^2) + 2(e^2 + f^2) + (a + c)^2 + (b + d)^2 \\ &= -2(a^2 + b^2 + c^2 + d^2) + 2(e^2 + f^2) + 2(a + c)^2 \\ &= -2(b^2 + d^2) + 2(e^2 + f^2) + 4a \cdot c, \end{aligned} \quad (2.20)$$

it follows that

$$e^2 + f^2 = b^2 + d^2 - 2a \cdot c \leq b^2 + d^2 + 2|a| \cdot |c|. \quad (2.21)$$

(2.18) is proved. Equality of (2.18) holds if and only if $\angle(\overrightarrow{AB}, \overrightarrow{CD}) = \angle(a, c) = \pi$. This completes the proof. $\square$

Lemma 2.4. *Let Γ_N ($N \geq 4$) be a polygon with sides N in R^3 . Setting*

$$S_k := \sin \frac{k\pi}{N}, \quad L_k := \left[N(2S_k)^2 \right]^{-1} \sum_{i=1}^N \left| \overrightarrow{A_i A_{i+k}} \right|^2, \quad k = 1, 2, \dots, N-1, \quad (2.22)$$

we get the following inequality:

$$L_k \leq \left(\frac{S_1}{S_k} \right)^2 L_1 + \frac{S_{k-1} S_{k+1}}{2S_k^2} (L_{k+1} + L_{k-1}), \quad (2.23)$$

for $2 \leq k \leq N-2$. A sufficient condition of equality is that Γ_N is a plane regular N -polygon in R^3 .

Proof. Consider the quadrilateral $A_{i-1} A_i A_{i-1+k} A_{i+k}$. From

$$\left(S_{k-1}^{-1} \left| \overrightarrow{A_i A_{i-1+k}} \right| - S_{k+1}^{-1} \left| \overrightarrow{A_{i-1} A_{i+k}} \right| \right)^2 \geq 0, \quad (2.24)$$

we obtain

$$2 \left| \overrightarrow{A_i A_{i-1+k}} \right| \cdot \left| \overrightarrow{A_{i-1} A_{i+k}} \right| \leq \frac{S_{k+1}}{S_{k-1}} \left| \overrightarrow{A_i A_{i-1+k}} \right|^2 + \frac{S_{k-1}}{S_{k+1}} \left| \overrightarrow{A_{i-1} A_{i+k}} \right|^2. \quad (2.25)$$

It follows from Lemma 2.3 and (2.25) that

$$\begin{aligned} \left| \overrightarrow{A_i A_{i+k}} \right|^2 + \left| \overrightarrow{A_{i-1} A_{i-1+k}} \right|^2 &\leq \left| \overrightarrow{A_{i-1} A_i} \right|^2 + \left| \overrightarrow{A_{i-1+k} A_{i+k}} \right|^2 + 2 \left| \overrightarrow{A_i A_{i-1+k}} \right| \left| \overrightarrow{A_{i-1} A_{i+k}} \right| \\ &\leq \left| \overrightarrow{A_{i-1} A_i} \right|^2 + \left| \overrightarrow{A_{i-1+k} A_{i+k}} \right|^2 + \frac{S_{k+1}}{S_{k-1}} \left| \overrightarrow{A_i A_{i-1+k}} \right|^2 + \frac{S_{k-1}}{S_{k+1}} \left| \overrightarrow{A_{i-1} A_{i+k}} \right|^2, \end{aligned} \quad (2.26)$$

which implies that

$$\begin{aligned} &\sum_{i=1}^N \left(\left| \overrightarrow{A_i A_{i+k}} \right|^2 + \left| \overrightarrow{A_{i-1} A_{i-1+k}} \right|^2 \right) \\ &\leq \sum_{i=1}^N \left(\left| \overrightarrow{A_{i-1} A_i} \right|^2 + \left| \overrightarrow{A_{i-1+k} A_{i+k}} \right|^2 + \frac{S_{k+1}}{S_{k-1}} \left| \overrightarrow{A_i A_{i-1+k}} \right|^2 + \frac{S_{k-1}}{S_{k+1}} \left| \overrightarrow{A_{i-1} A_{i+k}} \right|^2 \right). \end{aligned} \quad (2.27)$$

Since

$$\begin{aligned}
\sum_{i=1}^N \left| \overrightarrow{A_i A_{i+k}} \right|^2 &= \sum_{i=1}^N \left| \overrightarrow{A_{i-1} A_{i-1+k}} \right|^2 = 4NS_k^2 L_k, \\
\sum_{i=1}^N \left| \overrightarrow{A_{i-1} A_i} \right|^2 &= \sum_{i=1}^N \left| \overrightarrow{A_{i-1+k} A_{i+k}} \right|^2 = 4NS_1^2 L_1, \\
\sum_{i=1}^N \left| \overrightarrow{A_i A_{i-1+k}} \right|^2 &= 4NS_{k-1}^2 L_{k-1}, \quad \sum_{i=1}^N \left| \overrightarrow{A_{i-1} A_{i+k}} \right|^2 = 4NS_{k+1}^2 L_{k+1},
\end{aligned} \tag{2.28}$$

the inequality (2.27) is equivalent to

$$\begin{aligned}
8NS_k^2 L_k &\leq 8NS_1^2 L_1 + 4NS_{k+1} S_{k-1} L_{k-1} + 4NS_{k-1} S_{k+1} L_{k+1} \\
&\iff L_k \leq \left(\frac{S_1}{S_k} \right)^2 L_1 + \frac{S_{k-1} S_{k+1}}{2S_k^2} (L_{k+1} + L_{k-1}).
\end{aligned} \tag{2.29}$$

Inequality (2.23) is proved. From this proof and Lemma 2.3, we know that a sufficient condition of equality is that Γ_N is a regular polygon with N sides in R^3 . $\square$

Remark 2.5. A sufficient condition of equality of (2.23) is that Γ_N is a regular polygon with N sides in R^3 . This condition is not necessary. For example, when $N = 4$, the equality holds in (2.23) if and only if Γ_4 is a parallelogram in R^3 .

Remark 2.6. If Γ_N is a regular N -polygon, L_k defined by Lemma 2.4 is equal to R_0^2 , where R_0 denotes the circumradius of Γ_N . Namely,

$$L_k = R_0^2, \quad k = 1, 2, \dots, N-1. \tag{2.30}$$

Lemma 2.7. Suppose L_k is defined by Lemma 2.4, for any positive integer $k, j : k \geq 2, k+j \leq N-1$, there exist constants $C_{k+j,j}, C_{k-1,j}, C_{1,j}$ which is only related to k, j, N such that

$$\begin{aligned}
L_k &\leq C_{k+j,j} L_{k+j} + C_{k-1,j} L_{k-1} + C_{1,j} L_1, \\
C_{k+j,j} + C_{k-1,j} + C_{1,j} &= 1.
\end{aligned} \tag{2.31}$$

A sufficient condition of equality is that Γ_N is a regular N -polygon in R^3 .

