Research Article

About Zero Torsion Linear Maps on Lie Algebras

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We prove that any zero torsion linear map on a nonsolvable real Lie algebra \( g_0 \) is an extension of some \( CR \)-structure. We then study the cases of \( sl(2, \mathbb{R}) \) and the 3-dimensional Heisenberg Lie algebra \( n \). In both cases, we compute up to equivalence all zero torsion linear maps on \( g_0 \), and deduce an explicit description of the equivalence classes of integrable complex structures on \( g_0 \times g_0 \).

1. Introduction

Given a real Lie algebra \( g_0 \), the determination up to equivalence of zero torsion linear maps from \( g_0 \) to \( g_0 \) plays an important role in the computation of complex structures on direct products involving \( g_0 \) [1]. The direct computation of those maps can be difficult for semisimple \( g_0 \), so there is a point in exploring alternative ways, particularly their relation to \( CR \)-structures. For compact \( g_0 \), maximal rank \( CR \)-structures have been classified up to equivalence in [2]. In the case of \( su(2) \), all zero torsion linear maps are extensions of certain \( CR \)-structures (see [1]). One can then ask the natural question whether or not any zero torsion linear map on a nonabelian \( g_0 \) is necessarily an extension of some \( CR \)-structure. In the present note, we answer the question in the positive for nonsolvable Lie algebras. Then we make a detailed study of two basic examples: \( g_0 = sl(2, \mathbb{R}) \) in the positive case, and \( g_0 = n \) the 3-dimensional Heisenberg Lie algebra in the negative. In both cases, we compute (up to equivalence) all zero torsion linear maps, and the result is used to exhibit a complete set of representatives of equivalence classes of complex structures on \( g_0 \times g_0 \).

An interesting direction for future research could be to investigate zero torsion linear maps and \( CR \)-structures on various constructions of Lie algebras, for example like those considered in [3] (see also [4]).
2. Zero Torsion Linear Maps and Extension of CR-Structures

A CR-structure on a smooth real manifold $M$ is a subbundle $\mathcal{U}$ of the complexified tangent bundle $\mathbb{C}T(M)$ of $M$ such that $[\mathcal{U}, \mathcal{U}] \subset \mathcal{U}$ (i.e., the space of smooth sections of $\mathcal{U}$ is closed under commutators) and $\mathcal{U} \cap \overline{\mathcal{U}} = \{0\}$ ($\overline{\cdot}$ denoting here conjugation in $\mathbb{C}T(M)$). The rank or CR-dimension is the complex dimension of $\mathcal{U}$. For general background on CR-structures we refer the reader to [5].

Throughout this section, $g_0$ will denote any finite-dimensional real Lie algebra, $\mathfrak{g}$ its complexification, and $\sigma$ or simply $\overline{\cdot}$ the conjugation in $\mathfrak{g}$ with respect to $g_0$.

If $G_0$ is a connected finite dimensional real Lie group, with Lie algebra $g_0$, left invariant CR-structures on $G_0$ are identified to CR-structures on $g_0$ in the following sense [6, 7].

**Definition 2.1.** A rank $r$ $(1 \leq r \leq [(\dim g_0)/2])$ CR-structure on $g_0$ is a $r$-dimensional complex subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{m} \cap \overline{\mathfrak{m}} = \{0\}$.

If $\dim g_0$ is even, a CR-structure of maximal rank is an (integrable) complex structure. Now one has the following straightforward lemma.

**Lemma 2.2.** Let $p$ be vector subspace of $g_0$ and $J_p : p \rightarrow p$ a linear map. Consider the real vector subspace $\mathfrak{m} = \{\tilde{X}; X \in p\}$ of $\mathfrak{g}$, where $\tilde{X} = X - iJ_pX$ for $X \in p$. Denote $m' = \text{im} = \{J_pX + iX; X \in p\}$. Then

(i) $\mathfrak{m}$ is a complex vector subspace of $\mathfrak{g}$ if and only if $J_p^2 = -1_p$;

(ii) $\mathfrak{m}$ is stable with respect to the bracket if and only if for any $X,Y \in p$

\[ [X,Y] - [J_pX, J_pY] \in p, \]

\[ J_p([X,Y] - [J_pX, J_pY]) = [J_pX, Y] + [X, J_pY]. \]  

In that case, $[\tilde{X}, \tilde{Y}] = \tilde{Z}$ with $Z = [X,Y] - [J_pX, J_pY]$.

(iii) $\mathfrak{m}'$ is stable with respect to the bracket if and only if for any $X,Y \in p$

\[ [J_pX, Y] + [X, J_pY] \in p, \]

\[ J_p([J_pX, Y] + [X, J_pY]) = [J_pX, J_pY] - [X,Y]. \]  

In that case, $[\tilde{X}, \tilde{Y}] = -i\tilde{Z}$ with $Z = [J_pX, Y] + [X, J_pY]$ for $X,Y \in p$.

**Remark 2.3.** When (i) is satisfied, (ii) and (iii) are trivially equivalent since then $m = m'$.

**Lemma 2.4.** A rank $r$ CR-structure on $g_0$ can be defined in an alternative way as $(p, J_p)$ where $p$ is a $2r$-dimensional $(1 \leq r \leq [(\dim g_0)/2])$ vector subspace of $g_0$ and $J_p : p \rightarrow p$ is a linear map satisfying the 3 conditions

\[ J_p^2 = -1_p, \]

\[ [J_pX, Y] + [X, J_pY] \in p \quad \forall X,Y \in p, \]

\[ [J_pX, J_pY] - [X,Y] - J_p([J_pX, Y] + [X, J_pY]) = 0 \quad \forall X,Y \in p. \]
Let $\mathfrak{g}$ be a rank $r$ CR-structure on $\mathfrak{g}_0$. Note first that the taking of the real part is a bijective linear map of the real algebra $\mathfrak{m}$ onto its image $\mathfrak{p} = 9\mathfrak{m}$, $\dim \mathfrak{p} = 2r$, and there exists a unique linear map $J_p : \mathfrak{p} \to \mathfrak{g}_0$ such that $\mathfrak{m} = \{ \tilde{X} = X - ij_pX; X \in \mathfrak{p} \}$. Now, for $X \in \mathfrak{p}$, $i\tilde{X} = J_pX + iX \in \mathfrak{m} = \mathfrak{m}$ hence $J_pX \in \mathfrak{p}$, so that $J_p : \mathfrak{p} \to \mathfrak{p}$. Then (2.3), (2.4), (2.5) follow from Lemma 2.2 and Remark 2.3.

