Research Article

On Projective Modules and Computation of Dimension of a Module over Laurent Polynomial Ring

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We give a procedure and describe an algorithm to compute the dimension of a module over Laurent polynomial ring. We prove the cancellation theorems for projective modules and also prove the *qualitative version* of Laurent polynomial analogue of Horrocks' Theorem.

1. Introduction

Hilbert [1] introduced free resolution on iterated syzygies of a finitely generated graded module M over polynomial ring $R = K[x_1, \dots, x_n]$. A choice of $a_0 = \dim(M \otimes_R K)$ homogeneous generators of M defines a surjective homomorphism $R^{a_0} \rightarrow M$, and its kernel is the first syzygy module of M . Hilbert proved that the syzygies are also finitely generated (Hilbert basis theorem). Hilbert's syzygy theorem states that "every finitely generated module M over $K[x_1, \dots, x_n]$ has finite free resolution, that is, the l th syzygy is R^{a_l} for some a_l ." The number of generators a_i of the syzygies is chosen minimally, they are independent of the choice of generators. These a_i are called the Betti numbers of M and the minimal length of free resolution (i.e., i) is dimension of a module M . When M is projective, the dimension of M is called projective dimension of M (denoted by $\text{pd}(M)$). A projective dimension is a measure of how far the module is being projective.

Kaplansky [2] described how homological dimension changes when one passes from a ring R to a quotient ring $R/(X)$, where (X) is the ideal generated by a nonzero divisor, without using Ext or Tor. Shanuel noticed that there is an elegant relation between different projective resolutions of the same module. Using the ideas of Shanuel, Kaplansky defined the projective dimension of a module.

The global dimension (or homological dimension) of a ring R denoted by $\text{gldim}(R)$, is a non-negative integer or infinity, and is a homological invariant of the ring. It is defined to be the supremum of the set of projective dimensions of all (finite or cyclic) R -modules. Global dimension is an important technical notion in the dimension theory of Noetherian rings. Eisenbud [3] proved that a commutative Noetherian local ring R is regular if and only if it has finite global dimension, in which case the global dimension coincides with the Krull dimension of R . This theorem opened the door for application of homological methods to commutative algebra. Now we state a theorem of Eisenbud [3] which implies that every module has a finite free resolution of length at most $l < \infty$.

Theorem 1.1 (see [3]). *The following conditions on a ring R are equivalent:*

- (1) $\text{gldim}(R) \leq l$, that is, $\text{pd}(M) \leq l$ for every R -module M ,
- (2) $\text{pd}(M) \leq l$ for every finitely generated module M .

Serre [4] taught a course on multiplicities at the college de France. Part of that course focussed upon the simple inequality $\text{pd}_R(M) \leq \text{pd}_R(S) + \text{pd}_S(M)$ for module M over an R -algebra S . Auslander and Buchsbaum [5] realized that Serre's method could be used to study the close connection between the dimension and multiplicity over a local ring. This led them to the Auslander-Buchsbaum equality [6]: let M be a finitely generated module over a Noetherian local ring $(R, \mathfrak{m})$. If $\text{pd}(M) < \infty$, then $\text{depth}(R) = \text{depth}(M) + \text{pd}(M)$. Gago-Vargas [7] computed the projective dimension of a module over Weyl algebra. Projective dimension is a main ingredient to compute the tilted algebra which nowadays plays a very important role in the representation theory of algebras. In this paper, we give a procedure and describe an algorithm to compute the projective dimension of a module which is valid over the Laurent polynomial ring.

We also prove the cancellation theorems for projective modules over the Laurent polynomial ring [8, Section 5] motivated by Swan's work. Horrocks proved the following theorem.

Theorem 1.2 (see [9]). *Let R be a local ring, $A = R[x]$ and S be the set of monic polynomials of A . If P is a finitely generated projective A -module such that P_S is free over A_S , then P is a free A -module.*

This theorem draw the attention of several mathematicians, as a result several analogues of Horrocks' theorem can be found in the literature [10, 11]. Mandal proved the Laurent polynomial version of Horrocks' theorem in [12] whereas Nashier [13] proved the *qualitative version* of the same theorem over polynomial ring. But it is observed that the *qualitative version* of Horrocks' theorem over Laurent polynomial ring is not found anywhere in the literature. One of the objectives of this paper is to fill this gap. In this paper, we prove the *qualitative version* of Horrocks' theorem over the Laurent polynomial ring.

