Research Article

Some Generalizations of Fixed Point Results for Multivalued Contraction Mappings

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We establish some fixed point results for multivalued contraction type mappings in terms of a w -distance in a complete metric space. Our results generalize very recent results of some authors (Ćirić, 2008, 2009; Feng and Liu 2006; Klim and Wardowski 2007; and Latif and Abdou 2011).

1. Introduction

The Banach contraction principle [1] plays an important role in nonlinear analysis. Following the Banach contraction principle, Nadler [2] first initiated the study of fixed point theorems for multivalued contraction mappings and inspired by his results, the fixed point theory of multi-valued contraction has been further developed in different directions by many authors, in particular, by Reich, Mizoguchi-Takahashi, Feng-Liu, and many others.

The aim of this paper is to present results which are generalizations of the very recent results of Klim and Wardowski [3], Ćirić [4, 5], and Latif and Abdou [6], as well as of the result of Mizoguchi and Takahashi [7] and many others.

Let (X, d) be a metric space. We denote the collection of all nonempty closed bounded subsets of X by $CB(X)$ and the collection of all nonempty closed subsets and all nonempty compact subsets of X by $C(X)$ and $K(X)$, respectively. Throughout this paper, we assume that $\mathbb{R}$, $\mathbb{N}$, $\mathbb{N}_e$, and $\mathbb{N}_o$ denote the sets of all real numbers, positive integers, even positive integers, and odd positive integers, respectively. Let H be the Hausdorff metric induced by d , that is,

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}, \quad (1.1)$$

for all $A, B \in CB(X)$, where $d(x, B) = \inf\{d(x, y) : y \in B\}$. An element $x \in X$ is said to be a fixed point of a multi-valued mapping $T : X \rightarrow C(X)$ if $x \in T(x)$. A map $f : X \rightarrow \mathbb{R}$ is called lower semicontinuous if for any sequence $\{x_n\}$ in X and $x \in X$ such that $x_n \rightarrow x$, we have $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$.

The following theorem is an extension of Banach contraction principle for multi-valued mappings, which was obtained by Nadler in [2].

Theorem 1.1 (see [2]). *Let (X, d) be a complete metric space, and let $T : X \rightarrow CB(X)$ be a multi-valued mapping. Assume that there exists $r \in [0, 1)$ such that for all $x, y \in X$,*

$$H(T(x), T(y)) \leq rd(x, y), \quad (1.2)$$

then there exists $z \in X$ such that $z \in T(z)$.

A generalization of Theorem 1.1 was proved by Mizoguchi and Takahashi that is, in fact, a partial answer of question of Reich [8].

Theorem 1.2 (see [7]). *Let (X, d) be a complete metric space, and let $T : X \rightarrow CB(X)$ be a multi-valued mapping. If there exists a function $\varphi : (0, \infty) \rightarrow [0, 1)$ such that $\limsup_{r \rightarrow t^+} \varphi(r) < 1$ for each $t \in [0, \infty)$, and if for all $x, y \in X$,*

$$H(T(x), T(y)) \leq \varphi(d(x, y))d(x, y), \quad (1.3)$$

then there exists $z \in X$ such that $z \in T(z)$.

Recently, an interesting result had been obtained by Feng and Liu [9]. They extended Theorem 1.1 in another direction different from that of Mizoguchi-Takahashi's theorem.

Theorem 1.3 (see [9]). *Let (X, d) be a complete metric space, and let $T : X \rightarrow C(X)$ be a multi-valued mapping. If there exist constants $\alpha, \beta \in (0, 1)$, $\beta < \alpha$, such that for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions:*

- (i) $\alpha d(x, y) \leq d(x, T(x))$,
- (ii) $d(y, T(y)) \leq \beta d(x, T(x))$,

then there exists $z \in X$ such that $z \in T(z)$ provided the function $f(x) = d(x, T(x))$ is lower semicontinuous.

By using the ideas of Mizoguchi-Takahashi and Feng-Liu, Klim and Wardowski proved the following two theorems that are different from Theorem 1.2.

