Research Article

The Cycle-Complete Graph Ramsey Number $r(C_9, K_8)$

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1. Introduction

All graphs considered in this paper are undirected and simple. $C_m, P_m, K_m$ and $S_m$ stand for cycle, path, complete, and star graphs on $m$ vertices, respectively. The graph $K_1 + P_n$ is obtained by adding an additional vertex to the path $P_n$ and connecting this new vertex to each vertex of $P_n$. The number of edges in a graph $G$ is denoted by $E(G)$. Further, the minimum degree of a graph $G$ is denoted by $\delta(G)$. An independent set of vertices of a graph $G$ is a subset of the vertex set $V(G)$ in which no two vertices are adjacent. The independence number of a graph $G$, $\alpha(G)$, is the size of the largest independent set. The neighborhood of the vertex $u$ is the set of all vertices of $G$ that are adjacent to $u$, denoted by $N(u)$. $N[u]$ denote to $N(u) \cup \{u\}$. For vertex-disjoint subgraphs $H_1$ and $H_2$ of $G$ we let $E(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$. Let $H$ be a subgraph of the graph $G$ and $U \subseteq V(G)$, $N_H(U)$ is defined as $\left(\bigcup_{u \in U} N(u)\right) \cap V(H)$. Suppose that $V_1 \subseteq V(G)$ and $V_1$ is nonempty, the subgraph of $G$ whose vertex set is $V_1$ and whose edge set is the set of those edges of $G$ that have both ends in $V_1$ is called the subgraph of $G$ induced by $V_1$, denoted by $\langle V_1 \rangle_G$.

The cycle-complete graph Ramsey number $r(C_m, K_n)$ is the smallest integer $N$ such that for every graph $G$ of order $N$, $G$ contains $C_m$ or $\alpha(G) \geq n$. The graph $(n - 1)K_{m-1}$ shows that $r(C_m, K_n) \geq (m - 1)(n - 1) + 1$. In one of the earliest contributions to graphical Ramsey theory, Bondy and Erdős [1] proved the following result: for all $m \geq n^2 - 2$, $r(C_m, K_n) = (m - 1)(n - 1) + 1$. The above restriction was improved by Nikiforov [2] when he proved the equality for $m \geq 4n + 2$. Erdős et al. [3] gave the following conjecture.
Conjecture 1. \( r(C_m, K_n) = (m - 1)(n - 1) + 1, \) for all \( m \geq n \geq 3 \) except \( r(C_5, K_3) = 6. \)

The conjecture was confirmed by Faudree and Schelp [4] and Rosta [5] for \( n = 3 \) in early work on Ramsey theory. Yang et al. [6] and Bollobas et al. [7] proved the conjecture for \( n = 4 \) and \( n = 5 \), respectively. The conjecture was proved by Schiermeyer [8] for \( n = 6. \) Jaradat and Baniabdelrahman [9, 10] proved the conjecture for \( n = 7 \) and \( m = 7, 8. \) Later on, Chena et. al. [11] proved the conjecture for \( n = 7. \) Recently, Jaradat and Al-Zaleq [12] and Y. Zhang and K. Zhang [13], independently, proved the conjecture in the case \( n = m = 8. \) In a related work, Radziszowski and Tse [14] showed that \( r(C_4, K_7) = 22 \) and \( r(C_5, K_8) = 26. \) In [15] Jayawardene and Rousseau proved that \( r(C_5, K_6) = 21. \) Also, Schiermeyer [16] proved that \( r(C_5, K_7) = 25. \) For more results regarding the Ramsey numbers, see the dynamic survey [17] by Radziszowski.

Until now, the conjecture is still open. Researchers are interested in determining all the values of the Ramsey number \( r(C_m, K_n) \). In this paper our main purpose is to determine the values of \( r(C_9, K_8) \) which confirm the conjecture in the case \( m = 9 \) and \( n = 8. \) The following known theorem will be used in the sequel.

**Theorem 1.1.** Let \( G \) be a graph of order \( n \) without a path of length \( k (k \geq 1) \). Then

\[
\mathcal{L}(G) \leq \frac{k - 1}{2} n.
\]

Further, equality holds if and only if its components are complete graphs of order \( k. \)

### 2. Main Result

In this paper we confirm the Erdős, Faudree, Rousseau, and Schelp conjecture in the case \( C_9 \) and \( K_8. \) In fact, we prove that \( r(C_9, K_8) = 57. \) It is known, by taking \( G = (n - 1)K_{m-1} \), that \( r(C_m, K_n) \geq (m - 1)(n - 1) + 1. \) In this section we prove that this bound is exact in the case \( m = 9 \) and \( n = 8. \) Our proof depends on a sequence of 8 lemmas.