2. Preliminary Notes

In this section we define some terms used in this paper and state certain standard results without proof. We hope that this will improve the readability and understanding of the proof of the paper.

A polynomial f in the Laurent polynomial ring $R[X, X^{-1}]$ is said to be doubly monic polynomial if coefficient of the highest degree term and the lowest degree term are unit.

We need the following result of Kunz [14] to compute the projective resolution in Section 3.

Definition 2.1 (see [14]). An exact sequence

$$F_{i+1} \xrightarrow{\alpha_i} F_i F_1 \xrightarrow{\alpha_0} F_0 \xrightarrow{\epsilon} M \rightarrow 0 \quad (2.1)$$

with only free (resp., projective) modules $F_i (i = 0, 1, 2, \dots)$ is called a free (resp., projective) resolution of M .

Projective modules over the Laurent polynomial ring $R[X, X^{-1}]$ are stably free. In [15] it is proved that a projective module is stably free provided it possesses a finite free resolution. The minimum length of free resolution is called the *projective dimension* of a module M .

Let R be a ring and M be an R -module. $M[X, X^{-1}]$ means the analogously defined Laurent polynomial module. We identify $M \otimes_R R[X, X^{-1}]$ with $M[X, X^{-1}]$, where $M[X, X^{-1}] = \{\sum_{n=-\infty}^{\infty} m_n X^n \mid m_n \in M \text{ and } m_n = 0 \text{ for all but finite } n\}$. If $f : M \rightarrow N$ is a homomorphism of R -modules, then this induces a homomorphism

$$\begin{aligned} \psi : M[X, X^{-1}] &\rightarrow N[X, X^{-1}], \\ \text{defined by } \psi\left(\sum m_n X^n\right) &= \sum f(m_n) X^n, \end{aligned} \quad (2.2)$$

where ψ can be identified with $f \otimes_R R[X, X^{-1}]$. Given any R -module M and $f \in \text{End}_R(M)$, we can make M as $R[X, X^{-1}]$ -module whose scalar multiplication is defined as $m(\sum a_n X^n) = \sum a_n \sum f^n(m)$. We denote this $R[X, X^{-1}]$ -module by ${}_f M$. There is a canonical $R[X, X^{-1}]$ -epimorphism $\varphi_f : M[X, X^{-1}] \rightarrow {}_f M$ defined by $\varphi_f(\sum m_n X^n) = \sum f^n(m_n)$. The characteristic sequence of f is an exact sequence which proved in Proposition 4.3.

The following is the Laurent polynomial version of a Horrocks Theorem which we state as follows.

Theorem 2.2 (see [12]). *Suppose $R[X, X^{-1}]$ is a Laurent polynomial ring over a local Noetherian commutative ring R , and P is a projective $R[X, X^{-1}]$ -module. If P_f is free for some doubly monic Laurent polynomial f , then P is free.*

In the proof of the Theorem 5.1, we need the following theorem.

Theorem 2.3 (see [12]). *Suppose that $R[X, X^{-1}]$ is a Laurent polynomial ring over a local Noetherian commutative ring R . Also suppose that P and P' are two projective $R[X, X^{-1}]$ -modules with $\text{Rank } P' < \text{Rank } P$. If P'_f is a direct summand of P_f for some doubly monic polynomial f , then P' is also a direct summand of P .*

3. Projective Dimension of Module

Lemma 3.1. *Let M_1, M_2, M_3 be R -modules and*

$$0 \longrightarrow M_1 \xrightarrow{\alpha_1} M_2 \xrightarrow{\alpha_2} M_3 \longrightarrow 0 \quad (3.1)$$

be a split short exact sequence, and β_1 and β_2 are the splittings corresponding to α_1 and α_2 , respectively. Then the following sequence

$$0 \longrightarrow M_3 \xrightarrow{\beta_2} M_2 \xrightarrow{\beta_1} M_1 \longrightarrow 0 \quad (3.2)$$

is an exact sequence.