The converse comes again from Lemma 2.2 and Remark 2.3. □

Remark 2.5. The condition (2.3) implies det $J_p = 1$ and Trace $(J_p) = 0$, hence if $r = 1$, (2.4) and (2.5) follow from (2.3) and can be omitted.

Definition 2.6. A linear map $J : \mathfrak{g}_0 \to \mathfrak{g}_0$ is said to have zero torsion if it satisfies the condition

$$[JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0 \quad \forall X, Y \in \mathfrak{g}_0.$$  \hfill (2.6)

If $J$ has zero torsion and satisfies in addition $J^2 = -1$, $J$ is an (integrable) complex structure on $\mathfrak{g}_0$. That means that $G_0$ can be given the structure of a complex manifold with the same underlying real structure and such that the canonical complex structure on $G_0$ is the left invariant almost complex structure $\tilde{J}$ associated to $J$ (for more details, see [8]).

When computing the matrices of the zero torsion maps in some fixed basis $(x_i)_{1 \leq i \leq n}$ of $\mathfrak{g}_0$, we will denote by $ijk \mid k$ ($1 \leq i, j, k \leq n$) the torsion equation obtained by projecting on $x_k$ the equation (2.6) with $X = x_i, Y = x_j$.

The automorphism group Aut $\mathfrak{g}_0$ of $\mathfrak{g}_0$ acts on the set of all zero torsion linear maps and on the set of all complex structures on $\mathfrak{g}_0$ by $J \mapsto \Phi \circ J \circ \Phi^{-1}$ for all $\Phi \in$ Aut $\mathfrak{g}_0$. It acts also on the set of CR-structures by $(p, J_p) \mapsto (\Phi p, \Phi \circ J_p \circ \Phi^{-1})$. Two $J, J'$ (resp., $(p, J_p), (p', J_p')$) are said to be equivalent (notation: $J \equiv J'$ (resp., $J_p \equiv J_p'$)) if they are on the same Aut $\mathfrak{g}_0$ orbit. This means that the corresponding left invariant CR-structures on the connected simply connected real Lie group associated to $\mathfrak{g}_0$ are intertwined by some Lie group automorphism. It is a stronger notion than CR-diffeomorphism, where the intertwining is simply required to be a diffeomorphism.

Lemma 2.7. Let $J : \mathfrak{g}_0 \to \mathfrak{g}_0$ be a linear map, $\mathfrak{m} = \{ \tilde{X} = X - ijX; X \in \mathfrak{g}_0 \}$ and $\mathfrak{m}' = \mathfrak{im} = \{ JX + iX; X \in \mathfrak{g}_0 \}$;

(i) $\mathfrak{m} \cap \mathfrak{m}' = \{ \tilde{X}; X \in \ker(J^2 + 1) \}$,

(ii) $\mathfrak{m}'$ is a real subalgebra of $\mathfrak{g}$ if and only if $J$ has zero torsion,

(iii) if $J$ has zero torsion, $\mathfrak{m} \cap \mathfrak{m}'$ is a complex subalgebra of $\mathfrak{g}$.

Proof. (i) For any $Z \in \mathfrak{g}$ one has

$$\begin{align*}
Z \in \mathfrak{m} \cap \mathfrak{m}' & \iff \exists X, Y \in \mathfrak{g}_0, \; Z = X - iJX = JY + iY \\
& \iff \exists X, Y \in \mathfrak{g}_0, \; Z = X - iJX, X = JY, Y = -JX \\
& \iff \exists X \in \mathfrak{g}_0, \; Z = X - iJX, X = -J^2X \\
& \iff \exists X \in \ker(J^2 + 1), \; Z = \tilde{X}.
\end{align*}$$ \hfill (2.7)
(ii) The result follows from Lemma 2.2(iii) since the first condition in (2.2) (with \( p = g \) and \( J_p = J \)) is trivially satisfied and the second condition is the zero torsion condition.

(iii) From (ii), \([m', m'] \subset m'\), hence \([m, m'] \subset m\) and \([m \cap m', m \cap m'] \subset m \cap m'.\) Clearly \(m \cap m'\) is stable by multiplication by \(i\).

**Definition 2.8.** Let \( J : g_0 \to g_0 \) be a zero torsion linear map. We say that \( J \) is an extension of a CR-structure if there exists a vector subspace \( p \neq \{0\} \) of \( g_0 \) such that \( p \) equipped with the restriction \( J_p \) of \( J \) to \( p \) is a CR-structure on \( g_0 \).

**Definition 2.9.** A real form \( u \) of \( g \) is said to be of type I (with respect to \( g_0 \)) if \( g_0 \cap u \neq \{0\} \). \( g_0 \) is said to be type I if any real form \( u \) of \( g \) is of type I.

**Remark 2.10.** Introduce the real linear projections \( \pi_1 : g \to g_0, \pi_2 : g \to g_0 \) defined by \( z = \pi_1(z) + i\pi_2(z) \) for \( z \in g \). Then a real form \( u \) of \( g \) is of type I if and only if \( \ker \pi_2 |_u \neq \{0\} \).

**Proposition 2.11.** Let \( J : g_0 \to g_0 \) be a zero torsion linear map, \( m = \{X = X - iJX; X \in g_0\} \) and \( m' = \text{im} \). \( J \) is an extension of a CR-structure if and only if \( m \cap m' \neq \{0\} \).

**Proof.** From Lemma 2.7, \( m \cap m' \) is a complex subalgebra of \( g \) and \( m \cap m' = \{X; X \in \ker (J^2 + 1)\} \). If \( J \) is an extension of a CR-structure, one has \( \{0\} \neq p \subset \ker (J^2 + 1) \) hence \( m \cap m' \neq \{0\} \). Conversely, if \( m \cap m' \neq \{0\} \), let \( p = \ker (J^2 + 1) \). Then \( p \) is stable under \( J \), and if \( J_p \) denotes the restriction of \( J \) to \( p \), conditions (2.3), (2.5) are trivially satisfied. Condition (2.4) holds true since, from Lemma 2.7(iii), \( m \cap m' \) is a subalgebra of \( g \). Precisely, for \( X, Y \in p \), \( \tilde{X}, \tilde{Y} \in m \cap m' \) hence \( [i\tilde{X}, \tilde{Y}] = [J_p X + iX, Y - iJ_p Y] \in m \cap m' \) and (2.4) follows.