Proof. We prove it by mathematical induction with respect to j .

(i) When $j = 1$, let $C_{k+1,1} = C_{k-1,1} = S_{k-1} S_{k+1} / (2S_k^2) (> 0)$, $C_{1,1} = (S_1/S_k)^2 (> 0)$ from Lemma 2.4, we have

$$L_k \leq C_{k+1,1} L_{k+1} + C_{k-1,1} L_{k-1} + C_{1,1} L_1. \tag{2.32}$$

Let Γ_N be a regular N -polygon. By Remark 2.6, we know $L_k = L_{k+1} = L_{k-1} = L_1 = R_0^2 > 0$; it follows from Lemma 2.4 that the equality of (2.32) holds, thus, $C_{k+1,1} + C_{k-1,1} + C_{1,1} = 1$.

(ii) Assume that Lemma 2.7 holds for $j = n \geq 1$. Now we want to prove that Lemma 2.7 holds for $j = n + 1$. By the hypothesis $k, n + 1 : k \geq 2, k + n + 1 \leq N - 1$, we know that $k, n : k \geq 2, k + n \leq N - 2 \leq N - 1$. Thus, from the induction hypothesis, there exist constants $C_{k+n,n}, C_{k-1,n}, C_{1,n}$ such that

$$C_{k+n,n} + C_{k-1,n} + C_{1,n} = 1, \quad (2.33)$$

$$L_k \leq C_{k+n,n}L_{k+n} + C_{k-1,n}L_{k-1} + C_{1,n}L_1. \quad (2.34)$$

A sufficient condition of equality of (2.34) is that Γ_N is a regular polygon with N sides. Since $k + 1, n : k + 1 \geq 3 \geq 2, (k + 1) + n \leq N - 1$, by induction hypothesis, substitute k by $k + 1$ in (2.33) and (2.34), in other words, there exist constants $C_{k+1+n,n}^*, C_{k,n}^*, C_{1,n}^*$ such that

$$C_{k+1+n,n}^* + C_{k,n}^* + C_{1,n}^* = 1, \quad (2.35)$$

$$L_{k+1} \leq C_{k+1+n,n}^*L_{k+1+n} + C_{k,n}^*L_k + C_{1,n}^*L_1. \quad (2.36)$$

Substituting (2.36) into (2.32), we get the following inequality:

$$L_k \leq C_{k+1,1} \left(C_{k+1+n,n}^*L_{k+1+n} + C_{k,n}^*L_k + C_{1,n}^*L_1 \right) + C_{k-1,1}L_{k-1} + C_{1,1}L_1. \quad (2.37)$$

Notice $0 < C_{k+1,1} < 1, 0 < C_{k,n}^* < 1, 1 - C_{k+1,1}C_{k,n}^* > 0$. Solving inequality (2.37) with respect to L_k , we obtain

$$L_k \leq C_{k+n+1,n+1}^{**}L_{k+n+1} + C_{k-1,n+1}^{**}L_{k-1} + C_{1,n+1}^{**}L_1, \quad (2.38)$$

where

$$\begin{aligned} C_{k+n+1,n+1}^{**} &= \frac{C_{k+1,1}C_{k+n+1,n}^*}{1 - C_{k+1,1}C_{k,n}^*} > 0, \\ C_{k-1,n+1}^{**} &= \frac{C_{k-1,1}}{1 - C_{k+1,1}C_{k,n}^*} > 0, \\ C_{1,n+1}^{**} &= \frac{C_{k+1,1}C_{1,n}^* + C_{1,1}}{1 - C_{k+1,1}C_{k,n}^*} > 0. \end{aligned} \quad (2.39)$$

Let Γ_N be a regular N -polygon. From Remark 2.6, we know $L_k = L_{k+n+1} = L_{k-1} = L_1 = R_0^2 > 0$. It follows from Lemma 2.4 and induction hypothesis that the equality of (2.38) holds. Thus, $C_{k+n+1,n+1}^{**} + C_{k-1,n+1}^{**} + C_{1,n+1}^{**} = 1$, that is, Lemma 2.7 holds for $j = n + 1$. This completes the proof. $\square$

Lemma 2.8. Suppose L_k is defined by Lemma 2.4, we have the inequality as follows:

$$L_k \leq L_1, \quad k = 2, 3, \dots, N-2. \quad (2.40)$$

A sufficient condition of equality is that Γ_N is a regular polygon with N sides in R^3 .

Proof. Setting $k + j = N - 1$ in (2.31), we get

$$L_k \leq C_{N-1, N-1-k} L_{N-1} + C_{k-1, N-1-k} L_{k-1} + C_{1, N-1-k} L_1. \quad (2.41)$$

Since $A_i = A_j \Leftrightarrow i \equiv j \pmod{N}$,

$$L_{N-1} := \left[N(2S_{N-1})^2 \right]^{-1} \sum_{i=1}^{N-1} \left| \overrightarrow{A_i A_{i+N-1}} \right|^2 = \left[N(2S_1)^2 \right]^{-1} \sum_{i=1}^{N-1} \left| \overrightarrow{A_i A_{i-1}} \right|^2 = L_1. \quad (2.42)$$

It follows from (2.41) and (2.42) that there exist constants $C_{k-1}, C_1 : C_{k-1} + C_1 = 1$ such that

$$L_k \leq C_{k-1} L_{k-1} + C_1 L_1, \quad (2.43)$$

where $C_{k-1} = C_{k-1, N-1-k} > 0$, $C_1 = C_{N-1, N-1-k} + C_{1, N-1-k} > 0$, for any $k \in \{2, 3, \dots, N-2\}$. Using (2.43) repeatedly, we get

$$\begin{aligned} L_k &\leq C_{k-1} L_{k-1} + C_1 L_1 \leq C_{k-1} (C_{k-2}^* L_{k-2} + C_1^* L_1) + C_1 L_1 \\ &= C_{k-2}^{**} L_{k-2} + C_1^{**} L_1 \leq C_{k-3}^{***} L_{k-3} + C_1^{***} L_1 \leq \dots \leq C L_1, \end{aligned} \quad (2.44)$$

therefore,

$$L_k \leq C L_1. \quad (2.45)$$

Set Γ_N is a regular polygon with N sides in R^3 . By Remark 2.6, we know $L_k = L_1 = R_0^2 > 0$; from Lemma 2.7, we have the equality of (2.45) holds, which implies that $C = 1$. A sufficient condition of equality of (2.40) is that Γ_N is a regular N -polygon in R^3 . This completes the proof. $\square$