Proof. Since β_1 and β_2 are the splittings corresponding to α_1 and α_2 , respectively, then $\beta_1 \circ \alpha_1 = I_{M_1}$ and $\alpha_2 \circ \beta_2 = I_{M_3}$. Let $x \in M_2$. Then $\alpha_2(x - \beta_2(\alpha_2(x))) = \alpha_2(x) - \alpha_2(\beta_2(\alpha_2(x))) = \alpha_2(x) - \alpha_2(x) = 0$. Hence $x - \beta_2(\alpha_2(x)) \in \text{Ker}(\alpha_2) = \text{Im}(\alpha_1)$. Since α_1 is injective, there exists a unique $\beta_1(x) \in M_1$ such that $\alpha_1(\beta_1(x)) = x - \beta_2(\alpha_2(x))$. We need to show that $\text{Ker}(\beta_1) \subset \text{Im}(\beta_2)$. Let $x \in \text{Ker}(\beta_1)$. Then $\beta_1(x) = 0$ and $\alpha_1(\beta_1(x)) = \alpha_1(0) = 0$, by injectivity of α_1 . Hence $x - \beta_2(\alpha_2(x)) = 0 \Rightarrow x = \beta_2(\alpha_2(x))$. Thus there exists $\alpha_2(x) \in M_3$ such that $\beta_2(\alpha_2(x)) = x$. Hence $x \in \text{Im}(\beta_2)$. Therefore $\text{Ker} \beta_1 \subset \text{Im}(\beta_2)$. Conversely, let $x \in \text{Im}(\beta_2)$. Then there exists $y \in M_3$ such that $\beta_2(y) = x$. Applying α_2 on both sides, we get $y = \alpha_2(x)$. Now applying β_2 on both sides, we get $\beta_2 \alpha_2(x) = x$ and therefore $x - \beta_2 \alpha_2(x) = 0$. Using the defining property of β_1 and injectivity of α_1 , $\alpha_1(\beta_1(x)) = 0 = \alpha_1(0)$. Therefore $\beta_1(x) = 0$. Hence $x \in \text{Ker}(\beta_1)$. Thus $\text{Im}(\beta_2) \subset \text{Ker}(\beta_1)$. Therefore $\text{Im}(\beta_2) = \text{Ker}(\beta_1)$ $\square$

By applying the technique discussed in [7] for Weyl algebra, we give a procedure to calculate the projective dimension of a module over the Laurent polynomial ring. We also describe an algorithm to compute the projective dimension of a module.

Procedure. The first step is to define the Laurent polynomial ring $R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ in terms of quotient ring as

$$R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}] \cong \frac{R[x_1, y_1, x_2, y_2, \dots, x_n, y_n]}{(x_1 y_1 - 1, x_2 y_2 - 1, \dots, x_n y_n - 1)}. \quad (3.3)$$

The second step is to test whether the $R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ -module M is projective or not. If M is projective, then again we test whether M is stably free or not, that is, we find a matrix that defines an isomorphism $M \oplus R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^r \cong R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^s$, for some positive integers r and s . If M is stably free, then the next step is to find finite free resolution of the projective module M . We denote the homomorphisms with their matrices to simplify the notation.

Given an $R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ -module M defined by a system of generators in some $R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^t$, choose a free $R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ -module P_0 and a surjection $\sigma : P_0 \rightarrow M$ with kernel C_0 , then we get an exact sequence

$$0 \longrightarrow C_0 \xrightarrow{\psi_0} P_0 \xrightarrow{\sigma} M \longrightarrow 0. \quad (3.4)$$

Now, choose a free $R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ -module P_1 together with a surjection map $\phi_1 : P_1 \rightarrow C_0$ with kernel C_1 . Again we choose a free $R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ -module P_2 together with a surjection $\phi_2 : P_2 \rightarrow C_1$ with kernel C_2 , then we get an exact sequence

$$0 \longrightarrow C_2 \xrightarrow{\psi_2} P_2 \xrightarrow{\phi_2} C_1 \xrightarrow{\psi_1} P_1 \xrightarrow{\phi_1} C_0 \xrightarrow{\psi_0} P_0 \xrightarrow{\sigma=\alpha_0} M \longrightarrow 0. \quad (3.5)$$