**Corollary 2.12.** There is a one-to-one correspondence between non-type I real forms of \( g \) and zero torsion linear maps \( J : g_0 \to g_0 \) which are no extension of a CR-structure.

**Proof.** Let \( J : g_0 \to g_0 \) be a zero torsion linear map that is no extension of a CR-structure. Then \( m' \cap m' = \{0\} \), hence \( m' = \{JX + iX; X \in g_0\} \) is a real form of \( g \) which is non-type I. Conversely, if \( u \) is a non-type I real form of \( g \), then \( \pi_2(u) = \{0\} \) implies \( u = \{JX + iX; X \in g_0\} \) for some linear map \( J : g_0 \to g_0 \), and as \( m' \) is a real subalgebra, \( J \) has zero torsion from Lemma 2.7(ii). Now \( m \cap m' = \{0\} \) since \( u \) is a real form hence \( J \) is not an extension of a CR-structure.

**Corollary 2.13.** If \( g_0 \) is of type I, then any zero torsion linear map \( J : g_0 \to g_0 \) is an extension of a CR-structure.

**Proposition 2.14.** Let \( u \) be any real form of \( g \), \( \tau, \sigma \) the conjugations with respect to \( u \), \( g_0 \), and \( N = \sigma \tau \in \text{Aut } g \). If \( N \) has a nonzero fixed point, then \( u \) is type I.

**Proof.** Let \( Z \) be a fixed point of \( N \). \( NZ = Z \) reads \( \sigma Z = \tau Z \). Consider \( V = \sigma Z = \tau Z \). Then \( \sigma V = \tau V = Z \). Hence \( W = V + Z \) has \( \tau W = Z + \tau Z = Z + V = W \) and similarly \( \sigma W = W \). Hence \( W \in g_0 \cap u \). Now, \( W = 0 \) if and only if \( \sigma Z = \tau Z = -Z \), that is, \( iZ \in g_0 \cap u \).

**Corollary 2.15.** If \( g_0 \) is nonsolvable, then it is type I.

**Proof.** If \( g_0 \) is nonsolvable, so is \( g \). Now, it is known that any automorphism of a nonsolvable Lie algebra over a characteristic 0 field has a nonzero fixed point ([9]). Hence any real form \( u \) of \( g \) is type I.
3. Case of $\mathfrak{sl}(2, \mathbb{R})$

$G = SL(2, \mathbb{R})$ denotes the Lie group of real $2 \times 2$ matrices with determinant 1

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1.$$  \hfill (3.1)

Its Lie algebra $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R})$ consists of the zero trace real $2 \times 2$ matrices

$$X = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = xH + yX_+ + zX_-,$$  \hfill (3.2)

with basis $H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $X_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and commutation relations

$$[H, X_+] = 2X_+, \quad [H, X_-] = -2X_-, \quad [X_+, X_-] = H.$$  \hfill (3.3)

Beside the basis $(H, X_+, X_-)$, we will also make use of the basis $(Y_1, Y_2, Y_3)$ where $Y_1 = (1/2)H, Y_2 = (1/2)(X_+ - X_-), Y_3 = (1/2)(X_+ + X_-)$, with commutation relations

$$[Y_1, Y_2] = Y_3, \quad [Y_1, Y_3] = Y_2, \quad [Y_2, Y_3] = Y_1.$$  \hfill (3.4)

The adjoint representation of $G$ on $\mathfrak{g}_0$ is given by $\text{Ad}(\sigma)X = \sigma X \sigma^{-1}$. The matrix $\Phi$ of $\text{Ad}(\sigma)$ ($\sigma$ as in (3.1)) in the basis $(H, X_+, X_-)$ is

$$\Phi = \begin{pmatrix} 1 + 2bc & -ac & bd \\ -2ab & a^2 & -b^2 \\ 2cd & -c^2 & d^2 \end{pmatrix}.$$  \hfill (3.5)

The adjoint group $\text{Ad}(G)$ is the identity component of $\text{Aut} \, \mathfrak{g}_0$ and one has

$$\text{Aut} \, \mathfrak{g}_0 = \text{Ad}(G) \cup \Psi_0 \text{Ad}(G), \quad \Psi_0 = \text{diag}(1, -1, -1).$$  \hfill (3.6)

The adjoint action of $G$ on $\mathfrak{g}_0$ preserves the form $x^2 + yz$. The orbits are as follows:

(i) the trivial orbit $\{0\}$;

(ii) the upper sheet $z > 0$ of the cone $x^2 + yz = 0$ (orbit of $X_-$);

(iii) the lower sheet $z < 0$ of the cone $x^2 + yz = 0$ (orbit of $-X_-$);

(iv) for all $s > 0$ the one-sheet hyperboloid $x^2 + yz = s^2$ (orbit of $sH$);

(v) for all $s > 0$ the upper sheet $z > 0$ of the hyperboloid $x^2 + yz = -s^2$ (orbit of $s(-X_+ + X_-)$);

(vi) for all $s > 0$ the lower sheet $z < 0$ of the hyperboloid $x^2 + yz = -s^2$ (orbit of $s(X_+ - X_-)$).
The orbits of $g_0$ under the whole $\text{Aut} g_0$ are as follows, beside $\{0\}$:

(I) the cone $x^2 + yz = 0$ (orbit of $X_-$);

(II) the one-sheet hyperboloid $x^2 + yz = s^2$ (orbit of $sH$) ($s > 0$);

(III) the two-sheet hyperboloid $x^2 + yz = -s^2$ (orbit of $s(X_+ - X_-)$) ($s > 0$).

**Proposition 3.1.** Let $g_0 = \mathfrak{sl}(2, \mathbb{R})$, and $J : g_0 \to g_0$ any linear map. $J$ has zero torsion if and only if it is equivalent to the endomorphism defined in the basis $(Y_1, Y_2, Y_3)$ (resp., $(H, X_+, X_-)$) by

$$J_*(\lambda) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \lambda & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

(3.7)

$J_*(\lambda) \neq J_*(\mu)$ for $\lambda \neq \mu$ (resp.,

$$J(\alpha) = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & \alpha & -\alpha \\ 1 & -\alpha & \alpha \end{pmatrix}, \quad \alpha \in \mathbb{R},$$

(3.8)

$J(\alpha) \neq J(\beta)$ for $\alpha \neq \beta$).