**Lemma 2.1.** Let \( G \) be a graph of order \( \geq 57 \) that contains neither \( C_9 \) nor an 8-element independent set. Then \( \delta(G) \geq 8. \)

**Proof.** Suppose that \( G \) contains a vertex of degree less than 8, say \( u. \) Then \( |V(G - N[u])| \geq 49. \) Since \( r(C_9, K_7) = 49, \) as a result \( G - N[u] \) has independent set consists of 7 vertices. This set with the vertex \( u \) is an 8-element independent set of vertices of \( G. \) That is a contradiction.

Throughout all Lemmas 2.2 to 2.8, we let \( G \) be a graph with minimum degree \( \delta(G) \geq 8 \) that contains neither \( C_9 \) nor an 8-element independent set.

**Lemma 2.2.** If \( G \) contains \( K_8, \) then \( |V(G)| \geq 72. \)

**Proof.** Let \( U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\} \) be the vertex set of \( K_8. \) Let \( R = G - U \) and \( U_i = N(u_i) \cap V(R) \) for each \( 1 \leq i \leq 8. \) Since \( \delta(G) \geq 8, \) \( U_i \neq \emptyset \) for all \( 1 \leq i \leq 8. \) Since there is a path of order \( 8 \) joining any two vertices of \( U, \) as a result \( U_i \cap U_j = \emptyset \) for all \( 1 \leq i < j \leq 8 \) (otherwise, if \( w \in U_i \cap U_j \) for some \( 1 \leq i < j \leq 8, \) then the concatenation of the \( u_iu_j \)-path of order \( 8 \) with \( u_iwu_i, \) is a cycle of order \( 9, \) a contradiction). Similarly, since there is a path of order \( 7 \) joining any two vertices of \( U, \) as a result for all \( 1 \leq i < j \leq 8 \) and for all \( x \in U_i \) and \( y \in U_j \)
we have that $xy \notin E(G)$ (otherwise, if there are $1 \leq i < j \leq 8$ such that $x \in U_i$, $y \in U_j$ and $xy \in E(G)$, then the concatenation of the $u_iu_j$-path of order 7 with $u_ixwyu_j$, is a cycle of order 9, a contradiction). Also, since there is a path of order 6 joining any two vertices of $U$, as a result, $N_R(U_i) \cap N_R(U_j) = \emptyset$, $1 \leq i < j \leq 8$ (otherwise, if there are $1 \leq i < j \leq 8$ such that $w \in N_R(U_i) \cap N_R(U_j)$, then the concatenation of the $u_iu_j$-path of order 6 with $u_ixwyu_j$, is a cycle of order 9 where $x \in U_i$, $y \in U_j$ and $xw, wy \in E(G)$, a contradiction). Therefore $|U_i \cup N_R(U_i) \cup \{u_i\}| \geq \delta(G) + 1$. Thus, $|V(G)| \geq 8(\delta(G) + 1) \geq (8)(9) = 72$. 

\section*{Lemma 2.3.} If $G$ contains $K_8 - S_6$, then $G$ contains $K_8$.

\textbf{Proof.} Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ be the vertex set of $K_8 - S_6$ where the induced subgraph of $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ is isomorphic to $K_7$. Without loss of generality we may assume that $u_1u_8u_2u_7 \in E(G)$. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 8$. Then, as in Lemma 2.2, we have the following:

(1) $U_i \cap U_j = \emptyset$ for all $1 \leq i < j \leq 8$ except possibly for $i = 1$ and $j = 2$.

(2) $E(U_i, U_j) = \emptyset$ for all $1 \leq i < j \leq 8$.

(3) $N_R(U_i) \cap N_R(U_j) = \emptyset$ for all $1 \leq i < j \leq 8$.

(4) $E(N_R(U_i), N_R(U_j)) = \emptyset$ for all $1 \leq i < j \leq 8$.

Since $\alpha(G) \leq 7$, as a result at least five of the induced subgraphs $\langle U_i \cup N_R(U_i) \rangle_G$, $3 \leq i \leq 8$ are complete. Since $\delta(G) \geq 8$, it implies that these complete graphs contain $K_8$. Hence, $G$ contains $K_8$. 

\section*{Lemma 2.4.} If $G$ contains $K_7$, then $G$ contains $K_8 - S_6$ or $K_8$.

\textbf{Proof.} Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be the vertex set of $K_7$. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 7$. Since $\delta(G) \geq 8$, $U_i \neq \emptyset$ for all $1 \leq i \leq 7$. Now we consider the following two cases.

\textbf{Case 1.} $U_i \cap U_j \neq \emptyset$, for some $1 \leq i < j \leq 7$, say $w \in U_i \cap U_j$. Then it is clear that $G$ contains $K_8 - S_6$. In fact, the induced subgraph $\langle U \cup \{w\} \rangle_G$ contains $K_8 - S_6$.