Theorem 1.4 (see [3]). *Let (X, d) be a complete metric space, and let $T : X \rightarrow C(X)$ be a multi-valued mapping. Assume that the following conditions hold:*

- (i) *the map $f : X \rightarrow \mathbb{R}$, defined by $f(x) = d(x, T(x))$, $x \in X$, is lower semi-continuous;*
- (ii) *there exists a constant $\alpha \in (0, 1)$ and a function $\varphi : [0, \infty) \rightarrow [0, \alpha)$ satisfying $\limsup_{r \rightarrow t^+} \varphi(r) < \alpha$, for each $t \in [0, \infty)$, and for any $x \in X$ there is $y \in T(x)$ satisfying*

$$\alpha d(x, y) \leq d(x, T(x)), \quad d(y, T(y)) \leq \varphi(d(x, y))d(x, y). \quad (1.4)$$

Then there exists $z \in X$ such that $z \in T(z)$.

Theorem 1.5 (see [3]). Let (X, d) be a complete metric space, and let $T : X \rightarrow K(X)$ be a multi-valued mapping. Assume that the following conditions hold:

- (i) the map $f : X \rightarrow \mathbb{R}$, defined by $f(x) = d(x, T(x))$, $x \in X$, is lower semi-continuous;
- (ii) there exists a function $\varphi : [0, \infty) \rightarrow [0, 1)$ satisfying $\limsup_{r \rightarrow t^+} \varphi(r) < 1$, for each $t \in [0, \infty)$, and for any $x \in X$ there is $y \in T(x)$ satisfying

$$d(x, y) = d(x, T(x)), \quad d(y, T(y)) \leq \varphi(d(x, y))d(x, y). \quad (1.5)$$

Then there exists $z \in X$ such that $z \in T(z)$.

In [4, 5], Ćirić proved a few interesting theorems which are generalizations of the above-mentioned theorems, one of which is as follows.

Theorem 1.6 (see [4]). Let (X, d) be a complete metric space, and let $T : X \rightarrow C(X)$ be a multi-valued mapping. If there exist a function $\varphi : [0, \infty) \rightarrow [0, 1)$ and a nondecreasing function $\psi : [0, \infty) \rightarrow [\alpha, 1)$, $\alpha > 0$, such that

$$\varphi(t) < \psi(t), \limsup_{r \rightarrow t^+} \varphi(r) < \limsup_{r \rightarrow t^+} \psi(r), \quad \forall t \in [0, \infty), \quad (1.6)$$

and for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions:

$$\varphi(d(x, y))d(x, y) \leq d(x, T(x)), \quad d(y, T(y)) \leq \psi(d(x, y))d(x, y), \quad (1.7)$$

then T has a fixed point in X provided $f(x) = d(x, T(x))$ is lower semi-continuous.

Very recently the fixed point theorems of Ćirić were extended by Liu et al. [10] and by Nicolae [11] in a new direction. Also Latif and Abdou [6] improve two theorems of Ćirić with respect to w -distance. The concept of w -distance was introduced by Kada et al. [12] on a metric space as follows.

Let (X, d) be a metric space, then a function $v : X \times X \rightarrow [0, \infty)$ is called a w -distance on X if the following axioms are satisfied:

- (1) $v(x, z) \leq v(x, y) + v(y, z)$, for any $x, y, z \in X$;
- (2) for any $x \in X$, $v(x, \cdot) : X \rightarrow [0, \infty)$ is lower semi-continuous;
- (3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $v(z, x) \leq \delta$ and $v(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

The metric d is a w -distance on X . One can see other examples of w -distances in [12]. One of theorems, which was proved by Latif and Abdou, is as follows.

Theorem 1.7 (see [6]). Let (X, d) be a complete metric space with a w -distance v . Let $T : X \rightarrow C(X)$ be a multi-valued map. Assume that the following conditions hold:

- (i) there exist a constant $\alpha \in (0, 1)$ and a function $\varphi : [0, \infty) \rightarrow [\alpha, 1)$ such that $\limsup_{r \rightarrow t^+} \varphi(r) < 1$, for each $t \in [0, \infty)$;
- (ii) the map $f : X \rightarrow \mathbb{R}$, defined by $f(x) = v(x, T(x))$, $x \in X$, is lower semi-continuous;

(iii) for any $x \in X$, there is $y \in T(x)$ satisfying

$$\sqrt{\varphi(f(x))}v(x, y) \leq f(x), \quad f(y) \leq \varphi(f(x))v(x, y). \quad (1.8)$$

Then there exists $z \in X$ such that $f(z) = 0$. Further, if $v(z, z) = 0$, then $z \in T(z)$.