Continuing this process, we have an exact sequence

$$\phi : 0P_l \xrightarrow{\alpha_l} P_{l-1} \xrightarrow{\alpha_{l-1}} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\sigma=\alpha_0} M \longrightarrow 0, \quad (3.6)$$

which is a free resolution of the module M , with $\text{rank}(P_i) = t_i$, where $\alpha_l = \psi_{l-1}\phi_l$. For $l = 1$

$$0 \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0 \quad (3.7)$$

is a free resolution of module M . Since M is a projective module, this sequence splits, so there exists $\beta_1 : P_0 \rightarrow P_1$ such that $\beta_1\alpha_1 = I_{t_1}$. We can compute the matrix β_1 from the rows of the matrix α_1 . We express each vector of the canonical basis of P_1 as a linear combination of the rows of α_1 , and with these coefficients we construct the matrix β_1 . Using Lemma 3.1, the exact sequence in (3.7) splits, giving another exact sequence

$$0 \longrightarrow M \xrightarrow{\beta_0} P_0 \xrightarrow{\beta_1} P_1 \longrightarrow 0. \quad (3.8)$$

Then $P_1 \oplus M \cong P_0 \cong \text{Ker}(\beta_1) \oplus P_1$. Since M is a projective module, the short exact sequence

$$0 \longrightarrow \text{Ker}(\alpha_0) \longrightarrow P_0 \xrightarrow{\alpha_0} M \longrightarrow 0 \quad (3.9)$$

splits, so $\text{Ker}(\alpha_0) = \text{Im}(\alpha_1)$ is projective. Hence $\text{pd}(M) = 0$. Suppose M is not projective, then the short exact sequence (3.9) does not split. Therefore $\text{Im}(\alpha_1) = \text{Ker}(\alpha_0)$ is not projective. For $l = 2$, the exact sequence (3.6) does not split, therefore $\text{Im}(\alpha_2) = \text{Ker}(\alpha_1)$ is not projective. Continuing this process $\text{Im}(\alpha_1), \text{Im}(\alpha_2), \dots, \text{Im}(\alpha_{l-1})$ are not projective and $\text{Im}(\alpha_l)$ is projective. In this way after performing a finite number of steps, we obtain the minimum length of free resolution of M which is the projective dimension of M such that $\text{pd}(M) = l$ see Algorithm 1.

Remark 3.2. Now we give an algorithm to calculate the projective dimension of a module over the Laurent polynomial ring. We follow the technique discussed in [16] over polynomial ring.

Example 3.3. Let

$$A = R[x, x^{-1}, y, y^{-1}] \cong \frac{R[x, y, p, q]}{(xp - 1, yq - 1)} \quad (3.10)$$

ring $R = 0, (x, y, p, q), dp;$

ideal $I = xp - 1, yq - 1;$

$I = \text{std}(I);$

$I;$

$I[1] = yq - 1;$

Let

$$A = R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}] \cong \frac{R[x_1, y_1, x_2, y_2, \dots, x_n, y_n]}{(x_1 y_1 - 1, x_2 y_2 - 1, \dots, x_n y_n - 1)}.$$

Objective: Computation of the projective dimension of a module.

Input: A finitely generated A -module $M = \langle f_1, f_2, \dots, f_m \rangle \subset A^l$ and a positive integer l .

Output: A projective dimension of M and a list of matrices $\alpha_1, \alpha_2, \dots, \alpha_l$ with $\alpha_i \in \text{Mat}(l_{i-1} \times l_i, A)$, where $i = 1, 2, \dots, l$, such that

$$\partial : 0 \rightarrow P_l \xrightarrow{\alpha_l} P_{l-1} \xrightarrow{\alpha_{l-1}} P_{l-2} \xrightarrow{\alpha_{l-2}} \dots \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} P_0/M \rightarrow 0$$

is a free resolution of P_0/M . If $\text{pd}(M) = 0$, then M is projective. The algorithm returns $M \oplus A^r \cong A^l$.