\textbf{Case 2.} $U_i \cap U_j = \emptyset$. for all $1 \leq i < j \leq 7$. Note that between any two vertices of $U$ there are paths of order 5, 6 and 7. Thus, as in Lemma 2.2, for all $1 \leq i < j \leq 7$, we have the following.

(1) $E(U_i, U_j) = \emptyset$.

(2) $N_R(U_i) \cap N_R(U_j) = \emptyset$.

(3) $E(N_R(U_i), N_R(U_j)) = \emptyset$.

Since $\alpha(G) \leq 7$, we have that the induced subgraphs $\langle U_i \cup N_R(U_i) \rangle_G$, $1 \leq i \leq 7$ are complete. Since $\delta(G) \geq 8$, it implies that these complete graphs contain $K_8$. Hence, $G$ contains $K_8$. 

\section*{References}
Lemma 2.5. If $G$ contains $K_1 + P_7$, then $G$ contains $K_2$.

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ be the vertex set of $K_1 + P_7$ where $K_1 = u_1$ and $P_7 = u_2u_3u_4u_5u_6u_7u_8$. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 8$. Now we have the following two cases.

Case 1. $U_4 \cap U_6 = \emptyset$. Since $\delta(G) \geq 8$, $U_i \neq \emptyset$ for all $1 \leq i \leq 8$. Now we have the following.

1. $U_i \cap U_j = \emptyset$ for all $2 \leq i < j \leq 8$ except possibly for $(i, j) \in \{(3, 5), (3, 6), (3, 7), (4, 7), (5, 7)\}$ since otherwise a cycle of order 9 is produced, a contradiction.

2. $E(U_i, U_j) = \emptyset$ for all $2 \leq i < j \leq 8$.

3. $N_R(U_i) \cap N_R(U_j) = \emptyset$ for all $2 \leq i < j \leq 8$.

4. $E(N_R(U_i), N_R(U_j)) = \emptyset$ for all $2 \leq i < j \leq 8$.

{(2), (3), and (4) follows easily from being that $K_1 + P_7$ contains paths of order 7, 6, and 5 between any two vertices $u_i$ and $u_j$, $2 \leq i < j \leq 8$. Since $\alpha(G) \leq 7$, as a result at least three of the induced subgraphs $(U_i \cup N_R(U_i))_C, i = 2, 4, 5, 6, 8$ are complete graphs. Now we have the following two assertions.

(i) $|N_R(U_i)| \geq 7$ and so $|U_i \cup N_R(U_i)| \geq 8$ for each $i = 2, 8$. The following is the proof of assertion (i) for $i = 8$.

Since $\delta(G) \geq 8$, $|U_8| \geq 1$. Let $y \in U_8$ and $y$ is adjacent to $x \in \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$.

Then we have the following.

(i) If $x = u_1$, then $u_8yu_1u_2u_3u_4u_5u_6u_7u_8$ is a $C_9$, this is a contradiction.

(ii) If $x = u_2$, then $u_8yu_2u_3u_4u_5u_6u_7u_1u_8$ is a $C_9$, this is a contradiction.

(iii) If $x = u_3$, then $u_8yu_3u_2u_1u_4u_5u_6u_7u_8$ is a $C_9$, this is a contradiction.

(iv) If $x = u_4$, then $u_8yu_4u_3u_2u_1u_5u_6u_7u_8$ is a $C_9$, this is a contradiction.

(v) If $x = u_5$, then $u_8yu_5u_4u_3u_2u_1u_6u_7u_8$ is a $C_9$, this is a contradiction.

(vi) If $x = u_6$, then $u_8yu_6u_5u_4u_3u_2u_1u_7u_8$ is a $C_9$, this is a contradiction.

(vii) If $x = u_7$, then $u_8yu_7u_6u_5u_4u_3u_2u_1u_8$ is a $C_9$, this is a contradiction.

Since $\delta(G) \geq 8$, $|N_R(y)| \geq 7$, and so $|\{y\} \cup N_R(y)| \geq 8$. Hence, $|U_8 \cup N_R(U_8)| \geq 8$. By a similar argument as above and using the symmetry of $P_7 + K_1$, one can easily show that $|U_2 \cup N_R(U_2)| \geq 8$.

(ii) If there is $i \in \{4, 5, 6\}$ such that $|N_R(U_i)| < 6$, then $|N_R(U_i)| \geq 6$ and so $|U_i \cup N_R(U_i)| \geq 7$ for any $j \in \{4, 5, 6\}$ with $i \neq j$. The following is the proof of assertion (ii).

Assume that $|N_R(U_4)| < 6$. By (1) $U_4 \cap U_i = \emptyset$ for all $2 \leq i \leq 8$ except possibly $i = 4, 7$. Thus, for $y \in U_4$, $y$ is adjacent to $u_4$ and to at most $u_1$ and $u_7$. Now we show that $|N_R(U_5)| \geq 6$. Assume $|N_R(U_5)| < 6$. By (1) $U_5 \cap U_i = \emptyset$ for all $2 \leq i \leq 8$ except possibly $i = 3, 5, 7$. Thus, for any $w \in U_5$, $w$ is adjacent to $u_5$ and to at most $u_1$, $u_3$ and $u_7$. Now, we have the following.