Lemma 2.9. Let $\Gamma = \Gamma(AB) : r = r(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} (a \leq t \leq b)$ be a smooth curve with the end points A and B in R^3 . If the function $f : \Gamma(AB) \rightarrow R$ is Riemann integrable on $\Gamma(AB)$, considering a partition of $\Gamma(AB)$ by means of $N-1$ ($N \geq 3$) points $A = A_0, A_1, \dots, A_{i-1}, A_i, \dots, A_N = B$ such that $|\overrightarrow{A_0 A_1}| = |\overrightarrow{A_1 A_2}| = \dots = |\overrightarrow{A_{i-1} A_i}| = \dots = |\overrightarrow{A_{N-1} A_N}| = |\Gamma_N|/N$, where $\Gamma_N : A_0 A_1 \dots A_{N-1} A_N$ is a broken line, we have

$$M(f, \Gamma(AB)) := \frac{1}{|\Gamma(AB)|} \int_{\Gamma(AB)} f ds = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(A_i). \quad (2.46)$$

Proof. Since $\Gamma(AB)$ is a smooth curve with the end points A and B in R^3 ,

$$|\Gamma(AB)| = \int_a^b \sqrt{x'^2(t) + y'^2(t) + z'^2(t)} dt \quad (2.47)$$

exists and $|\Gamma(AB)| > 0$. And $f : \Gamma(AB) \rightarrow R$ is Riemann integrable on $\Gamma(AB)$, it follows that $f : \Gamma(AB) \rightarrow R$ is bounded on $\Gamma(AB)$, that is, there exists a constant $M_1 > 0$ such that $|f(P)| \leq M_1$ for any $P \in \Gamma(AB)$. For any $i \in \{0, 1, 2, \dots, N-1\}$,

$$\lim_{N \rightarrow \infty} |\Gamma(A_i A_{i+1})| = \lim_{N \rightarrow \infty} \left| \overrightarrow{A_i A_{i+1}} \right| = \lim_{N \rightarrow \infty} \frac{|\Gamma_N|}{N} = \frac{|\Gamma(AB)|}{\lim_{N \rightarrow \infty} N} = 0. \quad (2.48)$$

It follows from the definition of the curve integral

$$\int_{\Gamma(AB)} f ds = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(P_i) |\Gamma(A_i A_{i+1})|, \quad (\forall P_i \in \Gamma(A_i A_{i+1})) \quad (2.49)$$

that

$$\int_{\Gamma(AB)} f ds = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(A_i) |\Gamma(A_i A_{i+1})|. \quad (2.50)$$

Since $\lim_{N \rightarrow \infty} (|\Gamma(A_i A_{i+1})| / |\overrightarrow{A_i A_{i+1}}|) = 1 (\forall i \in \{0, 1, 2, \dots, N-1\})$, for any $\varepsilon > 0$, there exists $m \in \mathbb{Z} (m > 1)$ such that $1 \leq |\Gamma(A_i A_{i+1})| / |\overrightarrow{A_i A_{i+1}}| < 1 + \varepsilon$, for $N > m$, therefore

$$0 \leq |\Gamma(A_i A_{i+1})| - |\overrightarrow{A_i A_{i+1}}| < \varepsilon |\overrightarrow{A_i A_{i+1}}| = \frac{\varepsilon |\Gamma_N|}{N} \leq \frac{\varepsilon |\Gamma(AB)|}{N}. \quad (2.51)$$

Thus,

$$\begin{aligned} \left| \sum_{i=0}^{N-1} f(A_i) |\Gamma(A_i A_{i+1})| - \sum_{i=0}^{N-1} f(A_i) |\overrightarrow{A_i A_{i+1}}| \right| &= \left| \sum_{i=0}^{N-1} f(A_i) (|\Gamma(A_i A_{i+1})| - |\overrightarrow{A_i A_{i+1}}|) \right| \\ &\leq \sum_{i=0}^{N-1} |f(A_i)| \cdot (|\Gamma(A_i A_{i+1})| - |\overrightarrow{A_i A_{i+1}}|) \\ &\leq \sum_{i=0}^{N-1} M_1 \cdot \frac{\varepsilon |\Gamma(AB)|}{N} \\ &= M_1 |\Gamma(AB)| \varepsilon. \end{aligned} \quad (2.52)$$

It follows that

$$\lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(A_i) |\Gamma(A_i A_{i+1})| = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(A_i) |\overrightarrow{A_i A_{i+1}}|. \quad (2.53)$$

In view of (2.50), we get

$$\begin{aligned}
 M(f, \Gamma(AB)) &:= \frac{1}{|\Gamma(AB)|} \int_{\Gamma} f ds = \frac{1}{|\Gamma(AB)|} \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(A_i) |\Gamma(A_i A_{i+1})| \\
 &= \frac{1}{|\Gamma(AB)|} \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(A_i) \left| \overrightarrow{A_i A_{i+1}} \right| = \lim_{N \rightarrow \infty} \frac{1}{|\Gamma(AB)|} \sum_{i=0}^{N-1} f(A_i) \frac{|\Gamma_N|}{N} \quad (2.54) \\
 &= \lim_{N \rightarrow \infty} \frac{|\Gamma_N|}{|\Gamma(AB)|} \cdot \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(A_i) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(A_i).
 \end{aligned}$$

This completes the proof of Lemma 2.9. $\square$

Lemma 2.10. *Let Γ be a smooth closed curve in R^3 . If the points A, B move continuously on Γ and keep $|\Gamma(AB)|$ invariable, where $0 < l := |\Gamma(AB)| < |\Gamma|/2$, we obtain*

$$M^{[2]}(\overrightarrow{AB}, \Gamma) := \sqrt{\frac{1}{|\Gamma|} \oint_{\Gamma} \left| \overrightarrow{AB} \right|^2 \cdot ds} \leq \frac{|\Gamma|}{\pi} \sin \frac{l\pi}{|\Gamma|}. \quad (2.55)$$

A sufficient condition of equality is that Γ is a circle in R^3 .