**Proof.** Let $J = (\xi_i^j)_{1 \leq i,j \leq 3}$ in the basis $(H, X_+, X_-)$. The 9 torsion equations are in the basis $(H, X_+, X_-)$:

12 | 1 \ 2\left(\xi_2^2 + \xi_1^1\right)\xi_2^1 + \left(\xi_2^2 - \xi_1^1\right)\xi_3^1 - \left(\xi_2^2 + 2\xi_1^1\right)\xi_2^3 = 0,

12 | 2 \ 2\left(\xi_3^1\xi_2^1 + 1 + \left(\xi_2^2\right)^2\right) - \xi_3^1\xi_2^2 - 2\xi_3^2\xi_2^2 = 0,

12 | 3 \left(\xi_3^1 + 2\xi_2^1\right)\xi_3^1 - 2\left(\xi_3^1 + 2\xi_1^1\right)\xi_2^3 + 2\xi_3^2\xi_2^3 = 0,

13 | 1 \left(\xi_2^2 - 2\xi_3^1\right)\xi_3^1 + 2\xi_3^2\xi_2^2 + \xi_1^1\xi_3^1 - \left(\xi_2^1 + 2\xi_3^1\right)\xi_2^3 = 0,

13 | 2 \ 2\left(\xi_2^2 - 2\xi_3^1\right)\xi_2^3 + \left(\xi_1^1 + 2\xi_3^1\right)\xi_3^1 - 2\xi_3^3\xi_3^3 = 0,

13 | 3 \xi_3^1\xi_3^1 - 2\xi_3^2\xi_2^2 - 2 + 2\xi_3^2\xi_3^2 - 2\left(\xi_3^3\xi_3^3\right)^2 = 0,

23 | 1 \ 4\xi_3^1\xi_2^1 - 1 - \xi_2^1\xi_3^1 - \xi_3^1\xi_3^1 + \left(\xi_2^2 - \xi_1^1\right)\xi_3^3 = 0,

23 | 2 \ 4\xi_3^1\xi_2^1 - \left(\xi_2^2 + \xi_3^3\right)\xi_3^1 = 0,

23 | 3 \ 4\xi_3^1\xi_2^1 - \left(\xi_2^2 + \xi_3^3\right)\xi_3^1 = 0.
\( J \) has at least one real eigenvalue \( \lambda \). Let \( v \in g_0, v \neq 0 \), an eigenvector associated to \( \lambda \). From the classification of the Aut \( g_0 \) orbits of \( g_0 \), we then get 3 cases according to whether \( v \) is on the orbit (I), (II), (III) (in the cases (II), (III) one may choose \( v \) so that \( s = 1 \)).

**Case 1.** There exists \( \varphi \in \text{Aut} \ g_0 \) such that \( v = \varphi(X_\pm) \). Then, replacing \( J \) by \( \varphi^{-1} J \varphi \), we may suppose \( \xi^3_1 = \xi^3_2 = 0 \). That case is impossible from 13 \( | 2 \) and 13 \( | 3 \).

**Case 2.** There exists \( \varphi \in \text{Aut} \ g_0 \) such that \( v = \varphi(H) \). Then we may suppose \( \xi^2_1 = \xi^2_3 = 0 \). Then from 12 \( | 2 \), \( \xi^2_1 \xi^2_2 \neq 0 \), and 23 \( | 2 \), 23 \( | 3 \) yield \( \xi^2_1 = \xi^2_3 = 0 \). Then 12 \( | 3 \) and 13 \( | 2 \) successively give \( \xi^3_3 = \xi^2_2 + 2\xi^1_1 \) and \( \xi^1_1 = 0 \). Now 12 \( | 2 \) and 23 \( | 1 \) read, respectively, \( -\xi^2_2 \xi^3_1 + (\xi^2_2)^2 + 1 = 0 \), and \( \xi^2_3 \xi^2_2 - (\xi^2_2)^2 + 1 = 0 \). Hence that case is impossible.

**Case 3.** There exists \( \varphi \in \text{Aut} \ g_0 \) such that \( v = \varphi(X_+ - X_-) \). Then we may suppose that \( v = X_+ - X_- \). Now instead of the basis \((H,X_+,X_-)\), we consider the basis \((Y_1,Y_2,Y_3)\). The matrix of \( J \) in the basis \((Y_1,Y_2,Y_3)\) has the form

\[
J_\ast = \begin{pmatrix}
\eta^1_1 & 0 & \eta^1_3 \\
\eta^2_1 & \lambda & \eta^2_3 \\
\eta^3_1 & 0 & \eta^3_3
\end{pmatrix}
\]  
(3.10)

Then the 9 torsion equations \(*ij \mid k\) (the star is to underline that the new basis is in use) for \( J \) in that basis are as follows:

\[
\begin{align*}
*12 \mid 1 & \quad (\eta^3_1 + \eta^3_3) \lambda - (\eta^3_1 - \eta^3_3) \eta^1_1 = 0, \\
*12 \mid 2 & \quad (\eta^1_1 + \lambda) \eta^2_1 - \eta^2_1 \eta^1_1 = 0, \\
*12 \mid 3 & \quad \eta^1_1 \lambda - 1 + (\eta^2_1)^2 - (\eta^1_1 + \lambda) \eta^3_3 = 0, \\
*13 \mid 1 & \quad \eta^2_3 \eta^3_1 + \eta^2_3 \eta^1_1 + \eta^2_3 \eta^3 - \eta^2_1 \eta^3_3 = 0, \\
*13 \mid 2 & \quad \eta^1_1 \lambda + 1 + (\eta^3_1)^2 + (\eta^3_1)^2 + \eta^3_1 \eta^1_1 - (\eta^1_1 - \lambda) \eta^3_3 = 0, \\
*13 \mid 3 & \quad \eta^3_1 \eta^1_1 - \eta^3_1 (\eta^3_1 + \eta^3_3) - \eta^3_1 \eta^3_3 = 0, \\
*23 \mid 1 & \quad \eta^3_1 \lambda + 1 - (\eta^3_1)^2 + (\eta^3_1)^2 = 0, \\
*23 \mid 2 & \quad \eta^3_1 \eta^3_1 - (\eta^3_1 + \lambda) \eta^1_1 = 0, \\
*23 \mid 3 & \quad (\eta^3_1 + \eta^3_3) \lambda + (\eta^3_1 - \eta^3_3) \eta^1_1 = 0.
\end{align*}
\]  
(3.11)

From \(*12 \mid 1 \) and \(*23 \mid 3 \),

\[
\eta^1_1 (\eta^3_1 - \eta^3_3) = -\eta^3_1 (\eta^3_1 - \eta^3_3).
\]  
(3.12)
Case 1. Suppose first that $\eta^3_1 = \eta^1_3$. Then $\lambda \eta^3_1 = 0$.