(A) If $w$ adjacent to both $u_1$ and $u_3$, then $u_2u_3wu_4u_5u_6u_7u_8y_1u_2$ is a $C_9$.

(B) If $w$ adjacent to both $u_1$ and $u_7$, then $u_2u_3wu_4u_5u_6u_7u_8u_2$ is a $C_9$.

(C) If $w$ adjacent to both $u_3$ and $u_7$, then $u_2u_3wu_4u_5u_6u_7u_8u_1u_2$ is a $C_9$.

Thus, $w$ is adjacent to at most one of $u_1, u_3$, and $u_7$, and so $|N_R(U_5)| \geq 6$. We now show that $|N_R(U_6)| \geq 6$. As above assume $|N_R(U_6)| < 6$. By (1), $U_6 \cap U_i = \emptyset$ for all $2 \leq i \leq 8$ except
possibly \( i = 3, 6 \). Thus, for \( w \in U_9 \), \( w \) is adjacent to \( u_1, u_3, \) and \( u_6 \). Hence, \( u_6u_7yu_4u_3wu_6u_5u_1u_8 \) is a \( C_9 \), which implies that \( w \) is adjacent to at most one of \( u_1 \) and \( u_3 \) and so \( |N_R(U_6)| \geq 6 \).

Now, by using the same argument as above and taking into account that \( P_7 + K_1 \) is symmetric, we can easily see that if \( |N_R(U_6)| < 6 \), then both of \( |N_R(U_4)| \) and \( |N_R(U_5)| \) are greater than or equal 6. So we need to consider the case when \( |N_R(U_5)| < 6 \). As above, \( U_5 \cap U_i = \emptyset \) for all \( 2 \leq i \leq 8 \) except possibly \( i = 5, 3 \) and 7. Thus, for any \( w \in U_5 \), \( w \) is adjacent to \( u_5 \) and to at most \( u_1, u_3 \) and \( u_7 \). Now, assume that \( |N_R(U_4)| < 6 \). By using (A), (B) and (C) as above and using the same arguments to get the same contradiction. Similarly, by symmetry we get that \( |N_R(U_6)| \geq 6 \).

Therefore, from (i) and (ii), at least four of the induced subgraphs \( (U_i \cup N_R(U_i)) \), \( i = 2, 4, 5, 6 \) contain 7 vertices and so at least two of them contain \( K_7 \). Thus, \( G \) contains \( K_7 \).

Case 2. \( U_4 \cap U_6 \neq \emptyset \), say \( w_0 \in U_4 \cap U_6 \). For simplicity, in the rest of this case we consider \( U'_i = N(u_i) \cap V(R') \) where \( R' = G - U \cup \{u_6\} \) and let \( J = \{2, 3, 5, 7, 8, 9\} \). Then \( u_2u_9, u_3u_9, u_5u_9, u_7u_9 \notin E(G) \) (otherwise, \( G \) contains \( C_9 \)) and \( \delta(G) \geq 8 \). Hence \( U'_i \neq \emptyset \), for all \( i \in J \). Now we have the following assertions (see the Appendix).

\[
\begin{align*}
(1) & \quad U'_i \cap U'_j = \emptyset \text{ for all } i, j \in J \text{ and } i \neq j. \\
(2) & \quad E(U'_i, U'_j) = \emptyset \text{ for all } i, j \in J \text{ and } i \neq j. \\
(3) & \quad N_R(U'_i) \cap N_R(U'_j) = \emptyset \text{ for all } i, j \in J \text{ and } i \neq j. \\
(4) & \quad E(N_R(U'_i), N_R(U'_j)) = \emptyset \text{ for all } i, j \in J \text{ and } i \neq j. \\
\end{align*}
\]

Since \( \alpha(G) \leq 7 \), as a result at least five of the induced subgraphs \( (U'_i \cup N_R(U'_i)) \), \( i = 2, 3, 5, 7, 8, 9 \) are complete graphs. Since \( \delta(G) \geq 8 \) and \( G \) contains no \( C_9 \), \( |N_R(U'_i)| \geq 6 \) and so \( |U'_i \cup N_R(U'_i)| \geq 7 \) for each \( i = 2, 5, 8, 9 \). Therefore at least three of the induced subgraphs \( (U'_i \cup N_R(U'_i)) \), \( i = 2, 3, 5, 7, 8, 9 \) contain \( K_7 \). Thus, \( G \) contains \( K_7 \).