Proof. By the theory of real number, there exists an increasing sequence of positive integers $\{k_n\}$ such that

$$\lim_{n \rightarrow \infty} \frac{k_n}{10^n} = \frac{l}{|\Gamma|}. \quad (2.56)$$

Write $k := k_n$, $N := 10^n$. Consider a partition of Γ by means of N ($N \geq 10$) points $A_1, A_2, \dots, A_{i-1}, A_i, \dots, A_N$ such that $|\overrightarrow{A_1 A_2}| = |\overrightarrow{A_2 A_3}| = \dots = |\overrightarrow{A_{N-1} A_N}| = |\overrightarrow{A_N A_1}| = |\Gamma_N|/N$, where $\Gamma_N : A_1 A_2 \dots A_{N-1} A_N A_1$ is a polygon with N sides.

First, we give the following fact: if the point A moves to some point A_i along Γ , the point B moves to A_i^* along Γ , in other words, $A = A_i \in \Gamma$, $B = A_i^* \in \Gamma$, $|\Gamma(A_i A_i^*)| = l$, then

$$\lim_{n \rightarrow \infty} \left| \overrightarrow{A_i^* A_{i+k}} \right| = 0. \quad (2.57)$$

In fact, from $\lim_{n \rightarrow \infty} \Gamma_N(A_i A_{i+k}) \subset \Gamma$, $|\Gamma(A_i A_i^*)| = l$, we only need to prove

$$\lim_{n \rightarrow \infty} |\Gamma_N(A_i A_{i+k})| = l. \quad (2.58)$$

Since

$$\begin{aligned} \lim_{n \rightarrow \infty} |\Gamma_N(A_i A_{i+k})| &= \lim_{n \rightarrow \infty} \left(k \left| \overrightarrow{A_i A_{i+1}} \right| \right) = \lim_{n \rightarrow \infty} \left(k \frac{|\Gamma_N|}{N} \right) \\ &= \lim_{n \rightarrow \infty} (|\Gamma_N|) \lim_{n \rightarrow \infty} \frac{k}{N} = |\Gamma| \frac{l}{|\Gamma|} = l, \end{aligned} \quad (2.59)$$

(2.58) holds, it follows that (2.57) holds.

From (2.57), we know for any $\varepsilon > 0$, there exists $m \in \mathbb{Z} (m > 1)$, when $N > m$, such that

$$\begin{aligned} \left| \left| \overrightarrow{A_i A_i^*} \right|^2 - \left| \overrightarrow{A_i A_{i+k}} \right|^2 \right| &< \varepsilon, \\ \left| \frac{1}{N} \sum_{i=1}^N \left| \overrightarrow{A_i A_i^*} \right|^2 - \frac{1}{N} \sum_{i=1}^N \left| \overrightarrow{A_i A_{i+k}} \right|^2 \right| &= \frac{1}{N} \left| \sum_{i=1}^N \left(\left| \overrightarrow{A_i A_i^*} \right|^2 - \left| \overrightarrow{A_i A_{i+k}} \right|^2 \right) \right| \\ &\leq \frac{1}{N} \sum_{i=1}^N \left| \left| \overrightarrow{A_i A_i^*} \right|^2 - \left| \overrightarrow{A_i A_{i+k}} \right|^2 \right| < \frac{1}{N} \sum_{i=1}^N \varepsilon = \varepsilon. \end{aligned} \quad (2.60)$$

By Lemma 2.9, we get

$$\frac{1}{|\Gamma|} \oint_{\Gamma} \left| \overrightarrow{AB} \right|^2 ds = \frac{1}{|\Gamma|} \oint_{\Gamma} \left| \overrightarrow{AB} \right|^2 ds_A = \lim_{n \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \left| \overrightarrow{A_i A_i^*} \right|^2 = \lim_{n \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \left| \overrightarrow{A_i A_{i+k}} \right|^2. \quad (2.61)$$

It follows from inequality (2.40) that

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \left| \overrightarrow{A_i A_{i+k}} \right|^2 &\leq \frac{1}{N} \left(\frac{\sin(k\pi/N)}{\sin(\pi/N)} \right)^2 \sum_{i=1}^N \left| \overrightarrow{A_i A_{i+1}} \right|^2 = \frac{1}{N} \left(\frac{\sin(k\pi/N)}{\sin(\pi/N)} \right)^2 \sum_{i=1}^N \left(\frac{|\Gamma_N|}{N} \right)^2 \\ &= \frac{1}{N} \left(\frac{\sin(k\pi/N)}{\sin(\pi/N)} \right)^2 \frac{|\Gamma_N|^2}{N} = \left(\frac{\sin(k\pi/N)}{N \sin(\pi/N)} \right)^2 |\Gamma_N|^2. \end{aligned} \quad (2.62)$$

It follows from (2.61) and (2.62) that

$$\begin{aligned} \frac{1}{|\Gamma|} \oint_{\Gamma} \left| \overrightarrow{AB} \right|^2 ds &= \lim_{n \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \left| \overrightarrow{A_i A_{i+k}} \right|^2 \leq \lim_{n \rightarrow \infty} \left(\frac{\sin(k\pi/N)}{N \sin(\pi/N)} \right)^2 |\Gamma_N|^2 \\ &= \left(\frac{\sin \lim_{n \rightarrow \infty} (k\pi/N)}{\lim_{n \rightarrow \infty} N \sin(\pi/N)} \right)^2 \lim_{n \rightarrow \infty} |\Gamma_N|^2 = \left(\frac{\sin(l\pi/(|\Gamma|))}{\pi} \right)^2 |\Gamma|^2 = \left(\frac{|\Gamma|}{\pi} \sin \frac{l\pi}{|\Gamma|} \right)^2. \end{aligned} \quad (2.63)$$

Namely, (2.55) holds. From Lemma 2.8, a sufficient condition of equality of (2.55) is that Γ is a circle in R^3 . This completes the proof. $\square$

Lemma 2.11 (Cauchy inequality [1, page 6]). *Let Γ be a smooth closed curve in R^3 . If the function $f : \Gamma \rightarrow R$ and $g : \Gamma \rightarrow R$ are smooth, we have the inequality as follows:*

$$\left| \oint_{\Gamma} f \cdot g ds \right| \leq \sqrt{\oint_{\Gamma} f^2 ds} \cdot \sqrt{\oint_{\Gamma} g^2 ds}. \quad (2.64)$$

Lemma 2.12. *Assume Γ is a closed Jordan curve in R^3 and $O \in R^3$, then $D(\Gamma) \subset \Omega[O, \Gamma]$.*

Proof. Let $A, B \in R^3, A \neq B$, AB denote the straight line through the points A and B , and let $\overline{AB}$ denote the oriented line segment with the initial point A and terminal point B .

