International Scholarly Research Network
ISRN Algebra
Volume 2011, Article ID 930189, 16 pages
doi:10.5402/2011/930189

Research Article

About Zero Torsion Linear Maps on Lie Algebras

Louis Magnin

Institut de Mathématiques de Bourgogne, UMR CNRS 5584, Université de Bourgogne, BP 47870, 21078 Dijon Cedex, France

Correspondence should be addressed to Louis Magnin, magnin@u-bourgogne.fr

Received 6 May 2011; Accepted 31 May 2011

Academic Editors: A. De Sole, A. Isaev, A. V. Kelarev, and A. Kiliçman

We prove that any zero torsion linear map on a nonsolvable real Lie algebra $\mathfrak{g}_0$ is an extension of some CR-structure. We then study the cases of $\mathfrak{sl}(2,\mathbb{R})$ and the 3-dimensional Heisenberg Lie algebra $\mathfrak{n}$. In both cases, we compute up to equivalence all zero torsion linear maps on $\mathfrak{g}_0$, and deduce an explicit description of the equivalence classes of integrable complex structures on $\mathfrak{g}_0 \times \mathfrak{g}_0$.

1. Introduction

Given a real Lie algebra $\mathfrak{g}_0$, the determination up to equivalence of zero torsion linear maps from $\mathfrak{g}_0$ to $\mathfrak{g}_0$ plays an important role in the computation of complex structures on direct products involving $\mathfrak{g}_0$ [1]. The direct computation of those maps can be difficult for semisimple $\mathfrak{g}_0$, so there is a point in exploring alternative ways, particularly their relation to CR-structures. For compact $\mathfrak{g}_0$, maximal rank CR-structures have been classified up to equivalence in [2]. In the case of $\mathfrak{su}(2)$, all zero torsion linear maps are extensions of certain CR-structures (see [1]). One can then ask the natural question whether or not any zero torsion linear map on a nonabelian $\mathfrak{g}_0$ is necessarily an extension of some CR-structure. In the present note, we answer the question in the positive for nonsolvable Lie algebras. Then we make a detailed study of two basic examples: $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R})$ in the positive case, and $\mathfrak{g}_0 = \mathfrak{n}$ the 3-dimensional Heisenberg Lie algebra in the negative. In both cases, we compute (up to equivalence) all zero torsion linear maps, and the result is used to exhibit a complete set of representatives of equivalence classes of complex structures on $\mathfrak{g}_0 \times \mathfrak{g}_0$.

An interesting direction for future research could be to investigate zero torsion linear maps and CR-structures on various constructions of Lie algebras, for example like those considered in [3] (see also [4]).

2. Zero Torsion Linear Maps and Extension of CR-Structures

A CR-structure on a smooth real manifold M is a subbundle $\mathcal{V}$ of the complexified tangent bundle $\mathbb{C}T(M)$ of M such that $[\mathcal{V},\mathcal{V}] \subset \mathcal{V}$ (i.e., the space of smooth sections of $\mathcal{V}$ is closed under commutators) and $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$ ($^{-}$ denoting here conjugation in $\mathbb{C}T(M)$). The rank or CR-dimension is the complex dimension of $\mathcal{V}$. For general background on CR-structures we refer the reader to [5].

Throughout this section, $\mathfrak{g}_0$ will denote any finite-dimensional real Lie algebra, $\mathfrak{g}$ its complexification, and σ or simply $^{-}$ the conjugation in $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

If G_0 is a connected finite dimensional real Lie group, with Lie algebra $\mathfrak{g}_0$, left invariant CR-structures on G_0 are identified to CR-structures on $\mathfrak{g}_0$ in the following sense [6, 7].

Definition 2.1. A rank r $(1 \leqslant r \leqslant [(\dim \mathfrak{g}_0)/2])$ CR-structure on $\mathfrak{g}_0$ is a r-dimensional complex subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{m} \cap \overline{\mathfrak{m}} = \{0\}$.

If $\dim \mathfrak{g}_0$ is even, a CR-structure of maximal rank is an (integrable) complex structure. Now one has the following straightforward lemma.