**Lemma 2.6.** If \( G \) contains \( K_1 + P_6 \), then \( G \) contains \( K_1 + P_7 \) or \( K_7 \).

**Proof.** Let \( U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\} \) be the vertex set of \( K_1 + P_6 \) where \( K_1 = u_1 \) and \( P_6 = u_2u_3u_4u_5u_6u_7 \). Let \( R = G - U \) and \( U_i = N(u_i) \cap V(R) \) for each \( 1 \leq i \leq 7 \). Since \( \delta(G) \geq 8 \), \( |U_i| \geq 2 \) for all \( 1 \leq i \leq 7 \). Now we have the following cases.

Case 1. \( U_i \cap U_j = \emptyset \) for all \( 2 \leq i < j \leq 7 \). Then we have the following.

\[
\begin{align*}
(1) & \quad E(U_i, U_j) = \emptyset \text{ for all } 2 \leq i < j \leq 7 \text{ except possibly for } (i, j) \in \{(3, 5), (3, 6), (4, 6)\}. \\
(2) & \quad N_R(U_i) \cap N_R(U_j) = \emptyset \text{ for all } 2 \leq i < j \leq 7. \\
(3) & \quad E(N_R(U_i), N_R(U_j)) = \emptyset \text{ for all } 2 \leq i < j \leq 7. \\
\end{align*}
\]

Since \( \alpha(G) \leq 7 \), as a result at least one of the induced subgraphs \( (U_i \cup N_R(U_i)) \), \( i = 2, 4, 5, 7 \) is complete. Since \( \delta(G) \geq 8 \), it implies that this complete graph contains \( K_7 \).
Case 2. \( U_i \cap U_j \neq \emptyset \) for some \( 2 \leq i < j \leq 7 \), say \( u_6 \in U_i \cap U_j \). In the rest of this case we have the following subcases:

Subcase 2.1. \( (r, s) \in \{(6, 7), (5, 7), (4, 7), (7, 3), (2, 7), (5, 6), (4, 6), (6, 3), (4, 5)\} \). For simplicity, in the rest of this subcase we consider \( U_i' = N(u_i) \cap V(R') \) where \( R' = G - U \cup \{u_6\} \) and let \( J = \{m : 2 \leq m \leq 8 \} \) and \( m \notin \{r, s, [(r + s) / s] + 1\} \). Since \( \delta(G) \geq 8 \), then \( U_i' \neq \emptyset \), for all \( 2 \leq i \leq 8 \). Now we have the following assertions.

1. \( U_i' \cap U_j' = \emptyset \) for all \( i, j \in J \) with \( i \neq j \).
2. \( E(U_i', U_j') = \emptyset \) for all \( i, j \in J \) with \( i \neq j \).
3. \( N_R(U_i') \cap N_R(U_j') = \emptyset \) for all \( i, j \in J \) with \( i \neq j \).
4. \( E(N_R(U_i'), N_R(U_j')) = \emptyset \) for all \( i, j \in J \) with \( i \neq j \).

Since \( \alpha(G) \leq 7 \), as a result at least one of the induced subgraphs \( \langle U_i' \cup N_R(U_i') \rangle \rangle, i \in J \) is complete. Since \( \delta(G) \geq 8 \) and \( |U_i'| \geq 2 \) for each \( i \in J \) (because otherwise \( G \) contains \( K_1 + P_7 \)), it implies that this complete graph contains \( K_7 \).

Subcase 2.2. \( (r, s) \notin \{(6, 7), (5, 7), (4, 7), (7, 3), (2, 7), (5, 6), (4, 6), (6, 3), (4, 5)\} \). Then, by the symmetry, we have a subcase similar to Subcase 2.1. \( \qed \)

**Lemma 2.7.** If \( G \) contains \( K_6 \), then \( G \) contains \( K_1 + P_6 \) or \( K_7 \).

**Proof.** Let \( U = \{u_1, u_2, u_3, u_4, u_5, u_6\} \) be the vertex set of \( K_6 \). Let \( R = G - U \) and \( U_i = N(u_i) \cap V(R) \) for each \( 1 \leq i \leq 6 \). Since \( \delta(G) \geq 8 \), \( |U_i| \geq 3 \) for all \( 1 \leq i \leq 6 \). Now we split our work into the following two cases.

**Case 1.** There are \( 1 \leq i < j \leq 6 \) such that \( U_i \cap U_j \neq \emptyset \), then \( G \) contains \( K_1 + P_6 \).

**Case 2.** \( U_i \cap U_j = \emptyset \) for all \( 1 \leq i < j \leq 6 \). Then we consider the following subcases.

**Subcase 2.1.** \( E(U_i, U_j) = \emptyset \) for all \( 1 \leq i < j \leq 6 \). Since between any two vertices of \( U \) there are paths of order 5 and 6, as a result \( N_R(U_i) \cap N_R(U_j) = \emptyset \) and \( E(N_R(U_i), N_R(U_j)) = \emptyset \) for each \( 1 \leq i < j \leq 6 \). Therefore, since \( \alpha(G) \leq 7 \), at least five of the induced subgraphs \( \langle U_i' \cup N_R(U_i') \rangle \rangle, 1 \leq i \leq 6 \) are complete graphs. Since \( \delta(G) \geq 8 \), these complete graphs contain \( K_7 \). Thus, \( G \) contains \( K_7 \).