START:

initialize $i = 1$;

if $(\alpha_1$ does not split)

$\alpha_1 = \text{matrix}(f_1, f_2, \dots, f_m) \in \text{Mat}(l \times m, A)$ and

$\text{pd}(M) = 1$;

else

Let β_1 be the split of α_1 . Then $\text{pd}(M) = 0$ and

$P_1 \oplus M \cong P_0 \cong \ker(\beta_1) \oplus P_1$;

end if

if $(i = l)$

return $\text{pd}(M)$;

else

while $(i < l)$ do

$i++$;

$\alpha_i = \text{syz}(\alpha_{i-1})$;

end loop

end if

if $(\alpha_l$ does not split)

$\text{pd}(M) = l$;

else

Let β_l be the split of α_l . Then $\text{pd}(M) = l - 1$;

end if

return $\text{pd}(M)$;

STOP.

Algorithm 1

$I[2] = xp - 1$;

qring $A = I$;

A ;

****quotient ring from ideal****

$[1] = yq - 1$;

$[2] = xp - 1$;

Poly $f_1 = x^3p + (x^3p)^{-1} + y^4q$;

Poly $f_2 = xp + (yq)^{-1} + yq$;

Ideal $M = (f_1, f_2)$;

Resolution $L = \text{mres}(M, 0)$;

We find a resolution of P_0/M over Laurent polynomial ring:

$$0 \rightarrow P_0 \xrightarrow{\alpha_0} P_0/M \rightarrow 0.$$

Hence, projective dimension of M is 0.

Example 3.4. Let

$$\begin{aligned} A &= R[x, x^{-1}, y, y^{-1}, z, z^{-1}, s, s^{-1}] \\ &\cong \frac{R[x, y, z, p, q, u, s, t]}{(xp - 1, yq - 1, zu - 1, st - 1)} \end{aligned} \quad (3.11)$$

```

ring R = 0, (x, y, z, p, q, u, s, t), dp;
ideal I = xp - 1, yq - 1, zu - 1, st - 1;
I = std(I);
I;
I[1] = st - 1
I[2] = zu - 1
I[3] = yq - 1
I[4] = xp - 1
qring A = I;
A;
**quotient ring from ideal**
[1] = st - 1
[2] = zu - 1
[3] = yq - 1
[4] = xp - 1
poly f = x^3p + (x^3p)^{-1} + y^4q + zu^2;
poly g = x^6p^2 + (yq)^{-1} + yq + s^3t + yq;
ideal M = (f, g);
resolution L = mres(M, 0).

```

We find a resolution of P_0/M over Laurent polynomial ring:

$$\begin{aligned} 0 \rightarrow P_{32}^3 \oplus P_{33}^6 \xrightarrow{\alpha_{11}} P_{30} \oplus P_{31}^5 \xrightarrow{\alpha_{10}} P_{28}^2 \oplus P_{29}^3 \xrightarrow{\alpha_9} P_{26}^2 \oplus P_{27}^2 \xrightarrow{\alpha_8} P_{20}^{-1} \oplus P_{21} \oplus P_{22}^3 \oplus P_{23} \oplus P_{24} \oplus P_{25} \xrightarrow{\alpha_7} \\ P_{15}^2 \oplus P_{16}^2 \oplus P_{17}^2 \oplus P_{18} \oplus P_{19} \xrightarrow{\alpha_6} P_{11}^3 \oplus P_{12}^2 \oplus P_{13}^2 \oplus P_{14} \xrightarrow{\alpha_5} P_8^4 \oplus P_9^2 \oplus P_{10}^2 \xrightarrow{\alpha_4} P_5^3 \oplus P_6 \oplus P_7 \xrightarrow{\alpha_3} P_2^{-1} \oplus P_3 \oplus P_4 \xrightarrow{\alpha_2} \\ P_1 \xrightarrow{\alpha_1} P_0^0 \xrightarrow{\alpha_0} P_0/M \rightarrow 0. \end{aligned}$$

Hence, projective dimension of M is 11.

Remark 3.5. Examples 3.3 and 3.4 are verified using the singular software [17].