For any $P \in D(\Gamma)$, if $P = O$, then $P \in \Omega[O, \Gamma]$; if $P \neq O$, then

$$\begin{aligned} OP \cap D(\Gamma) &= \bigcup_{i \equiv 1 \pmod{2}, 1 \leq i \leq k-1} \overline{Q_i Q_{i+1}}, \quad Q_i \in \Gamma, \quad i = 1, 2, \dots, k, \quad k \geq 2, \\ P \in OP \cap D(\Gamma) &\implies \exists i : i \equiv 1 \pmod{2}, \quad 1 \leq i \leq k-1, \quad P \in \overline{Q_i Q_{i+1}}. \end{aligned} \quad (2.65)$$

If $O \in \overline{Q_i Q_{i+1}}$, then

$$\begin{aligned} P \in \overline{Q_i Q_{i+1}} &= \overline{Q_i O} \cup \overline{O Q_{i+1}} \implies P \in \overline{Q_i O} \subset \Omega[O, \Gamma] \vee P \in \overline{O Q_{i+1}} \subset \Omega[O, \Gamma] \\ &\implies P \in \Omega[O, \Gamma]. \end{aligned} \quad (2.66)$$

If $O \notin \overline{Q_i Q_{i+1}}$, without loss of generality, we set $|O Q_i| \leq |O Q_{i+1}|$, then

$$P \in \overline{Q_i Q_{i+1}} \subset \overline{O Q_{i+1}} \subset \Omega[O, \Gamma] \implies P \in \Omega[O, \Gamma]. \quad (2.67)$$

In all, for any $P \in D(\Gamma) \implies P \in \Omega[O, \Gamma]$, hence, $D(\Gamma) \subset \Omega[O, \Gamma]$. This completes the proof. $\square$

Lemma 2.13. *For $S\{O, \Gamma, \Gamma_N\}$, we have the following inequality:*

$$M_N^{[-2]}(r) := \left(\frac{1}{N} \sum_{i=1}^N r_i^{-2} \right)^{-1/2} \leq \frac{1}{2} \sqrt{\frac{1}{N} \sum_{i=1}^N |\overrightarrow{A_i A_{i+1}}|^2} \cdot \cot \frac{\pi}{N}, \quad (2.68)$$

with equality if and only if Γ_N is a regular N -polygon in R^3 and O is its central.

Proof. By Definition 1.5 and Lemma 2.12, for any $P \in V(\Gamma_N)$, we have $\Omega[O, \Gamma_N] = D(\Gamma_N) \subset \Omega[P, \Gamma_N]$. Thus,

$$\sum_{i=1}^N r_i |\overrightarrow{A_i A_{i+1}}| = 2|\Omega[O, \Gamma_N]| = 2|D(\Gamma_N)| \leq 2|\Omega[P, \Gamma_N]|. \quad (2.69)$$

Using Cauchy inequality $(\sum_{i=1}^N a_i b_i)^2 \leq (\sum_{i=1}^N a_i^2)(\sum_{i=1}^N b_i^2)$, (2.69) and (2.8), we get

$$\begin{aligned}
 |\Gamma_N|^2 &= \left(\sum_{i=1}^N |\overrightarrow{A_i A_{i+1}}| \right)^2 = \left(\sum_{i=1}^N \sqrt{r_i} |\overrightarrow{A_i A_{i+1}}| \cdot \sqrt{\frac{|\overrightarrow{A_i A_{i+1}}|}{r_i}} \right)^2 \\
 &\leq \left(\sum_{i=1}^N r_i |\overrightarrow{A_i A_{i+1}}| \right) \left(\sum_{i=1}^N \frac{|\overrightarrow{A_i A_{i+1}}|}{r_i} \right) \leq \left(\sum_{i=1}^N r_i |\overrightarrow{A_i A_{i+1}}| \right) \sqrt{\sum_{i=1}^N |\overrightarrow{A_i A_{i+1}}|^2} \sqrt{\sum_{i=1}^N r_i^{-2}} \\
 &\leq 2|\Omega[P, \Gamma_N]| \sqrt{\sum_{i=1}^N |\overrightarrow{A_i A_{i+1}}|^2} \sqrt{\sum_{i=1}^N r_i^{-2}} \leq \frac{|\Gamma_N|^2}{2N} \cot \frac{\pi}{N} \sqrt{\sum_{i=1}^N |\overrightarrow{A_i A_{i+1}}|^2} \sqrt{\sum_{i=1}^N r_i^{-2}} \\
 &= \frac{|\Gamma_N|^2}{2} \cot \frac{\pi}{N} \sqrt{\frac{1}{N} \sum_{i=1}^N |\overrightarrow{A_i A_{i+1}}|^2} [M_N^{[-2]}(r)]^{-1}.
 \end{aligned} \tag{2.70}$$

Therefore, (2.68) holds. The equality occurs if and only if Γ_N is a regular N -polygon in R^3 and O is its central. This completes the proof. $\square$

Lemma 2.14 (Power Mean Inequality, [1, 7, 10, 11]). *Let Γ be a smooth closed curve in R^3 . If the function $f : \Gamma \rightarrow]0, \infty[$ is smooth, we have the following inequality, for $p, q \in R$, $p < q$,*

$$M^{[p]}(f, \Gamma) \leq M^{[q]}(f, \Gamma). \tag{2.71}$$

Lemma 2.15. *For $S\{\Gamma, \Gamma_N\}$, we have*

$$\frac{1}{N} \sum_{i=1}^N \sin \frac{l_i \pi}{|\Gamma|} \leq \sin \frac{\pi}{N}. \tag{2.72}$$

If $N = 3$ or $N \geq 4$, $0 < l_i \leq |\Gamma|/4$, $i = 1, 2, \dots, N$, we get the inequality as follows:

$$\frac{1}{N} \sum_{i=1}^N \left(\sin \frac{l_i \pi}{|\Gamma|} \right)^2 \leq \left(\sin \frac{\pi}{N} \right)^2. \tag{2.73}$$

The equalities hold in (2.72) and (2.73) if and only if $l_1 = l_2 = \dots = l_N = |\Gamma|/N$.