**Subcase 2.2.** \( E(U_i, U_j) \neq \emptyset \) for some \( 1 \leq i < j \leq 6 \), say \( i = 1 \) and \( j = 2 \) and \( u_1u_2u_3u_4 \) is a path. For simplicity, in the rest of this subcase we consider \( U_i' = N(u_i) \cap V(R') \) where \( R' = G - U \cup \{u_7, u_8\} \). Since \( \delta(G) \geq 8 \), then \( U_i' \neq \emptyset \), for all \( 3 \leq i \leq 8 \). Now we have the following.

1. \( U_i' \cap U_j' = \emptyset \) for all \( 3 \leq i < j \leq 8 \).
2. \( E(U_i', U_j') = \emptyset \) for all \( 3 \leq i < j \leq 8 \).
3. \( N_R(U_i') \cap N_R(U_j') = \emptyset \) for all \( 3 \leq i < j \leq 8 \).
4. \( E(N_R(U_i'), N_R(U_j')) = \emptyset \) for all \( 3 \leq i < j \leq 8 \).

Therefore, since \( \alpha(G) \leq 7 \), at least five of the induced subgraphs \( \langle U_i' \cup N_R(U_i') \rangle \rangle, 3 \leq i \leq 8 \) are complete graphs. Since \( \delta(G) \geq 8 \), it implies that these complete graphs contain \( K_7 \). Thus, \( G \) contains \( K_7 \). \( \qed \)
Lemma 2.8. If \( G \) be a graph of order \( \geq 57 \), then \( G \) contains \( K_1 + P_6 \) or \( K_6 \).

Proof. Suppose that \( G \) contains neither \( K_1 + P_6 \) nor \( K_6 \). Then we have the following claims.

Claim 1. \( |N(u)| \leq 28 \) for any \( u \in V(G) \).

Proof. Suppose that \( u \) is a vertex with \( |\langle N_G(u) \rangle_G| \geq 29 \). Let \( \langle N_G(u) \rangle_G = \bigcup_{i=1}^r G_i \) where \( G_i \) is a component for each \( i \). \( \langle N_G(u) \rangle_G \) has minimum number of independent vertices if it has a maximum number of edges. Thus, by Theorem 1.1 \( G_i \) must be a complete graph for each \( i \). But \( \langle N_G(u) \rangle_G \) contains no \( P_6 \). Thus, \( G_i \) must be a complete graph of order at most 5. Also \( \langle N_G(u) \rangle_G \) contains no \( K_5 \), thus \( G_i \) must be a complete graph of order at most 4. Hence, the minimum number of independent vertices of \( \langle N_G(u) \rangle_G \) occurs only if \( \langle N_G(u) \rangle_G \) contains either a 7 tetrahedrons and an isolated vertex or 6 tetrahedrons, a triangle and a \( K_2 \) or 6 tetrahedrons and 2 triangles. In any of these cases \( \alpha(G) \geq 8 \). This is a contradiction. The proof of the claim is complete.

Claim 2. \( \alpha(G) = 7 \).

Proof. Since \( |V(G)| \geq 57 \) and \( G \) contains no \( C_9 \) and since \( r(C_9, K_7) = 49 \), \( \alpha(G) \geq 7 \). But \( G \) has no 8-element independent set, so \( \alpha(G) \leq 7 \). Thus, \( \alpha(G) = 7 \). The proof of the claim is complete.

Now, for any 7 independent vertices \( u_1, u_2, u_3, u_4, u_5, u_6, \) and \( u_7 \), set \( N_i[u_{i+1}] = \bigcup_{j=1}^i N[u_j], \) \( 1 \leq i \leq 6 \). Analogously, we set \( N_i[u_{i+1}], \) \( 1 \leq i \leq 6 \). Let \( A = \bigcup_{i=1}^6 N_i[u_{i+1}], \) \( B = \bigcup_{i=1}^6 N_i(u_{i+1}), \) and \( \beta = \alpha(\langle B \rangle_G) \).

Claim 3. \( |N(u_1) \cup B| \geq 50 \).

Proof. Suppose that \( |N(u_1) \cup B| \leq 49 \). Then \( |N[u_1] \cup A| \leq 56 \). And so \( |G-(N[u_1] \cup A)| \geq 57-56 = 1 \). But \( r(C_9, K_1) = 1 \), so \( G-(N[u_1] \cup A) \) contains a vertex, say \( u_8 \), which is not adjacent to any of \( u_1, u_2, u_3, u_4, u_5, u_6, \) and \( u_7 \). Thus, \( \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\} \) is an 8-element independent set. Therefore, \( \alpha(G) \geq 8 \). That is a contradiction. The proof of the claim is complete.