For the proof of the main results, we need the following crucial lemma [13].

Lemma 1.8. *Let (X, d) be a metric space, and let v be a w -distance on X . Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X , let $\{a_n\}$ and $\{b_n\}$ be sequences in $[0, \infty)$ converging to zero, and let $z \in X$. Then the following hold:*

- (1) *if $v(x_n, y_n) \leq a_n$ and $v(x_n, z) \leq b_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z ;*
- (2) *if $v(x_n, x_p) \leq a_n$ for any $n, p \in \mathbb{N}$ with $p > n$, then $\{x_n\}$ is a Cauchy sequence.*

In the present paper, using the concept of w -distance, in the same direction which has been used in [10, 11], we introduce some new contraction conditions for multi-valued mappings in complete metric spaces and three fixed point theorems for such contractions are proved. Our results generalize Theorems 1.6 and 1.7, and many other theorems.

2. Main Results

Recall that a sequence $\{x_n\}_{n \geq 0}$ in X is called an orbit of T at $x_0 \in X$ if $x_n \in T(x_{n-1})$, for all $n \geq 1$, and $v(x, G) = \inf\{v(x, y) : y \in G\}$ for any $x \in X$ and $G \in C(X)$.

Now, we shall prove a theorem which is a generalization of Theorem 1.7.

Theorem 2.1. *Let (X, d) be a complete metric space with a w -distance v , and let T be a multi-valued mapping from X into $C(X)$. Assume that the following conditions hold:*

- (i) *the map $f : X \rightarrow \mathbb{R}$, defined by $f(x) = v(x, T(x))$, $x \in X$, is lower semi-continuous;*
- (ii) *there exist the functions $\alpha : [0, \infty) \rightarrow (0, 1]$, $\beta : [0, \infty) \rightarrow [0, 1)$ which satisfy*

$$\begin{aligned} \exists k, \varepsilon \in (0, 1), \quad t^k \leq \alpha(t), \quad \text{for each } t \in [0, \varepsilon], \\ \limsup_{r \rightarrow t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \text{for each } t \in [0, \infty); \end{aligned} \quad (2.1)$$

(iii) for any $x \in X$, there is $y \in T(x)$ satisfying

$$\alpha(f(x))v(x, y) \leq f(x), \quad f(y) \leq \beta(f(x))v(x, y). \quad (2.2)$$

Then, there exists $z \in X$ such that $z \in T(z)$.

Proof. Let $x_0 \in X$. Then there exists $x_1 \in T(x)$ such that

$$\begin{aligned}\alpha(f(x_0))v(x_0, x_1) &\leq f(x_0), \\ f(x_1) &\leq \beta(f(x_0))v(x_0, x_1).\end{aligned}\tag{2.3}$$

Then, from (2.3), we have

$$f(x_1) \leq \frac{\beta(f(x_0))}{\alpha(f(x_0))} f(x_0).\tag{2.4}$$

Continuing this process, we can define an orbit $\{x_n\}$ of T in X , such that

$$\alpha(f(x_n))v(x_n, x_{n+1}) \leq f(x_n), \quad f(x_{n+1}) \leq \beta(f(x_n))v(x_n, x_{n+1}),\tag{2.5}$$

for all $n \geq 0$, which imply that

$$f(x_{n+1}) \leq \frac{\beta(f(x_n))}{\alpha(f(x_n))} f(x_n).\tag{2.6}$$

We can assume that $f(x_n) > 0$ for all $n \geq 0$, since $f(x_n) = 0$ for some n , then from (2.5), $v(x_n, x_{n+1}) = f(x_{n+1}) = 0$, and so for every $m \in \mathbb{N}$, there exists $y_m \in T(x_{n+1})$ such that $v(x_{n+1}, y_m) \leq 1/m$; consequently, $v(x_n, y_m) \leq v(x_n, x_{n+1}) + v(x_{n+1}, y_m) \leq 1/m$; so by Lemma 1.8, $y_m \rightarrow x_{n+1}$ and thus $x_{n+1} \in T(x_{n+1})$, that is, the assertion of the theorem is proved. Since $\beta(f(x))/\alpha(f(x)) < 1$ for all $x \in X$, then we have

$$f(x_{n+1}) < f(x_n).\tag{2.7}$$

Thus $\{f(x_n)\}$ is a decreasing sequence of positive real numbers and hence converges to a nonnegative number θ , $\theta \geq 0$. Let $\tau = \limsup_{n \rightarrow \infty} \beta(f(x_n))/\alpha(f(x_n)) < 1$. Then, for $s = (\tau + 1)/2$, we can take $\hat{n}_0 \in \mathbb{N}$ such that

$$\frac{\beta(f(x_n))}{\alpha(f(x_n))} < s, \quad \forall n \geq \hat{n}_0.\tag{2.8}$$