Proof. By Definition 1.4, we know $\sum_{i=1}^N l_i = |\Gamma|$. Since $\sin x$ is the concave function on $]0, \pi/2[$, it follows from Jensen inequality [17, 18] and $0 < l_i < |\Gamma|/2$ ($i = 1, 2, \dots, N$, $N \geq 3$) that (2.72) holds. When $N = 3$, $l_1 \pi / |\Gamma|$, $l_2 \pi / |\Gamma|$, $l_3 \pi / |\Gamma|$ are three inner angles of a triangle. See [1, page 205], (2.73) holds. Since $(\sin x)^2$ is the concave function on $]0, \pi/4[$, from $N \geq 4$, $0 < l_i \leq |\Gamma|/4$, $i = 1, 2, \dots, N$, (2.73) holds. In view of the condition of equality of Jensen inequality, the equalities of (2.72) and (2.73) hold if and only if $l_1 = l_2 = \dots = l_N = |\Gamma|/N$. This completes the proof. $\square$

3. Proof of Theorem 1.9

By (2.8),(2.64),(2.55), and (2.72) in order, we get

$$\begin{aligned}
& \oint_{\Gamma} |\Omega[P, \Gamma_N]| ds \\
& \leq \oint_{\Gamma} \frac{|\Gamma_N|^2}{4N} \cot \frac{\pi}{N} ds \\
& = \frac{1}{4N} \cot \frac{\pi}{N} \oint_{\Gamma} |\Gamma_N|^2 ds = \frac{1}{4N} \cot \frac{\pi}{N} \oint_{\Gamma} \left(\sum_{i=1}^N |\overrightarrow{A_i A_{i+1}}| \right)^2 ds \\
& = \frac{1}{4N} \cot \frac{\pi}{N} \oint_{\Gamma} \left(\sum_{i=1}^N |\overrightarrow{A_i A_{i+1}}|^2 + 2 \sum_{1 \leq i < j \leq N} |\overrightarrow{A_i A_{i+1}}| |\overrightarrow{A_j A_{j+1}}| \right) ds \\
& = \frac{1}{4N} \cot \frac{\pi}{N} \left(\sum_{i=1}^N \oint_{\Gamma} |\overrightarrow{A_i A_{i+1}}|^2 ds + 2 \sum_{1 \leq i < j \leq N} \oint_{\Gamma} |\overrightarrow{A_i A_{i+1}}| |\overrightarrow{A_j A_{j+1}}| ds \right) \\
& \leq \frac{|\Gamma|}{4N} \cot \frac{\pi}{N} \left(\sum_{i=1}^N \frac{\oint_{\Gamma} |\overrightarrow{A_i A_{i+1}}|^2 ds}{|\Gamma|} + 2 \sum_{1 \leq i < j \leq N} \sqrt{\frac{\oint_{\Gamma} |\overrightarrow{A_i A_{i+1}}|^2 ds}{|\Gamma|}} \sqrt{\frac{\oint_{\Gamma} |\overrightarrow{A_j A_{j+1}}|^2 ds}{|\Gamma|}} \right) \\
& \leq \frac{|\Gamma|}{4N} \cot \frac{\pi}{N} \left[\sum_{i=1}^N \left(\frac{|\Gamma|}{\pi} \sin \frac{l_i \pi}{|\Gamma|} \right)^2 + 2 \sum_{1 \leq i < j \leq N} \left(\frac{|\Gamma|}{\pi} \sin \frac{l_i \pi}{|\Gamma|} \right) \left(\frac{|\Gamma|}{\pi} \sin \frac{l_j \pi}{|\Gamma|} \right) \right] \\
& = \frac{|\Gamma|^3}{4\pi^2 N} \cot \frac{\pi}{N} \left(\sum_{i=1}^N \sin \frac{l_i \pi}{|\Gamma|} \right)^2 = \frac{|\Gamma|^3 N}{4\pi^2} \cot \frac{\pi}{N} \left(\frac{1}{N} \sum_{i=1}^N \sin \frac{l_i \pi}{|\Gamma|} \right)^2 \\
& \leq \frac{|\Gamma|^3 N}{4\pi^2} \cot \frac{\pi}{N} \left(\sin \frac{\pi}{N} \right)^2 = \frac{|\Gamma|^3 N}{8\pi^2} \sin \frac{2\pi}{N},
\end{aligned} \tag{3.1}$$

which implies that

$$\begin{aligned}
M(|\Omega[P, \Gamma_N]|, \Gamma) &:= \frac{1}{|\Gamma|} \oint_{\Gamma} |\Omega[P, \Gamma_N]| ds \leq \frac{N|\Gamma|^2}{8\pi^2} \sin \frac{2\pi}{N}, \\
\oint_{\Gamma} ds \oint_{\Gamma_N} |PQ| ds_Q &= 2 \oint_{\Gamma} |\Omega[P, \Gamma_N]| ds \leq \frac{N|\Gamma|^3}{4\pi^2} \sin \frac{2\pi}{N}.
\end{aligned} \tag{3.2}$$

Hence, (1.3) and (1.4) are proved. By this proof, a sufficient condition of equalities is that Γ is a circle and Γ_N is always a regular polygon with N sides. This completes the proof of Theorem 1.9.

Remark 3.1. We may extend the concepts of the bounded cone surface and the satellite system without any central to R^m ($m \geq 2$). And Theorem 1.9 also holds for the satellite system without any central in R^m .

4. Proof of Theorem 1.10

Since power mean $M_N^{[t]}(r)$ is increasing with respect to t and by (2.68), we have the following inequalities, for $t \leq -2$,

$$M_N^{[t]}(r) \leq M_N^{[-2]}(r) \leq \frac{1}{2} \sqrt{\frac{1}{N} \sum_{i=1}^N \left| \overrightarrow{A_i A_{i+1}} \right|^2} \cdot \cot \frac{\pi}{N}. \quad (4.1)$$

Write $f := (1/N) \sum_{i=1}^N \left| \overrightarrow{A_i A_{i+1}} \right|^2$. By (4.1), (2.71), (2.72) and (2.73), we get

$$\begin{aligned} \oint_{\Gamma} M_N^{[t]}(r) ds &\leq \frac{1}{2} \cot \frac{\pi}{N} \oint_{\Gamma} \sqrt{\frac{1}{N} \sum_{i=1}^N \left| \overrightarrow{A_i A_{i+1}} \right|^2} ds = \frac{|\Gamma|}{2} \cot \frac{\pi}{N} \frac{1}{|\Gamma|} \oint_{\Gamma} f^{1/2} ds \\ &= \frac{|\Gamma|}{2} \cot \frac{\pi}{N} \sqrt{M^{[1/2]}(f, \Gamma)} \leq \frac{|\Gamma|}{2} \cot \frac{\pi}{N} \sqrt{M^{[1]}(f, \Gamma)} \\ &= \frac{|\Gamma|}{2} \cot \frac{\pi}{N} \sqrt{\frac{1}{|\Gamma|} \oint_{\Gamma} f ds} = \frac{|\Gamma|}{2} \cot \frac{\pi}{N} \sqrt{\frac{1}{|\Gamma|} \oint_{\Gamma} \frac{1}{N} \sum_{i=1}^N \left| \overrightarrow{A_i A_{i+1}} \right|^2 ds} \\ &= \frac{|\Gamma|}{2} \cot \frac{\pi}{N} \sqrt{\frac{1}{N} \sum_{i=1}^N \frac{1}{|\Gamma|} \oint_{\Gamma} \left| \overrightarrow{A_i A_{i+1}} \right|^2 ds} \leq \frac{|\Gamma|}{2} \cot \frac{\pi}{N} \sqrt{\frac{1}{N} \sum_{i=1}^N \left(\frac{|\Gamma|}{\pi} \cdot \sin \frac{l_i \pi}{|\Gamma|} \right)^2} \\ &= \frac{|\Gamma|^2}{2\pi} \cot \frac{\pi}{N} \sqrt{\frac{1}{N} \sum_{i=1}^N \left(\sin \frac{l_i \pi}{|\Gamma|} \right)^2} \leq \frac{|\Gamma|^2}{2\pi} \cot \frac{\pi}{N} \cdot \sin \frac{\pi}{N} = \frac{|\Gamma|^2}{2\pi} \cos \frac{\pi}{N}. \end{aligned} \quad (4.2)$$