Now, by Lemma 2.1, \( \delta(G) \geq 8 \) and so by Claim 1, we have that \( 8 \leq |N(u_1)| \leq 28 \). Thus, if \( |N(u_1)| = r \), then \( |B| \geq 50 - r \). By a similar argument as in Claim 1, we have that \( \alpha(\langle N(u_1) \rangle_G) \geq \lceil r/4 \rceil \) and \( \beta \geq \lceil (50 - r)/4 \rceil \). Note that for any \( 8 \leq r \leq 21 \), \( \lceil (50 - r)/4 \rceil \) is greater than or equal to 8. And so \( \alpha(G) \geq 8 \). Now we have the following cases.

Case 1. \( 22 \leq |N(u_1)| \leq 25 \), then by a similar argument as in Claim 1, we have that \( \alpha(\langle N(u_1) \rangle_G) \geq 6 \) and \( \beta \geq 7 \). Then, \( \langle B \rangle_G \) has an independent set which consists of 7 vertices. This set with the vertex \( u_1 \) is an 8-element independent set of vertices of \( G \). That is a contradiction.

Case 2. \( |N(u_1)| = 26 \), then \( |B| \geq 24 \). By a similar argument as in Claim 1, we have that \( \alpha(\langle N(u_1) \rangle_G) \geq 7 \) and \( \beta \geq 6 \). Now we have the following two subcases.

Subcase 2.1. \( \beta \geq 7 \). Then we have a subcase similar to Case 1.

Subcase 2.2. \( \beta = 6 \). The best case of such subgraph is the graph that shown in Figure 1. Now we have the following two subcases.
Subcase 2.2.1. There is a vertex of $\bigcup_{i=1}^{6} N_i(u_{i+1})$, say $a_1$, that is not adjacent to at least one vertex of each $K^{(j)} (1 \leq j \leq 7)$, say $x_j$. Then $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, a_1\}$ is an 8-element independent set of vertices of $G$. And so $\alpha(G) \geq 8$. That is a contradiction.

Subcase 2.2.2. For each vertex of $\bigcup_{i=1}^{6} N_i(u_{i+1})$ there is $1 \leq j \leq 7$ such that this vertex is adjacent to all vertices of $K^{(j)}$. Then $G$ contains $K_1 + P_6$ or $C_9$. That is a contradiction.

Case 3. $27 \leq |N(u_1)| \leq 28$, Then by using the same argument as in Case 2, we have the same contradiction.

Theorem 2.9. $r(C_9, K_8) = 57$.

Proof. Suppose that there exists a graph $G$ of order 57 that contains neither $C_9$ nor an 8-elements independent set. Then by Lemma 2.1, $\delta(G) \geq 8$ and by Lemma 2.8, $G$ contains $K_1 + P_6$ or $K_6$. Thus, by Lemmas 2.7, 2.6, 2.5, 2.4, 2.3, and 2.2, we have that $|V(G)| \geq 72$. That is a contradiction. The proof is complete.

Appendix

To show that the assertions (1)–(4) of Case 2 of Lemma 2.5 are true, it suffices to show that for any two vertices of $\{u_2, u_3, u_5, u_7, u_8, u_9\}$ there are paths of order 8, 7, 6 and 5. The following
are paths of order 8 between vertices of \{u_3, u_5, u_7, u_8, u_9\}.

1- \text{u}_2-\text{u}_5 \text{ path: } u_2 u_3 u_4 u_5 u_6 u_7 u_8 u_9 \text{, by symmetry we find } u_7-\text{u}_8 \text{ path.}
2- \text{u}_2-\text{u}_5 \text{ path: } u_2 u_3 u_4 u_5 u_6 u_7 u_8 u_9 \text{, by symmetry we find } u_5-\text{u}_8 \text{ path.}
3- \text{u}_2-\text{u}_7 \text{ path: } u_2 u_3 u_4 u_5 u_1 u_8 \text{, by symmetry we find } u_3-\text{u}_8 \text{ path.}
4- \text{u}_2-\text{u}_8 \text{ path: } u_2 u_3 u_4 u_5 u_6 u_7 u_1 u_8.
5- \text{u}_2-\text{u}_9 \text{ path: } u_2 u_3 u_4 u_5 u_1 u_8 u_9 \text{, by symmetry we find } u_6-\text{u}_9 \text{ path.}
6- \text{u}_3-\text{u}_5 \text{ path: } u_3 u_4 u_5 u_6 u_7 u_8 u_1 u_5 \text{, by symmetry we find } u_5-\text{u}_7 \text{ path.}
7- \text{u}_3-\text{u}_7 \text{ path: } u_3 u_4 u_5 u_6 u_7 u_8 u_1 u_7.
8- \text{u}_3-\text{u}_9 \text{ path: } u_3 u_4 u_5 u_1 u_6 u_7 u_8 u_9 \text{, by symmetry we find } u_7-\text{u}_9 \text{ path.}
3- \text{u}_5-\text{u}_9 \text{ path: } u_5 u_3 u_2 u_1 u_7 u_8 u_9.