Then,

$$f(x_{n+1}) \leq s^{n-\hat{n}_0+1} f(x_{\hat{n}_0}), \quad \forall n \geq \hat{n}_0,\tag{2.9}$$

thus, $\theta = 0$. Therefore, by (ii) we can take $\tilde{n}_0 \in \mathbb{N}$ such that

$$f(x_n) \in [0, \varepsilon], \quad f(x_n)^k \leq \alpha(f(x_n)), \quad \forall n \geq \tilde{n}_0.\tag{2.10}$$

Now, from (2.5), (2.9), and (2.10), we have

$$\begin{aligned}v(x_n, x_{n+1}) &\leq \frac{f(x_n)}{\alpha(f(x_n))} \leq \frac{f(x_n)}{f(x_n)^k} = f(x_n)^{1-k} \\ &\leq q^{n-n_0} f(x_{n_0})^{1-k},\end{aligned}\tag{2.11}$$

for all $n \geq n_0$, where $n_0 = \max\{\hat{n}_0, \tilde{n}_0\}$ and $q = s^{1-k} < 1$. Then for any $p, n \in \mathbb{N}$, $p > n \geq n_0$, we have

$$\begin{aligned} v(x_n, x_p) &\leq \sum_{i=n}^{p-1} v(x_i, x_{i+1}) \leq \sum_{i=n}^{p-1} q^{i-n_0} f(x_{n_0})^{1-k} \\ &\leq \frac{q^{n-n_0}}{1-q} f(x_{n_0})^{1-k}. \end{aligned} \quad (2.12)$$

Hence, by Lemma 1.8, $\{x_n\}$ is a Cauchy sequence so there exists $z \in X$ such that $\lim_n x_n = z$. Since f is lower semi-continuous, we obtain

$$0 \leq f(z) \leq \liminf_{n \rightarrow \infty} f(x_n) = 0, \quad (2.13)$$

and thus,

$$f(z) = v(z, T(z)) = 0. \quad (2.14)$$

Now, we claim that $z \in T(z)$. Notice that the function $v(x, \cdot)$ is lower semi-continuous for all $x \in X$. Since $x_p \rightarrow z$, then by (2.12),

$$v(x_n, z) \leq \liminf_{p \rightarrow \infty} v(x_n, x_p) \leq \frac{q^{n-n_0}}{1-q} f(x_{n_0})^{1-k}, \quad (2.15)$$

for all $n \geq n_0$. On the other hand, from (2.14), for every $n \in \mathbb{N}$, there exists $y_n \in T(z)$ such that $v(z, y_n) \leq 1/n$. Then, for all $n \geq n_0$, we have

$$v(x_n, y_n) \leq v(x_n, z) + v(z, y_n) \leq \frac{q^{n-n_0}}{1-q} f(x_{n_0})^{1-k} + \frac{1}{n}. \quad (2.16)$$

Therefore, from (2.15), (2.16), we can find the sequences $\{a_n\}$ and $\{b_n\}$ in $[0, \infty)$ converging to zero, such that $v(x_n, y_n) \leq a_n$ and $v(x_n, z) \leq b_n$ for any $n \in \mathbb{N}$; then by Lemma 1.8 $y_n \rightarrow z$. The closedness of $T(z)$ implies $z \in T(z)$. $\square$

Now, we shall prove a theorem which is different from Theorem 2.1 and is a generalization of Theorem 1.6.