Subcase 1. Consider the subcase $\eta^3_1 = 0$. *13 | 1 and *13 | 3 read, respectively, $(\eta^3_1 - \eta^1_3)\eta^2_1 = 0$, $(\eta^3_1 - \eta^1_3)\eta^2_3 = 0$. Suppose $\eta^3_1 \neq \eta^1_3$. Then $\eta^1_3 = 0$, and *13 | 2 gives $\eta^1_1 + \lambda = (\eta^1_1 - \lambda)\eta^3_3$, which implies $\eta^3_1 = 0$ by *23 | 1. As *12 | 3 then reads $1 = 0$, this case $\eta^3_1 \neq \eta^1_3$ is not possible. Now, the case $\eta^3_1 = \eta^1_3$ is not possible either since then *23 | 1 would read $(\eta^1_1)^2 + 1 = 0$. We conclude that the Subcase 1 is not possible.

Subcase 2. Hence we are in the Subcase 2: $\eta^3_1 \neq 0$. Then $\lambda = 0$. From *13 | 2, $\eta^1_3\eta^1_3 \neq 0$. Then *23 | 1 yields $\eta^3_1 = (-1 + (\eta^1_1)^2)/\eta^1_1$ and *13 | 2 reads $(\eta^1_1)^2 + (\eta^3_3)^2 + 2 = 0$. This Subcase 2 is not possible either.

Case 2. Hence Case 1 is not possible, and we are necessarily in the Case 2: $\eta^3_1 \neq \eta^1_3$. From (3.12), $\eta^3_1 = -\eta^1_3$. Then *13 | 2 reads $(\eta^1_1)^2 + (\eta^3_3)^2 + 1 + \eta^3_3\eta^1_1 = 0$ hence $\eta^3_3 \neq 0$ and $\eta^3_1 = -(\eta^3_3)^2 + (\eta^1_1)^2 + 1)/\eta^1_1$. From *12 | 2, $\eta^1_3 = (\eta^3_3^2 + \lambda))/\eta^1_1$. Then *23 | 2 reads $\eta^1_3((\eta^3_3)^2 + (\eta^1_3 + \lambda)^2(\eta^3_3)^2 = 0$, that is, $\eta^3_1 = 0$, which implies $\eta^1_3 = 0$. Now *12 | 1 reads $\lambda(1 + (\eta^1_1)^2 - (\eta^3_3)^2) = -\eta^1_3(1 + (\eta^1_1)^2 + (\eta^3_3)^2)$. The subcase $\eta^3_1 \neq 0$ is not possible since then *12 | 3 would yield $\lambda = -(\eta^1_1)^2 + (\eta^3_3)^2 - 1)/2\eta^1_1$ and *12 | 1 would read $(\eta^3_3^2 + (\eta^3_3 + 1)^2)((\eta^1_1)^2 + (\eta^3_3 - 1)^2) = 0$. Hence $\eta^1_3 = 0$. Then *12 | 3 reads $(\eta^3_3)^2 = 1$. The condition $(\eta^3_3)^2 = 1$ now implies the vanishing of all the torsion equations. In that case

$$J_\varepsilon = \begin{pmatrix} 0 & 0 & -\varepsilon \\ 0 & \lambda & 0 \\ \varepsilon & 0 & 0 \end{pmatrix}, \quad \varepsilon = \pm 1.$$  \hfill (3.13)

Then in the basis $(H, X_+, X_-)$

$$J = \begin{pmatrix} 0 & -\varepsilon & -\varepsilon \\ \lambda & \varepsilon & -\varepsilon \\ \varepsilon & 0 & 0 \end{pmatrix}.$$  \hfill (3.14)

The cases $\varepsilon = \pm 1$ are equivalent under $\Psi_0$. \hfill \Box

Remark 3.2. $J_\varepsilon(\lambda)$ is an extension of a CR-structure, in agreement with Corollary 2.15.

4. Complex Structures on $\mathfrak{s}\mathfrak{l}(2, \mathbb{R}) \times \mathfrak{s}\mathfrak{l}(2, \mathbb{R})$

We consider the basis $(\chi_1^{(1)}, \chi_2^{(1)}, \chi_3^{(1)}, \chi_1^{(2)}, \chi_2^{(2)}, \chi_3^{(2)})$ of $\mathfrak{s}\mathfrak{l}(2, \mathbb{R}) \times \mathfrak{s}\mathfrak{l}(2, \mathbb{R})$, with the upper index referring to the first or second factor. The automorphisms of $\mathfrak{s}\mathfrak{l}(2, \mathbb{R}) \times \mathfrak{s}\mathfrak{l}(2, \mathbb{R})$ fall into 2 kinds: the first kind is comprised by the $\text{diag}(\Phi_1, \Phi_2)$, $\Phi_1, \Phi_2 \in \text{Aut}(\mathfrak{s}\mathfrak{l}(2, \mathbb{R}))$, and the second kind is comprised of the $\Gamma \circ \text{diag}(\Phi_1, \Phi_2)$, with $\Gamma$ the switch between the two factors of $\mathfrak{s}\mathfrak{l}(2, \mathbb{R}) \times \mathfrak{s}\mathfrak{l}(2, \mathbb{R})$. 
Proposition 4.1. Any integrable complex structure $J$ on $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$ is equivalent under some first kind automorphism to the endomorphism given in the basis $(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}, Y_1^{(2)}, Y_2^{(2)}, Y_3^{(2)})$ by the matrix

$$
\bar{J}_*(\xi_2^0, \xi_5^2) = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 \\
0 & \xi_2^0 & 0 & 0 & \xi_5^2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & -\left(\frac{\xi_2^0}{\xi_5^2}\right)^2 + 1 & 0 & 0 & -\xi_2^0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}, \quad \xi_2^0, \xi_5^2 \in \mathbb{R}, \ \xi_5^2 \neq 0. \tag{4.1}
$$

$\bar{J}_*(\xi_2^0, \xi_5^2)$ is equivalent to $\bar{J}_*(\xi_2^0, \xi_5^2)$ under some first (resp., second) kind automorphism if and only if $\xi_2^0 = \xi_2^0, \ \xi_5^2 = \xi_5^2$ (resp., $\xi_2^0 = -\xi_2^0, \ \xi_5^2 = -\left(\left(\frac{\xi_2^0}{\xi_5^2}\right)^2 + 1\right)/\xi_5^2$).