4. A Cancellation Theorem for Projective Modules over Laurent Polynomial Ring

Proposition 4.1. *Let P and P' be projective $R[X, X^{-1}]$ -modules and let $\varphi : P' \rightarrow P$, $\psi : P \rightarrow P'$ be injective homomorphisms. If $R[X, X^{-1}]$, $P/\varphi\psi P$, and $P'/\psi\varphi P'$ are projective over R , then $P/\varphi P'$ and $P'/\psi P$ are also projective over R .*

Proof. Since P and P' are $R[X, X^{-1}]$ -projective and $R[X, X^{-1}]$ is R -projective, then we have

$$\begin{aligned}
 P \oplus P_1 &\cong R[X, X^{-1}]^n \\
 P' \oplus P'_1 &\cong R[X, X^{-1}]^m \\
 R[X, X^{-1}] \oplus Q &\cong \sum R \\
 &\vdots \\
 R[X, X^{-1}]^n \oplus Q^n &\cong \sum R \\
 P \oplus P_1 \oplus Q^n &\cong \sum R \\
 P \oplus Q_1^n &\cong \sum R,
 \end{aligned} \tag{4.1}$$

where $P_1 \oplus Q^n = Q_1^n$. Therefore, P is a R -projective. Similarly P' is also a R -projective. Since

$$0 \longrightarrow P \xrightarrow{\beta} P' \longrightarrow P'/\psi P \longrightarrow 0, \tag{4.2}$$

$\text{pd}(P'/\psi P) \leq 1$ and similarly $\text{pd}(P/\varphi P') \leq 1$. Now isomorphism between P and ψP induces $P/\varphi P' \cong \psi P/\psi\varphi P'$. We have the exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & P & \longrightarrow & P' & \longrightarrow & P'/\psi P \longrightarrow 0 \\
 & & & & & & \parallel \\
 0 & \longrightarrow & \psi P/\psi\varphi P' & \longrightarrow & P'/\psi\varphi P' & \longrightarrow & P'/\psi P \longrightarrow 0
 \end{array} \tag{4.3}$$

By Schanuel's lemma, $P'/\psi\varphi P' \oplus P \cong \psi P/\psi\varphi P' \oplus P'$ so that $P'/\psi\varphi P' \oplus P \cong P/\varphi P' \oplus P'$. Since $P'/\psi\varphi P'$, P are R -projective and direct sum of projective module is also projective module, then $P'/\psi\varphi P' \oplus P$ is R -projective. By the isomorphism, $P/\varphi P' \oplus P'$ is an R -projective. $P/\varphi P' \oplus P' \oplus P_0 \cong R^n$ follows from the definition of R -projective module, where P_0 is an R -module. Let $\bar{P} = P' \oplus P_0$ be an R -module. Then $P/\varphi P' \oplus \bar{P} \cong R^n$. Hence $P/\varphi P'$ is an R -projective. Similarly $P'/\psi P$ is also an R -projective. $\square$

Corollary 4.2. *Let P and P' be projective $R[X, X^{-1}]$ -modules with $P \supset P' \supset fP$, where f is a doubly monic polynomial of Laurent polynomial ring. If $R[X, X^{-1}]$ and $R[X, X^{-1}]/fR[X, X^{-1}]$ are R -projective, then P/P' is also R -projective.*

Proof. Take φ the inclusion map $P' \hookrightarrow P$ and ψ the multiplication by f from $P \rightarrow P'$. Then $P/\varphi\psi P = P/fP$ and $P'/\psi\varphi P' = P'/fP'$. Since P, P' are $R[X, X^{-1}]$ -projective and $R[X, X^{-1}], R[X, X^{-1}]/fR[X, X^{-1}]$ are R -projective. Then P/fP and P'/fP' are $R[X, X^{-1}]/fR[X, X^{-1}]$ -projective. Therefore P/fP and P'/fP' are R -projective. Hence $P/\varphi\psi P$ and $P'/\psi\varphi P'$ are also R -projective. By Proposition 4.1, module $P/P' (= P/\varphi P')$ is R -projective. $\square$

Proposition 4.3. *Let M be an R -module and $f \in \text{End}_R(M)$. The characteristic sequence of f*

$$0 \longrightarrow M[X, X^{-1}] \xrightarrow{X.1_{M[X, X^{-1}]} - f[X, X^{-1}]} M[X, X^{-1}] \xrightarrow{\varphi_f} {}_fM \longrightarrow 0 \quad (4.4)$$

is an exact sequence in $R[X, X^{-1}]$ -module.