Theorem 2.2. *Let (X, d) be a complete metric space with a w -distance v , and let T be a multi-valued mapping from X into $C(X)$. Assume that the following conditions hold:*

- (i) *the map $f : X \rightarrow \mathbb{R}$, defined by $f(x) = v(x, T(x))$, $x \in X$, is lower semi-continuous;*
- (ii) *there exist the functions $\alpha : [0, \infty) \rightarrow (0, 1]$, $\beta : [0, \infty) \rightarrow [0, 1]$ which satisfy*

$$\begin{aligned} \exists k, \varepsilon \in (0, 1), \quad t^k \leq \alpha(t), \quad \text{for each } t \in [0, \varepsilon], \\ \limsup_{r \rightarrow t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \text{for each } t \in [0, \infty), \end{aligned} \quad (2.17)$$

and α is nondecreasing;

(iii) for any $x \in X$, there is $y \in T(x)$ satisfying

$$\alpha(v(x, y))v(x, y) \leq f(x), \quad f(y) \leq \beta(v(x, y))v(x, y). \quad (2.18)$$

Then, there exists $z \in X$ such that $z \in T(z)$.

Proof. By following the lines in the proof of Theorem 2.1, one can construct an orbit $\{x_n\}_{n=0}^{\infty}$ of T in X such that

$$\alpha(v(x_n, x_{n+1}))v(x_n, x_{n+1}) \leq f(x_n), \quad (2.19)$$

$$f(x_{n+1}) \leq \beta(v(x_n, x_{n+1}))v(x_n, x_{n+1}), \quad (2.20)$$

for all $n \geq 0$, which imply that

$$f(x_{n+1}) \leq \frac{\beta(v(x_n, x_{n+1}))}{\alpha(v(x_n, x_{n+1}))} f(x_n). \quad (2.21)$$

Thus, $\{f(x_n)\}$ is a decreasing sequence, and so there exists $\theta \geq 0$ such that $f(x_n) \rightarrow \theta$. From (2.19) and (2.20), we have

$$v(x_{n+1}, x_{n+2}) \leq \frac{f(x_{n+1})}{\alpha(v(x_{n+1}, x_{n+2}))} \leq \frac{\beta(v(x_n, x_{n+1}))}{\alpha(v(x_{n+1}, x_{n+2}))} v(x_n, x_{n+1}), \quad (2.22)$$

for all $n \geq 0$. Now we claim that $\{v(x_n, x_{n+1})\}$ is a nonincreasing sequence. Suppose not. Then there exists $n_0 \geq 0$ such that $v(x_{n_0+1}, x_{n_0+2}) > v(x_{n_0}, x_{n_0+1})$. Since α is nondecreasing, then from (2.22), we get that

$$\begin{aligned} v(x_{n_0}, x_{n_0+1}) &< v(x_{n_0+1}, x_{n_0+2}) \leq \frac{\beta(v(x_{n_0}, x_{n_0+1}))}{\alpha(v(x_{n_0+1}, x_{n_0+2}))} v(x_{n_0}, x_{n_0+1}) \\ &\leq \frac{\beta(v(x_{n_0}, x_{n_0+1}))}{\alpha(v(x_{n_0}, x_{n_0+1}))} v(x_{n_0}, x_{n_0+1}) \\ &< v(x_{n_0}, x_{n_0+1}), \end{aligned} \quad (2.23)$$

which is a contradiction. Then, $\{v(x_n, x_{n+1})\}$ is a nonincreasing sequence and so is convergent. Now by using the same argument as in the proof of Theorem 2.1, we obtain the existence of a real number $s \in (0, 1)$ and $\hat{n}_0 \in \mathbb{N}$ such that

$$f(x_{n+1}) \leq s^{n-\hat{n}_0+1} f(x_{\hat{n}_0}), \quad \forall n \geq \hat{n}_0, \quad (2.24)$$

thus $\theta = 0$. On the other hand, since α is nondecreasing, then by (2.19), we have

$$v(x_n, x_{n+1}) \leq \frac{f(x_n)}{\alpha(v(x_n, x_{n+1}))} \leq \frac{f(x_n)}{\alpha(f(x_n))}. \quad (2.25)$$

For the rest of the proof, we can go on as in the proof of Theorem 2.1. $\square$

In the same manner, we can present the following theorem.