Proof. Let $J = (\xi_{ij})_{1 \leq i, j \leq 6} = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 \\
0 & \xi_2^0 & 0 & 0 & \xi_5^2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -\left(\frac{\xi_2^0}{\xi_5^2}\right)^2 + 1 & 0 & 0 & -\xi_2^0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}$, $(J_1, J_2, J_3, J_4)$ is a 3 × 3 blocks, an integrable complex structure in the basis $(Y_1^{(k)})$. From Proposition 3.1, with some first kind automorphism, one may suppose $J_1 = \begin{pmatrix}
0 & 0 & -1 \\
0 & \xi_2^0 & 0 \\
1 & 0 & 0
\end{pmatrix}$, $J_4 = \begin{pmatrix}
0 & 0 & -1 \\
0 & \xi_2^0 & 0 \\
1 & 0 & 0
\end{pmatrix}$. As $\text{Tr}(J) = 0, \ \xi_5^2 = -\xi_2^0$. Then one is led to (4.1), and the result follows.

Remark 4.2. The complex subalgebra $m$ associated to $\bar{J}_*(\xi_2^0, \xi_5^2)$ has basis $\bar{Y}_1^{(1)} = Y_1^{(1)} - iY_3^{(1)}$, $\bar{Y}_1^{(2)} = Y_1^{(2)} - iY_3^{(2)}$, $\bar{Y}_2^{(2)} = -i\xi_2^0 Y_2^{(1)} + (1 + i\xi_2^0) Y_2^{(2)}$. The complexification $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$ has weight spaces decomposition with respect to the Cartan subalgebra $\mathfrak{h} = CY_2^{(1)} \oplus CY_2^{(2)}$:

$$
\mathfrak{h} \oplus \mathbb{C} \left( Y_1^{(1)} + iY_3^{(1)} \right) \oplus \mathbb{C} \left( Y_1^{(2)} + iY_3^{(2)} \right) \oplus \mathbb{C} \bar{Y}_1^{(1)} \oplus \mathbb{C} \bar{Y}_1^{(2)}. \tag{4.2}
$$

Then $m = (\mathfrak{h} \cap m) \oplus \mathbb{C} \bar{Y}_1^{(1)} \oplus \mathbb{C} \bar{Y}_1^{(2)}$, which is a special case of the general fact proved in [10] that any complex (integrable) structure on a reductive Lie group of class I is regular.

5. Case of $n$

Let $n$ be the real 3-dimensional Heisenberg Lie algebra with basis $(x_1, x_2, x_3)$ and commutation relations $[x_1, x_2] = x_3$.

Proposition 5.1. Let $J : n \to n$ any linear map. $J$ has zero torsion if and only if it is equivalent to one of the endomorphisms defined in the basis $(x_1, x_2, x_3)$ by the following:
(i)  \[
S(\xi^3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \xi^3_3 \end{pmatrix}, \quad \xi^3_3 \in \mathbb{R},
\]

(ii)  \[
D(\xi^1) = \begin{pmatrix} \xi^1_1 & 0 & 0 \\ 0 & \xi^1_1 & 0 \\ 0 & 0 & \frac{(\xi^1_1)^2 - 1}{2\xi^1_1} \end{pmatrix}, \quad \xi^1_1 \in \mathbb{R}, \xi^1_1 \neq 0,
\]

(iii)  \[
T(a, b) = \begin{pmatrix} 0 & -ab & 0 \\ 1 & b & 0 \\ 0 & 0 & \frac{ab - 1}{b} \end{pmatrix}, \quad a, b \in \mathbb{R}, \; b \neq 0.
\]

Any two distinct endomorphisms in the preceding list are nonequivalent. \(T(a, b)\) is equivalent to

\[
T'(a, b) = \begin{pmatrix} b & -b & 0 \\ a & 0 & 0 \\ 0 & 0 & \frac{ab - 1}{b} \end{pmatrix}.
\]

**Proof.** Let \(J = (\xi^1_{ij})_{1 \leq i, j \leq 3}\) in the basis \((x_1, x_2, x_3)\). The 9 torsion equations reduce to \(\xi^1_1 = \xi^2_1 = 0\) and equation 12 \(|3\) (with the general notation introduced after Definition 2.6) which reads

\[
\zeta^3 \text{Tr}(A) = \det(A) - 1,
\]

where \(A = \begin{pmatrix} \xi^1_{ij} \\ \xi^2_{ij} \end{pmatrix}\). Suppose first \(\text{Tr}(A) = 0\). Then \(A^2 = -I\), so that \(A\) is similar over \(\mathbb{C}\), hence over \(\mathbb{R}\), to \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\). Hence \(J \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\). Now, since \(\xi^3_3\) does not belong to the spectrum of \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\), taking the automorphism \(\begin{pmatrix} 1 & 0 \\ 0 & \alpha \beta \end{pmatrix}\) of \(n\) for suitable \(\alpha, \beta \in \mathbb{R}\), one gets \(J \equiv S(\xi^3_3)\). Suppose now \(\text{Tr}(A) \neq 0\). Then \(\zeta^3_3 = (\det(A) - 1) / \text{Tr}(A)\). If \(A\) is a scalar matrix, that is, \(A = \xi^1_1 I\), then \(J = \begin{pmatrix} \xi^1_{ij} & 0 & 0 \\ 0 & \xi^1_{ij} & 0 \\ \star & \star & ((\xi^1_1)^2 - 1)/2\xi^1_1 \end{pmatrix} \equiv D(\xi^1).\) If \(A\) is not a scalar matrix, then \(A\) is similar to \(\begin{pmatrix} 0 & -ab \\ 1 & b \end{pmatrix}\)
for some \( a, b \in \mathbb{R} \), and \( b \neq 0 \) from the trace. Then \( J \equiv T(a, b) \). Finally, \( T'(a, b) \equiv T(a, b) \) since the matrices \( \begin{pmatrix} 0 & -a \\ b & 0 \end{pmatrix} \) and \( \begin{pmatrix} b & -b \\ a & 0 \end{pmatrix} \) are similar for they have the same spectrum and are no scalar matrices.