Proof. Clearly φ_f is surjective, and we have

$$\begin{aligned} \varphi_f \left(X.1_{M[X, X^{-1}]} - f[X, X^{-1}] \right) \left(\sum m_n X^n \right) &= \varphi_f \left(\sum (m_n X^{n+1} - f(m_n) X^n) \right) \\ &= \sum (f^{n+1}(m_n) - f^n(f(m_n))) \\ &= \sum (f^{n+1}(m_n) - f^{n+1}(m_n)) \\ &= 0. \end{aligned} \quad (4.5)$$

$\text{Im}(X.1_{M[X, X^{-1}]} - f[X, X^{-1}]) \subseteq \text{Ker}(\varphi_f)$. Since, $X.1_{M[X, X^{-1}]} - f[X, X^{-1}]$ raises degree by one, and preserves leading coefficients, it is monomorphism. Finally, suppose

$$z = \sum m_n X^n \in \text{Ker}(\varphi_f) \implies \varphi_f \left(\sum m_n X^n \right) = 0, \text{ that is, } \sum f^n(m_n) = 0. \quad (4.6)$$

Then

$$\begin{aligned} z &= z - \sum f^n(m_n) \\ &= \sum (m_n X^n - f^n(m_n)) \\ &= \sum (X^n.1_{M[X, X^{-1}]} - f^n)(m_n) \\ &= \left(X.1_{M[X, X^{-1}]} - f[X, X^{-1}] \right) \left[\cdots - \frac{1}{X-f} \frac{X^n - f^n}{X^n f^n} - \frac{1}{X-f} \frac{X^{n-1} - f^{n-1}}{X^{n-1} f^{n-1}} \cdots \right. \\ &\quad \left. - \frac{1}{X-f} \frac{X-f}{Xf} + 0 + \frac{X-f}{X-f} + \frac{X^2 - f^2}{X-f} + \cdots \right. \\ &\quad \left. + \frac{X^n - f^n}{X-f} + \cdots \right] (m_n) \end{aligned}$$

$$\begin{aligned}
&= \left(X \cdot 1_{M[X, X^{-1}]} - f[X, X^{-1}] \right) \sum h_n(m_n) \\
\Rightarrow z &\in \text{Im} \left(X \cdot 1_{M[X, X^{-1}]} - f[X, X^{-1}] \right).
\end{aligned} \tag{4.7}$$

Therefore $\text{Ker}(\varphi_f) \subseteq \text{Im}(X \cdot 1_{M[X, X^{-1}]} - f[X, X^{-1}])$.
Hence $\text{Im}(X \cdot 1_{M[X, X^{-1}]} - f[X, X^{-1}]) = \text{Ker}(\varphi_f)$. $\square$

Theorem 4.4. *Let P and P' be finitely generated projective modules over $R[X, X^{-1}]$. Suppose that $P \supset P' \supset fP$ for some doubly monic polynomial $f \in R[X, X^{-1}]$. Then P and P' are stably isomorphic. In particular, if $P_f \cong P'_f$, then P and P' are stably isomorphic.*

Proof. Take $M = P/P'$. Since f is doubly monic polynomial, $R[X, X^{-1}]/(f)$ is free as an R -module. Therefore P/fP is R -projective, whence M is R -projective by Corollary 4.2. We have exact sequences of $R[X, X^{-1}]$ -modules

$$\begin{array}{ccccccc}
0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & M \longrightarrow 0 \\
& & & & & & \parallel \\
0 & \longrightarrow & M[X, X^{-1}] & \longrightarrow & M[X, X^{-1}] & \longrightarrow & M \longrightarrow 0
\end{array} \tag{4.8}$$

where $M[X, X^{-1}] = M \otimes_R R[X, X^{-1}]$. Since $P' \subset P$ and $M = P/P'$, the inclusion map i from P' to P and the surjective map π from P to M makes the first sequence an exact sequence. By Proposition 4.3, second sequence is also an exact sequence. Since M is an R -projective, $M[X, X^{-1}]$ is an $R[X, X^{-1}]$ -projective. Therefore by the Schanuel's lemma, $P \oplus M[X, X^{-1}] \cong P' \oplus M[X, X^{-1}]$. Hence P, P' are stably isomorphic. $\square$