Theorem 2.3. Let (X, d) be a complete metric space with a w -distance v , and let T be a multi-valued mapping from X into $C(X)$. Assume that the following conditions hold:

- (i) the map $f : X \rightarrow \mathbb{R}$, defined by $f(x) = v(x, T(x))$, $x \in X$, is lower semi-continuous,
- (ii) there exist the functions $\alpha : [0, \infty) \rightarrow (0, 1]$, $\beta : [0, \infty) \rightarrow [0, 1]$ which satisfy

$$\begin{aligned} \exists k, \varepsilon \in (0, 1), \quad t^k &\leq \beta(t), \quad \text{for each } t \in [0, \varepsilon], \\ \beta(t) &< \alpha(t), \quad \limsup_{r \rightarrow t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \text{for each } t \in [0, \infty), \end{aligned} \quad (2.26)$$

and also one of α and β is nondecreasing;

- (iii) for any $x \in X$, there is $y \in T(x)$ satisfying

$$\alpha(v(x, y))v(x, y) \leq f(x), \quad f(y) \leq \beta(v(x, y))v(x, y). \quad (2.27)$$

Then, there exists $z \in X$ such that $z \in T(z)$.

Proof. As in the proof of Theorem 2.1, one can construct an orbit $\{x_n\}_{n=0}^\infty$ of T in X such that (2.19), (2.20), (2.21), and (2.22) hold. Then, $\{f(x_n)\}$ is a decreasing sequence and so there exists $\theta \geq 0$ such that $f(x_n) \rightarrow \theta$. Now we claim that $\{v(x_n, x_{n+1})\}$ is a nonincreasing sequence. Suppose not. Then there exists $n_0 \geq 0$ such that $v(x_{n_0+1}, x_{n_0+2}) > v(x_{n_0}, x_{n_0+1})$. Since one of α and β is nondecreasing, it follows from (2.22) that

$$\begin{aligned} v(x_{n_0}, x_{n_0+1}) &< v(x_{n_0+1}, x_{n_0+2}) \leq \frac{\beta(v(x_{n_0}, x_{n_0+1}))}{\alpha(v(x_{n_0+1}, x_{n_0+2}))} v(x_{n_0}, x_{n_0+1}) \\ &\leq \max \left\{ \frac{\beta(v(x_{n_0}, x_{n_0+1}))}{\alpha(v(x_{n_0}, x_{n_0+1}))}, \frac{\beta(v(x_{n_0+1}, x_{n_0+2}))}{\alpha(v(x_{n_0+1}, x_{n_0+2}))} \right\} v(x_{n_0}, x_{n_0+1}) \\ &< v(x_{n_0}, x_{n_0+1}), \end{aligned} \quad (2.28)$$

which is a contradiction. Thus $\{v(x_n, x_{n+1})\}$ is a nonincreasing sequence and so is convergent. Now as in the proof of Theorem 2.1, we obtain the existence of a real number $s \in (0, 1)$ and $\hat{n}_0 \in \mathbb{N}$ such that

$$f(x_{n+1}) \leq s^{n-\hat{n}_0+1} f(x_{\hat{n}_0}), \quad \forall n \geq \hat{n}_0, \quad (2.29)$$

thus $\theta = 0$. If α is nondecreasing, then from assumption (ii) and (2.19), we have

$$v(x_n, x_{n+1}) \leq \frac{f(x_n)}{\alpha(v(x_n, x_{n+1}))} \leq \frac{f(x_n)}{\alpha(f(x_n))} \leq \frac{f(x_n)}{\beta(f(x_n))}, \quad (2.30)$$

and if β is nondecreasing then from (2.19) and (2.20), we have

$$v(x_n, x_{n+1}) \leq \frac{f(x_n)}{\beta(v(x_n, x_{n+1}))} \leq \frac{f(x_n)}{\beta(f(x_n))}, \quad (2.31)$$

for all $n \geq 0$. Therefore, in all cases we have shown

$$v(x_n, x_{n+1}) \leq \frac{f(x_n)}{\beta(f(x_n))}, \quad \forall n \geq 0, \quad (2.32)$$

and so, by following as in the proof of Theorem 2.1, we can take $n_0 \in \mathbb{N}$ such that

$$v(x_n, x_{n+1}) \leq q^{n-n_0} f(x_{n_0})^{1-k}, \quad (2.33)$$

for all $n \geq n_0$, where $q = s^{1-k} < 1$. The rest of the proof is similar to that of Theorem 2.1. $\square$

Remark 2.4. Theorem 2.1 is a generalization of Theorem 1.7. In fact, if we consider $\beta(t) = \varphi(t)$ and $\alpha(t) = \sqrt{\varphi(t)}$, then the assumptions of Theorem 2.1 are satisfied. Also, one can see that Theorem 2.1 generalizes Theorem 2.2 of Nicolae [11].