Therefore, (1.5) and (1.6) are proved. From this proof, the equalities of (1.5) and (1.6) occur if and only if Γ is a circle, O is the central of the circle, and Γ_N is a regular polygon with sides N . This completes the proof of Theorem 1.10.

5. Applications

In Theorem 1.9, if both Γ and Γ_N are simple closed curves in R^3 , and $\Omega[\Gamma_N]$ denotes the minimal surface [4, 5, 19–21] bounded by Γ_N , then for any $P \in V(\Gamma_N)$, $|\Omega[\Gamma_N]| \leq |\Omega[P, \Gamma_N]|$, where $|\Omega[\Gamma_N]|$ denotes the area of $\Omega[\Gamma_N]$. Thus, we get the theorem as follows.

Theorem 5.1. For $S\{\Gamma, \Gamma_N\}$, if both the curve Γ and the polygon Γ_N are simple closed curves in R^3 , we have

$$M(|\Omega[\Gamma_N]|, \Gamma) := \frac{1}{|\Gamma|} \oint_{\Gamma} |\Omega[\Gamma_N]| ds \leq \frac{N|\Gamma|^2}{8\pi^2} \sin \frac{2\pi}{N}, \quad (5.1)$$

where $\Omega[\Gamma_N]$ is a minimal surface bounded by Γ_N and $|\Omega[\Gamma_N]|$ denotes its area. A sufficient condition of equality in (5.1) is that Γ is a circle and Γ_N is always a regular N -polygon.

In Theorem 5.1, let $V(\Gamma_N) = \{A_1, A_2, \dots, A_N\}$ and satisfy $|\Gamma(A_1A_2)| = |\Gamma(A_2A_3)| = \dots = |\Gamma(A_{N-1}A_N)| = |\Gamma(A_NA_1)| = (|\Gamma|)/N$. Setting $N \rightarrow \infty$, we get the theorem as follows.

Theorem 5.2. If the curve Γ is a smooth simple closed curve in R^3 , we have

$$|\Omega[\Gamma]| \leq \frac{|\Gamma|^2}{4\pi}, \quad (5.2)$$

where $|\Omega[\Gamma]|$ denotes the area of the minimal surface $\Omega[\Gamma]$ bounded by Γ . A sufficient condition of equality is that Γ is a circle.

Remark 5.3. If Γ is a smooth Jordan closed curve in R^3 , then $\Omega[\Gamma] = D(\Gamma)$, thus, (5.2) is the well-known isoperimetric inequality [16, 22–33].

Definition 5.4. If Γ is a smooth Jordan closed curve in R^3 , O is a fixed point in the interior of Γ and the point P moves continuously on Γ , then $S\{O, \Gamma, P\}$ is called the satellite system of single point with a central.

For $S\{O, \Gamma, P\}$, write $\rho_P := |\overrightarrow{OP}|$. l_P denotes the tangent line of Γ at the point P , and $r_P := \text{Dis}(O, l_P)$ denotes the distance from the point O to the tangent line l_P .

Theorem 5.5. For $S\{O, \Gamma, P\}$, we have

$$\oint_{\Gamma} \rho_P ds \geq \oint_{\Gamma} r_P ds \geq 2|D(\Gamma)|, \quad (5.3)$$

that is,

$$M(\rho_P, \Gamma) \geq M(r_P, \Gamma) \geq \frac{2|D(\Gamma)|}{|\Gamma|}. \quad (5.4)$$

The first equalities in (5.3) and (5.4) hold if and only if Γ is a circle and the point O is the central of the circle; the second equalities hold if and only if $\Omega[O, \Gamma] = D(\Gamma)$.

Proof. Insert the inscribed polygon $\Gamma_N : A_1A_2 \cdots A_NA_1$ ($N \geq 3$) such that

$$|\overrightarrow{A_1A_2}| = |\overrightarrow{A_2A_3}| = \dots = |\overrightarrow{A_{N-1}A_N}| = |\overrightarrow{A_NA_1}| = \frac{|\Gamma_N|}{N}, \quad (5.5)$$

then the point O is in the inner of Γ_N when N is sufficiently large. In the following, we assume that O is in the inner of Γ_N .

According to Definition 1.8, we only need to prove (5.3). Since for any $P \in \Gamma$, $\rho_P \geq r_P$, the first inequality of (5.3) holds. Let $r_i := \text{Dis}(O, A_i A_{i+1})$ denote the distance from the point O to the line $A_i A_{i+1}$. We have

$$\lim_{N \rightarrow \infty} r_i = r_{A_i}, \quad i = 1, 2, 3, \dots, \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N r_{A_i} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N r_i. \quad (5.6)$$

By Lemma 2.12, we know $D(\Gamma_N) \subset \Omega[O, \Gamma_N]$. Thus,

$$|D(\Gamma_N)| \leq |\Omega[O, \Gamma_N]| = \sum_{i=1}^N |\Delta A_i O A_{i+1}|. \quad (5.7)$$

Equalities of (5.7) hold if and only if $\Omega[O, \Gamma_N] = D(\Gamma_N)$. By (5.6), (5.7), and Lemma 2.9, we get

$$\begin{aligned} |D(\Gamma)| &= \lim_{N \rightarrow \infty} |D(\Gamma_N)| \leq \lim_{N \rightarrow \infty} \sum_{i=1}^N |\Delta A_i O A_{i+1}| = \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{i=1}^N r_i \left| \overrightarrow{A_i A_{i+1}} \right| \\ &= \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{i=1}^N r_i \frac{|\Gamma_N|}{N} = \frac{\lim_{N \rightarrow \infty} |\Gamma_N|}{2} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N r_i = \frac{|\Gamma|}{2} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N r_{A_i} \\ &= \frac{|\Gamma|}{2} \frac{1}{|\Gamma|} \oint_{\Gamma} r_P ds = \frac{1}{2} \oint_{\Gamma} r_P ds. \end{aligned} \quad (5.8)$$

Observe that (5.8) implies

$$\oint_{\Gamma} r_P ds \geq 2|D(\Gamma)|. \quad (5.9)$$

The second inequality of (5.3) is proved.