The following are paths of order 7 between vertices of \{u_3, u_5, u_7, u_8, u_9\}.

1- \text{u}_2-\text{u}_5 \text{ path: } u_2 u_3 u_4 u_5 u_6 u_7 u_5 u_3 \text{, by symmetry we find } u_7-\text{u}_6 \text{ path.}
2- \text{u}_2-\text{u}_5 \text{ path: } u_2 u_3 u_4 u_5 u_6 u_7 u_5 u_9, \text{ by symmetry we find } u_5-\text{u}_6 \text{ path.}
3- \text{u}_2-\text{u}_7 \text{ path: } u_2 u_3 u_4 u_5 u_6 u_1 u_7, \text{ by symmetry we find } u_3-\text{u}_6 \text{ path.}
4- \text{u}_2-\text{u}_8 \text{ path: } u_2 u_3 u_4 u_5 u_6 u_7 u_1 u_8.
5- \text{u}_2-\text{u}_9 \text{ path: } u_2 u_3 u_4 u_5 u_7 u_6 u_9, \text{ by symmetry we find } u_8-\text{u}_9 \text{ path.}
2- \text{u}_3-\text{u}_5 \text{ path: } u_3 u_4 u_5 u_6 u_7 u_8 u_5 \text{, by symmetry we find } u_5-\text{u}_7 \text{ path.}
2- \text{u}_3-\text{u}_7 \text{ path: } u_3 u_4 u_5 u_6 u_7 u_8 u_7.
2- \text{u}_3-\text{u}_9 \text{ path: } u_3 u_4 u_5 u_7 u_6 u_8 u_9, \text{ by symmetry we find } u_7-\text{u}_9 \text{ path.}
3- \text{u}_5-\text{u}_9 \text{ path: } u_5 u_3 u_2 u_1 u_7 u_8 u_9.

The following are paths of order 6 between vertices of \{u_3, u_5, u_7, u_8, u_9\}.

1- \text{u}_2-\text{u}_5 \text{ path: } u_2 u_3 u_4 u_5 u_6 u_3 \text{, by symmetry we find } u_7-\text{u}_6 \text{ path.}
2- \text{u}_2-\text{u}_5 \text{ path: } u_2 u_3 u_4 u_5 u_6 u_5 \text{, by symmetry we find } u_5-\text{u}_6 \text{ path.}
3- \text{u}_2-\text{u}_7 \text{ path: } u_2 u_3 u_4 u_5 u_1 u_7, \text{ by symmetry we find } u_3-\text{u}_7 \text{ path.}
4- \text{u}_2-\text{u}_8 \text{ path: } u_2 u_3 u_4 u_5 u_6 u_1 u_8.
5- \text{u}_2-\text{u}_9 \text{ path: } u_2 u_3 u_4 u_1 u_5 u_9, \text{ by symmetry we find } u_6-\text{u}_9 \text{ path.}
2- \text{u}_3-\text{u}_5 \text{ path: } u_3 u_4 u_5 u_6 u_7 u_5 \text{, by symmetry we find } u_5-\text{u}_7 \text{ path.}
2- \text{u}_3-\text{u}_7 \text{ path: } u_3 u_4 u_5 u_6 u_7 u_5 u_7.
2- \text{u}_3-\text{u}_9 \text{ path: } u_3 u_4 u_5 u_7 u_6 u_8 u_9, \text{ by symmetry we find } u_7-\text{u}_9 \text{ path.}
3- \text{u}_5-\text{u}_9 \text{ path: } u_5 u_3 u_2 u_1 u_3 u_4 u_9.

The following are paths of order 5 between vertices of \{u_3, u_5, u_7, u_8, u_9\}.

1- \text{u}_2-\text{u}_5 \text{ path: } u_2 u_3 u_4 u_5 u_3 \text{, by symmetry we find } u_7-\text{u}_8 \text{ path.}
2- \text{u}_2-\text{u}_5 \text{ path: } u_2 u_3 u_4 u_5 u_5 \text{, by symmetry we find } u_5-\text{u}_8 \text{ path.}
3- \text{u}_2-\text{u}_7 \text{ path: } u_2 u_3 u_4 u_5 u_1 u_7, \text{ by symmetry we find } u_3-\text{u}_8 \text{ path.}
4- \text{u}_2-\text{u}_8 \text{ path: } u_2 u_3 u_4 u_1 u_8.
References


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