Remark 2.5. Theorem 2.2 essentially generalizes Theorem 1.6. Indeed, if we consider $\beta(t) = \varphi(t)$ and $\alpha(t) = \varphi(t)$, then all assumptions of Theorem 2.2 are satisfied.

The following example shows that there are mappings which satisfy the assumptions of Theorem 2.1 but do not satisfy the assumptions of Theorem 1.7.

Example 2.6. Consider $x_n = 1/n$, for $n \in \mathbb{N}$, and $x_0 = 0$. Then $X = \{x_0, x_1, x_2, \dots\}$ is a bounded complete subset of $\mathbb{R}$. Let $v(x, y) = d(x, y)$, for all $x, y \in X$. Define a mapping T from X into $C(X)$ by

$$T(x_n) = \begin{cases} x_0 & \text{if } n = 0, \\ x_1 & \text{if } n = 1, \\ x_1 & \text{if } n \in \{n \in \mathbb{N}_e : n \geq 2\}, \\ \{x_{n+1}, x_{n^2}\} & \text{if } n \in \{n \in \mathbb{N}_o : n > 2\}. \end{cases} \quad (2.34)$$

It is easy to verify that

$$f(x_n) = v(x_n, T(x_n)) = \begin{cases} 0 & \text{if } n = 0, 1, \\ 1 - \frac{1}{n} & \text{if } n \in \{n \in \mathbb{N}_e : n \geq 2\}, \\ \frac{1}{n} - \frac{1}{n+1} & \text{if } n \in \{n \in \mathbb{N}_o : n > 2\}, \end{cases} \quad (2.35)$$

is lower semi-continuous in X . Define $\alpha(t) : [0, \infty) \rightarrow (0, 1]$ and $\beta(t) : [0, \infty) \rightarrow [0, 1]$ by

$$\begin{aligned}\alpha(t) &= \begin{cases} 1 & \text{if } t \in \{0\} \cup [1, \infty), \\ \sqrt{t} & \text{if } t \in (0, 1), \end{cases} \\ \beta(t) &= \begin{cases} t & \text{if } t \in [0, 1), \\ 0 & \text{if } t \in [1, \infty). \end{cases}\end{aligned}\tag{2.36}$$

Since

$$\frac{\beta(t)}{\alpha(t)} = \begin{cases} \sqrt{t} & \text{if } t \in [0, 1), \\ 0 & \text{if } t \in [1, \infty), \end{cases}\tag{2.37}$$

then, we have

$$\limsup_{r \rightarrow t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \text{for each } t \in [0, \infty).\tag{2.38}$$

For $x = x_0, x_1$, there exists $y = x \in T(x)$ such that

$$\alpha(f(x))v(x, y) = 0 = f(x), \quad f(y) = 0 = \beta(f(x))v(x, y),\tag{2.39}$$

and for $x = x_n, n \geq 2$, if $n \in \mathbb{N}_e$, there exists $y = x_1 \in T(x)$ satisfying

$$\begin{aligned}\alpha(f(x))v(x, y) &= \alpha\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n}\right) < \left(1 - \frac{1}{n}\right) = f(x), \\ f(y) &= 0 < \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n}\right) = \beta(f(x))v(x, y),\end{aligned}\tag{2.40}$$

and, if $n \in \mathbb{N}_o$, there exists $y = x_{n^2} \in T(x)$ satisfying

$$\begin{aligned}\alpha(f(x))v(x, y) &= \left(\frac{1}{\sqrt{n(n+1)}}\right)\left(\frac{n-1}{n^2}\right) < \frac{1}{n(n+1)} = f(x), \\ f(y) &= \frac{1}{n^2(n^2+1)} < \left(\frac{1}{n(n+1)}\right)\left(\frac{n-1}{n^2}\right) = \beta(f(x))v(x, y).\end{aligned}\tag{2.41}$$