\[ \square \]

**Remark 5.2.** \( S(\xi^3_3) \) is an extension of a rank 1 CR-structure; however, \( D(\xi^1_1), T(a, b) \) are not.

### 6. CR-Structures on \( n \)

**Proposition 6.1.** (i) Any linear map \( J : n \to n \) which has zero torsion and is an extension of a rank 1 CR-structure on \( n \) such that \( p \) is nonabelian is equivalent to a unique

\[
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & \xi^3_3
\end{pmatrix}, \quad \xi^3_3 \in \mathbb{R}.
\]

(ii) Any linear map \( J : n \to n \) which is an extension of a rank 1 CR-structure on \( n \) such that \( p \) is abelian is equivalent to a unique

\[
\begin{pmatrix}
\xi^1_1 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}, \quad \xi^1_1 \in \mathbb{R}.
\]

\( J \) has nonzero torsion.

**Proof.** For any nonzero \( X \in p \), \( (X, J_p X) \) is a basis of \( p \). In case (i), \( [X, J_p X] \neq 0 \), since \( p \) is nonabelian. Then \( [X, J_p X] = \mu x_3, \mu \neq 0 \), and \( x_3 \notin p \) since otherwise \( p \) would be abelian. One may extend \( J_p \) to \( n \) in the basis \( (X, J_p X, \mu x_3) \) as

\[
J = \begin{pmatrix}
0 & -1 & \xi^1_3 \\
1 & 0 & \xi^2_3 \\
0 & 0 & \xi^3_3
\end{pmatrix},
\]

and \( J \) has zero torsion only if \( \xi^1_3 = \xi^2_3 = 0 \). In case (ii), necessarily \( x_3 \in p \) since \( p \) is abelian. Hence \( (x_3, J_p x_3) \) is a basis for \( p \). Take any linear extension \( J \) of \( J_p \) to \( n \). There exists some eigenvector \( y_1 \neq 0 \) of \( J \) associated to some eigenvalue \( \xi^1_1 \in \mathbb{R} \). Then \( y_1 \notin p \), which implies \( [y_1, J x_3] \neq 0 \), for otherwise \( y_1 \) would commute to the whole of \( n \) and then be some multiple of \( x_3 \in p \). Hence \( [y_1, J x_3] = \lambda x_3, \lambda \neq 0 \), and dividing \( y_1 \) by \( \lambda \) one may suppose \( \lambda = 1 \). In the basis \( y_1, y_2 = J x_3, y_3 = x_3 \) one has

\[
J = \begin{pmatrix}
\xi^1_1 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix},
\]

and (ii) follows. \( \square \)
Remark 6.2. Let \( N = \{ [x, y, z] \mid x, y, z \in \mathbb{R} \} \) denote the Heisenberg group, where 
\[
[x, y, z] = \begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}.
\]
\( N \) can also be realized \(([11, 12])\) as the boundary \( M_2 = \{ (\zeta, w) \mid \zeta = w\overline{w} \} \) of the Siegel half-space equipped with the multiplication \((\zeta_1, w_1)(\zeta_2, w_2) = (\zeta_1 + \zeta_2 + 2i\overline{w_1}w_2, w_1 + w_2)\). The map \( \Psi : N \to M_2 \) defined by 
\[
\Psi([x, y, z]) = (z - (1/2)xy + i(x^2 + y^2)/4, (1/2)(x - iy))
\]
is an isomorphism. If \( P, Q \) denote the left invariant vector fields associated, respectively, to \( x_1, x_2 \), then 
\[
(d\Psi)(P + iQ) = 2i\overline{w}(\partial/\partial \zeta) + (\partial/\partial w),
\]
hence the left invariant CR-structure on \( N \) associated to the CR-structure on \( n \) introduced in (i) is the CR-structure on \( M_2 \) induced from \( \mathbb{C}^2 \). The left invariant CR-structure on \( N \) associated to the CR-structure on \( n \) introduced in (ii) is not CR-diffeomorphic to the CR-structure on \( M_2 \) induced from \( \mathbb{C}^2 \), since the former has zero Levi form.

7. Complex Structures on \( n \times n \)

We will use for commutation relations 
\[
[x_1, x_2] = x_5, \quad [x_3, x_4] = x_6.
\]
The automorphisms of \( n \times n \) fall into 2 kinds. The first kind is comprised of the matrices 
\[
\Phi = 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b_3^2 & b_4^2 & 0 \\
0 & 0 & b_3^2 & b_4^2 & 0 & 0 \\
b_3^2 & b_4^2 & 0 & 0 & 0 & 0 \\
b_3^2 & b_4^2 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
\[
\left(b_1^2b_2 - b_1^2b_2\right)\left(b_3^2b_4 - b_3^2b_4\right) \neq 0. \tag{7.1}
\]

The second kind ones are \( \Psi = \Theta\Phi \) where \( \Phi \) is first kind and 
\[
\Theta = 
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \tag{7.2}
\]

Proposition 7.1. Any integrable complex structure \( J \) on \( n \times n \) is equivalent under some first kind automorphism to one of the following:
\( \tilde{S}_\varepsilon(\xi_5^5) = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \xi_5^5 & -\varepsilon \left( (\xi_5^5)^2 + 1 \right) \\
0 & 0 & 0 & 0 & \varepsilon & -\xi_5^5
\end{pmatrix}, \quad \varepsilon = \pm 1, \quad \xi_5^5 \in \mathbb{R}, \quad (7.3) \)

\( \tilde{S}_\varepsilon(\xi_5^5) \) is equivalent to \( \tilde{S}_\varepsilon(\xi_5^5) \) (\( \varepsilon, \varepsilon' = \pm 1; \xi_5^5, \xi_5^5 \in \mathbb{R} \)) under some first (resp., second) kind automorphism if and only if \( \varepsilon' = \varepsilon, \xi_5^5 = \xi_5^5 \) (resp., \( \varepsilon' = -\varepsilon, \xi_5^5 = -\xi_5^5 \)).