By this proof, the first equalities of (5.3) and (5.4) hold if and only if Γ is a circle and the point O is the central of the circle; the second equalities hold if and only if $\Omega[O, \Gamma] = D(\Gamma)$. This completes the proof. $\square$

Definition 5.6. Theorem 5.5 may be explained as follows: the point O denotes the central of earth; P is a satellite and Γ denotes the moving trajectory of satellites; $M(r_P, \Gamma)$ is the function mean of r_P on Γ and $M(\rho_P, \Gamma)$ is the function mean of ρ_P on Γ .

Definition 5.7. For the $S\{O, \Gamma, P_1, P_2\}$, which denotes the satellite system of two points with a central, in other words, there are only two satellites on same trajectory, we will discuss in another paper.

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References

- [1] J. C. Kuang, *Changyong Budengshi*, Hunan Education Press, Changsha, China, 2nd edition, 1993.
- [2] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, vol. 61 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.
- [3] H. Li and S. Xie, *Function of Complex Variable and Integral Transform*, China Higher Education Press, Beijing, China; Springer, Heidelberg, Germany, 2002.
- [4] W. Daren, *Differential Geometry*, China Higher Education Press, Beijing, China, 1986.
- [5] S. Xu, Y. Jin, and Q. Deng, "The isospectrum problem of compact minimal hypersurfaces in $S^{n+1}(1)$," *Mathematica Applicata*, vol. 19, no. 3, pp. 455–459, 2006.
- [6] J. Wen, R. Ke, and T. Lü, "A class of the geometric inequalities involving k -Brocard distance," *Chinese Quarterly Journal of Mathematics*, vol. 21, no. 2, pp. 210–219, 2006.
- [7] W. L. Wang, J. J. Wen, and H. N. Shi, "Optimal inequalities involving power means," *Acta Mathematica Sinica*, vol. 47, no. 6, pp. 1053–1062, 2004 (Chinese).
- [8] J. Wen, W. Wang, and Y. Lu, "The method of descending dimension for establishing inequalities," *Journal of Southwest University for Nationalities*, vol. 29, no. 5, pp. 527–532, 2003.
- [9] J. Pečarić, J. Wen, W.-L. Wang, and T. Lu, "A generalization of Maclaurin's inequalities and its applications," *Mathematical Inequalities & Applications*, vol. 8, no. 4, pp. 583–598, 2005.
- [10] J. Wen and W.-L. Wang, "The optimization for the inequalities of power means," *Journal of Inequalities and Applications*, vol. 2006, Article ID 46782, 25 pages, 2006.
- [11] R. J. Gardner, "The Brunn-Minkowski inequality," *Bulletin of the American Mathematical Society*, vol. 39, no. 3, pp. 355–405, 2002.
- [12] T. P. Lin, "The power mean and the logarithmic mean," *The American Mathematical Monthly*, vol. 81, pp. 879–883, 1974.
- [13] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, vol. 143 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1979.
- [14] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalities*, vol. 31 of *Mathematics and Its Applications*, D. Reidel, Dordrecht, The Netherlands, 1988.
- [15] J. Wen and Z. Zhang, "Vandermonde-type determinants and inequalities," *Applied Mathematics E-Notes*, vol. 6, pp. 211–218, 2006.
- [16] G. Pólya, *Mathematics and Plausible Reasoning, Vol.1: Induction and Analogy in Mathematics*, Princeton University Press, Princeton, NJ, USA, 1954.
- [17] R. Zhang, W. Wang, and J. Wen, "A class of reverse Jensen inequalities," *Journal of Southwest China Normal University. Natural Science*, vol. 29, no. 5, pp. 744–748, 2004.
- [18] J. Pečarić and D. Svrtan, "New refinements of the Jensen inequalities based on samples with repetitions," *Journal of Mathematical Analysis and Applications*, vol. 222, no. 2, pp. 365–373, 1998.
- [19] W. Blaschke and K. Leichtweiß, *Elementare Differentialgeometrie*, Springer, Berlin, Germany, 5th edition, 1973.
- [20] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1976.
- [21] H. W. Guggenheimer, *Differential Geometry*, McGraw-Hill, New York, NY, USA, 1963.
- [22] D. S. Mitrinović, J. E. Pečarić, and V. Volence, *Recent Advances in Geometric Inequalities*, Kluwer Academic, Dordrecht, The Netherlands, 1999.
- [23] W. Y. Hsiang, "An elementary proof of the isoperimetric problem," *Chinese Annals of Mathematics*, vol. 23, no. 1, pp. 7–12, 2002.
- [24] I. Chavel, *Isoperimetric Inequalities*, vol. 145 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 2001.

- [25] R. Osserman, "Isoperimetric inequalities," in *A Survey of Minimal Surfaces*, appendix 3, pp. 147–148, Dover, New York, NY, USA, 1986.
- [26] G. Pólya and G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, Annals of Mathematics Studies, no. 27, Princeton University Press, Princeton, NJ, USA, 1951.
- [27] J. L. Barbosa and M. do Carmo, "A proof of a general isoperimetric inequality for surfaces," *Mathematische Zeitschrift*, vol. 162, no. 3, pp. 245–261, 1978.
- [28] E. Heinz and S. Hildebrandt, "The number of branch points of surfaces of bounded mean curvature," *Journal of Differential Geometry*, vol. 4, pp. 227–235, 1970.
- [29] U. Dierkes, S. Hildebrandt, A. Küster, and O. Wohlrab, *Minimal Surfaces, Vol. I*, vol. 295 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 1992.
- [30] I. Chavel, *Isoperimetric Inequalities: Differential Geometric and Analytic Perspective*, vol. 145 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 2001.
- [31] F. R. K. Chung and P. Tetali, "Isoperimetric inequalities for Cartesian products of graphs," *Combinatorics, Probability and Computing*, vol. 7, no. 2, pp. 141–148, 1998.
- [32] R. Osserman, "The isoperimetric inequality," *Bulletin of the American Mathematical Society*, vol. 84, no. 6, pp. 1182–1238, 1978.
- [33] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, vol. 80 of *Monographs in Mathematics*, Birkhäuser, Basel, Switzerland, 1984.