Theorem 4.5. *Let F be a free $R[X, X^{-1}]$ -module of rank 2, P a projective $R[X, X^{-1}]$ -module and $F \supset P \supset XF$. Then $P \cong F$.*

Proof. Let $\{f_1, f_2\}$ be a free $R[X, X^{-1}]$ -basis for F , and let $F_1 = \{a_1 f_1 + a_2 f_2 \mid a_1, a_2 \in R\}$. Then F_1 is free over R of rank 2.

Claim. We show that each element of $F_1 \oplus XF \oplus (1/X)F$ can be written as elements of F .

$$\begin{aligned}
F_1 \oplus XF \oplus \frac{1}{X}F &\Longleftrightarrow a_1 f_1 + a_2 f_2 + X(g_1 f_1 + g_2 f_2) + \frac{1}{X}(h_1 f_1 + h_2 f_2) \\
&\Longleftrightarrow \left(a_1 + Xg_1 + \frac{1}{X}h_1\right)f_1 + \left(a_2 + Xg_2 + \frac{1}{X}h_2\right)f_2 \\
&\Longleftrightarrow Uf_1 + Vf_2 \\
&\Longleftrightarrow F(\text{direct sum as } R\text{-module}),
\end{aligned} \tag{4.9}$$

where g_1, g_2, h_1, h_2, U , and V are the elements of $R[X, X^{-1}]$. Thus with $P_1 = P \cap F_1$, $P = P_1 \oplus XF \oplus (1/X)F$ (direct sum as R -module). By Corollary 4.2, $F_1/P_1 = F/P$ is a projective over

R , so $F_1 = P_1 \oplus P'_1$ for some P'_1 . Now $F = P_1[X, X^{-1}] \oplus P'_1[X, X^{-1}]$ and $P = P_1 \oplus XF \oplus (1/X)F = P_1 \oplus X(P_1[X, X^{-1}] \oplus P'_1[X, X^{-1}]) \oplus (1/X)(P_1[X, X^{-1}] \oplus P'_1[X, X^{-1}]) = P_1[X, X^{-1}] \oplus P'_1[X, X^{-1}] \cong F$. $\square$

5. A Qualitative Version of Horrocks' Theorem

Theorem 5.1. *Let R be a local ring and P be a projective $R[X, X^{-1}]$ -module. Then, for any doubly monic polynomial f in $R[X, X^{-1}]$, the $R[X, X^{-1}]$ -module P and the $R[X, X^{-1}]_f$ -module P_f have the same minimal number of generators, that is, $\mu(P) = \mu(P_f)$.*

Proof. Let f be a doubly monic polynomial in $R[X, X^{-1}]$. Since there is a natural surjective map $\phi : P \rightarrow P_f$, we have $\mu(P_f) \leq \mu(P)$. To complete the proof it is enough to show $\mu(P_f) \geq \mu(P)$. We have the following two cases.

Case 1. P_f is a free module over $R[X, X^{-1}]_f$. By Theorem 2.2, P is free and the result follows.

Case 2. Suppose P_f is not free over $R[X, X^{-1}]_f$. Then $\text{Rank}(P_f) < \mu(P_f)$. Let $\mu(P_f) = n$. We can write P_f as a direct summand of $(R[X, X^{-1}]_f)^n \cong (R[X, X^{-1}]^n)_f$. Since $\text{Rank}(P) = \text{Rank}(P_f) < \text{Rank}(R[X, X^{-1}]^n)$. By Theorem 2.3, there exists an $R[X, X^{-1}]$ -module P' such that $P \oplus P' \cong R[X, X^{-1}]^n$. Therefore $\mu(P) \leq n = \mu(P_f)$. $\square$

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