\( \tilde{D}(\xi_1^1) = \begin{pmatrix}
\xi_1^1 & 0 & -((\xi_1^1)^2 + 1) & 0 & 0 & 0 \\
0 & \xi_1^1 & 0 & -((\xi_1^1)^2 + 1) & 0 & 0 \\
1 & 0 & -\xi_1^1 & 0 & 0 & 0 \\
0 & 1 & 0 & -\xi_1^1 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{(\xi_1^1)^2 - 1}{2 \xi_1^1} & -\frac{2 \xi_1^1}{2 \xi_1^1} \\
0 & 0 & 0 & 0 & \frac{1}{2 \xi_1^1} & -\frac{1 - (\xi_1^1)^2}{2 \xi_1^1}
\end{pmatrix}, \quad \xi_1^1 \in \mathbb{R} \setminus \{0\}, \quad (7.4) \)

\( \tilde{D}(\xi_1^1) \) is equivalent to \( \tilde{D}(\xi_1^1) \) (\( \xi_1^1, \xi_1^1 \in \mathbb{R} \)) under some first (resp., second) kind automorphism if and only if \( \xi_1^1 = \xi_1^1 \) (resp., \( \xi_1^1 = -\xi_1^1 \)).
\[ \bar{T}(\xi^3, \xi^4) = \begin{pmatrix} 0 & -\xi^3 \xi^4 & \phi \xi^3 \xi^4 & 0 & 0 \\ 1 & -\xi^3 \xi^4 & (\xi^3)^2 + 1 - \xi^4 \xi^3 & \xi^3 & 0 \\ 0 & \xi^3 & \xi^3 & -\xi^3 & 0 \\ 1 & 0 & \xi^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \]

where \( \xi^3 \in \mathbb{R} \setminus \{0\}, \xi^4 \in \mathbb{R}, \) (5.7). \( \bar{T}(\xi^3, \xi^4) \) is equivalent to \( \bar{T}(\xi_3^3, \xi_3^4) (\xi_3^3, \xi_3^4 \in \mathbb{R} \setminus \{0\}, \xi_3^4, \xi_3^4 \in \mathbb{R}) \) under some first (resp., second) kind automorphism if and only if \( \xi_3^3 = \xi_3^3, \xi_3^4 = \xi_3^4 \) (resp. \( \xi_3^3 = -\xi_3^3, \xi_3^4 = -\xi_3^4 \)).

Finally, the cases (i), (ii), (iii) are mutually nonequivalent, either under first or second kind automorphism.

**Proof.** Let \( J = (\xi_{ij})_{1 \leq i, j \leq 6} \) an integrable complex structure in the basis \((\chi_k)_{1 \leq k \leq 6} \). Denote \( J_1 = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{array} \right), J_2 = \left( \begin{array}{cc} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{array} \right), J_3 = \left( \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), J_4 = \left( \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \) and \( J_1^* = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{array} \right), J_2^* = \left( \begin{array}{cc} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{array} \right), \) are zero torsion linear maps from \( n \) to \( n \), hence equivalent to type (5.1), (5.2), or (5.3) in Proposition 5.1. It can be checked that their being different types would contradict with \( J^2 = -1 \). Hence, modulo equivalence under some first kind automorphism, we get 3 cases:

**Case 1.** \( J_1^* = \left( \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), J_2^* = \left( \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), \)

**Case 2.** \( J_1^* = D(\xi_1^3), J_2^* = D(\xi_1^3), (\xi_1^3, \xi_3^3 = 0), \) and

**Case 3.** \( J_1^* = \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right), J_2^* = \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right), (\xi_2^3, \xi_3^3 \neq 0). \)

Case 1 (resp., 2, 3) leads to (7.3) (resp., (7.4), (7.5)). The assertion about equivalence in Cases 1 and 2 are readily proved, as is equivalence under some first kind automorphism in Case 3 and the nonequivalence of the 3 types. Consider now \( \Theta \bar{T}(\xi_3^3, \xi_3^4) \Theta^{-1} \). It is equivalent under some first kind automorphism to some \( \bar{T}(\eta_3^3, \eta_3^4) \). That implies that the matrices \( \left( \begin{array}{ccc} \xi_3^3 & -\xi_3^4 \\ 0 & 0 \end{array} \right), \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right) \) are similar, which amounts to their having same trace and same determinant, that is, \( \eta_3^3 = -\xi_3^3, \eta_3^4 = -\xi_3^4 \). As \( \bar{T}(\xi_3^3, \xi_3^4) \) is equivalent to \( \bar{T}(\xi_3^3, \xi_3^4) \) under some second kind automorphism if and only if it is equivalent to \( \Theta \bar{T}(\xi_3^3, \xi_3^4) \Theta^{-1} \) under some first kind automorphism, the assertion about second kind equivalence in Case 3 follows. \( \square \)
Remark 7.2. In Case 3, had we used $J^*_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then we would have to separate further into 2 subcases: subcase $\xi^1_{2} \neq 0$,

$$
\tilde{T}\left(\xi^1_{2}, \xi^2_{2}\right) = \begin{pmatrix}
0 & \xi^1_{2} & -\frac{\xi^2_{2}}{\xi^1_{2}} & -\left(\xi^1_{2} + 1\right) & 0 & 0 \\
\xi^2_{2} & \xi^2_{2} + 1 & -\xi^2_{2} & 0 & 0 \\
0 & -\frac{1}{\xi^2_{2}} & \xi^1_{2} & 0 & 0 \\
1 & \xi^2_{2} & 1 & -\xi^2_{2} & 0 \\
0 & 0 & 0 & 0 & -\frac{\xi^2_{2} + 1}{\xi^2_{2}} \\
0 & 0 & 0 & 0 & 0 
\end{pmatrix}, \quad \xi^1_{2}, \xi^2_{2} \neq 0, \quad (7.6)
$$

subcase $\xi^1_{2} = 0$,

$$
\tilde{T}\left(\xi^2_{2}\right) = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 \\
1 & \xi^2_{2} & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{\xi^2_{2}} & 1 \\
0 & 0 & 0 & 0 & 1 & \frac{1}{\xi^2_{2}} \end{pmatrix}, \quad \xi^2_{2} \neq 0. \quad (7.7)
$$

Remark 7.3. $\tilde{S}(\xi^3_2)$ is abelian (i.e., the corresponding complex subalgebra $m$ is abelian).

Remark 7.4. If one looks for zero torsion linear maps instead of complex structures, then $J^*_1$ and $J^*_3$ may be of different types.

References