Therefore, all assumptions of Theorem 2.1 are satisfied and x_0, x_1 are two fixed points of T . Let us observe that T does not satisfy the assumptions of Theorem 1.7 provided that $v(x, y) = d(x, y)$, for all $x, y \in X$. Indeed, for any function $\varphi : [0, \infty) \rightarrow [\alpha, 1], \alpha \in (0, 1)$, there exists $n > 2, n \in \mathbb{N}_o$, such that for $x = x_n, y = x_{n^2} \in T(x)$, we have

$$f(x) = \frac{1}{n(n+1)} < \sqrt{\alpha}\left(\frac{n-1}{n^2}\right) \leq \sqrt{\varphi(f(x))}v(x, y),\tag{2.42}$$

and if $y = x_{n+1} \in T(x)$, we have

$$v(y, T(y)) = 1 - \frac{1}{n+1} > \frac{1}{n(n+1)} \geq \varphi(f(x))v(x, y), \quad (2.43)$$

that is, the assumptions of Theorem 1.7 are not satisfied. The next example is an application of Theorem 2.3.

Example 2.7. Let X be as in the Example 2.6, and let $v(x, y) = y$, for all $x, y \in X$. Note that v is a w -distance on X . Define a mapping T from X into $C(X)$ by

$$T(x_n) = \begin{cases} x_0 & \text{if } n = 0, \\ \{x_0, x_1\} & \text{if } n = 1, \\ x_1 & \text{if } n \in \{n \in \mathbb{N}_e : n \geq 2\}, \\ \{x_{n^4-1}, x_{n^3}\} & \text{if } n \in \{n \in \mathbb{N}_o : n > 2\}. \end{cases} \quad (2.44)$$

Clearly,

$$f(x_n) = v(x_n, T(x_n)) = \begin{cases} 0 & \text{if } n = 0, 1, \\ 1 & \text{if } n \in \{n \in \mathbb{N}_e : n \geq 2\}, \\ \frac{1}{n^4 - 1} & \text{if } n \in \{n \in \mathbb{N}_o : n > 2\} \end{cases} \quad (2.45)$$

is lower semi-continuous in X . Define $\alpha(t) : [0, \infty) \rightarrow (0, 1]$ and $\beta(t) : [0, \infty) \rightarrow [0, 1]$ by

$$\alpha(t) = \begin{cases} 1 & \text{if } t \in \{0\} \cup [1, \infty), \\ t^{1/3} & \text{if } t \in (0, 1), \end{cases} \quad (2.46)$$

$$\beta(t) = \begin{cases} t^{5/6} & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \left(\frac{1}{2}\right)^{5/6} & \text{if } t \in \left(\frac{1}{2}, \infty\right). \end{cases}$$

Note that $\beta(t)$ is nondecreasing and $t^{5/6} \leq \beta(t)$, for each $t \in [0, (1/2)]$. Since

$$\frac{\beta(t)}{\alpha(t)} = \begin{cases} t^{1/2} & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \left(\frac{1}{2}\right)^{5/6} t^{-1/3} & \text{if } t \in \left(\frac{1}{2}, 1\right), \\ \left(\frac{1}{2}\right)^{5/6} & \text{if } t \in [1, \infty), \end{cases} \quad (2.47)$$

then,

$$\limsup_{r \rightarrow t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \text{for each } t \in [0, \infty). \quad (2.48)$$

For $x = x_0, x_1$, there exists $y = x_0 \in T(x)$ such that

$$\alpha(v(x, y))v(x, y) = 0 = f(x), \quad f(y) = 0 = \beta(v(x, y))v(x, y), \quad (2.49)$$

and for $x = x_n, n \geq 2$, if $n \in \mathbb{N}_e$, there exists $y = x_1 \in T(x)$ satisfying

$$\begin{aligned} \alpha(v(x, y))v(x, y) &= 1 = f(x), \\ f(y) &= 0 < \beta(v(x, y))v(x, y), \end{aligned} \quad (2.50)$$

and, if $n \in \mathbb{N}_o$, there exists $y = x_{n^3} \in T(x)$ satisfying

$$\begin{aligned} \alpha(v(x, y))v(x, y) &= \left(\frac{1}{n}\right)\left(\frac{1}{n^3}\right) < \frac{1}{n^4 - 1} = f(x), \\ f(y) &= \frac{1}{n^{12} - 1} < \left(\frac{1}{n^3}\right)^{5/6} \left(\frac{1}{n^3}\right) = \beta(v(x, y))v(x, y). \end{aligned} \quad (2.51)$$

Then, all assumptions of Theorem 2.3 are satisfied and x_0, x_1 are two fixed points of T . Note that $v(x_1, x_1) \neq 0$.

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