Lemma 2.2. *Let $\mathfrak{p}$ be vector subspace of $\mathfrak{g}_0$ and $J_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$ a linear map. Consider the real vector subspace $\mathfrak{m} = \{\widetilde{X}; X \in \mathfrak{p}\}$ of $\mathfrak{g}$, where $\widetilde{X} = X - iJ_{\mathfrak{p}}X$ for $X \in \mathfrak{p}$. Denote $\mathfrak{m}' = i\mathfrak{m} = \{J_{\mathfrak{p}}X + iX; X \in \mathfrak{p}\}$. Then*

(i) *$\mathfrak{m}$ is a complex vector subspace of $\mathfrak{g}$ if and only if $J_{\mathfrak{p}}^2 = -1_{\mathfrak{p}}$;*

(ii) *$\mathfrak{m}$ is stable with respect to the bracket if and only if for any $X, Y \in \mathfrak{p}$*

$$\begin{gathered}[X,Y] - [J_{\mathfrak{p}}X, J_{\mathfrak{p}}Y] \in \mathfrak{p}, \\ J_{\mathfrak{p}}([X,Y] - [J_{\mathfrak{p}}X, J_{\mathfrak{p}}Y]) = [J_{\mathfrak{p}}X, Y] + [X, J_{\mathfrak{p}}Y].\end{gathered} \tag{2.1}$$

In that case, $[\widetilde{X}, \widetilde{Y}] = \widetilde{Z}$ with $Z = [X,Y] - [J_{\mathfrak{p}}X, J_{\mathfrak{p}}Y]$.

(iii) *$\mathfrak{m}'$ is stable with respect to the bracket if and only if for any $X, Y \in \mathfrak{p}$*

$$\begin{gathered}[J_{\mathfrak{p}}X, Y] + [X, J_{\mathfrak{p}}Y] \in \mathfrak{p}, \\ J_{\mathfrak{p}}([J_{\mathfrak{p}}X, Y] + [X, J_{\mathfrak{p}}Y]) = [J_{\mathfrak{p}}X, J_{\mathfrak{p}}Y] - [X,Y].\end{gathered} \tag{2.2}$$

In that case, $[\widetilde{X}, \widetilde{Y}] = -i\widetilde{Z}$ with $Z = [J_{\mathfrak{p}}X, Y] + [X, J_{\mathfrak{p}}Y]$ for $X, Y \in \mathfrak{p}$.

Remark 2.3. When (i) is satisfied, (ii) and (iii) are trivially equivalent since then $\mathfrak{m} = \mathfrak{m}'$.

Lemma 2.4. *A rank r CR-structure on $\mathfrak{g}_0$ can be defined in an alternative way as $(\mathfrak{p}, J_{\mathfrak{p}})$ where $\mathfrak{p}$ is a $2r$-dimensional $(1 \leqslant r \leqslant [(\dim \mathfrak{g}_0)/2])$ vector subspace of $\mathfrak{g}_0$ and $J_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$ is a linear map satisfying the 3 conditions*

$$J_{\mathfrak{p}}^2 = -1_{\mathfrak{p}}, \tag{2.3}$$

$$[J_{\mathfrak{p}}X, Y] + [X, J_{\mathfrak{p}}Y] \in \mathfrak{p} \quad \forall X, Y \in \mathfrak{p}, \tag{2.4}$$

$$[J_{\mathfrak{p}}X, J_{\mathfrak{p}}Y] - [X,Y] - J_{\mathfrak{p}}([J_{\mathfrak{p}}X, Y] + [X, J_{\mathfrak{p}}Y]) = 0 \quad \forall X, Y \in \mathfrak{p}. \tag{2.5}$$

Proof. Let $\mathfrak{m}$ be a rank r CR-structure on $\mathfrak{g}_0$. Note first that the taking of the real part is a bijective linear map of the real algebra $\mathfrak{m}$ onto its image $\mathfrak{p} = \Re\mathfrak{m}$, $\dim \mathfrak{p} = 2r$, and there exists a unique linear map $J_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{g}_0$ such that $\mathfrak{m} = \{\widetilde{X} = X - iJ_{\mathfrak{p}}X; X \in \mathfrak{p}\}$. Now, for $X \in \mathfrak{p}$, $i\widetilde{X} = J_{\mathfrak{p}}X + iX \in i\mathfrak{m} = \mathfrak{m}$ hence $J_{\mathfrak{p}}X \in \mathfrak{p}$, so that $J_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$. Then (2.3), (2.4), (2.5) follow from Lemma 2.2 and Remark 2.3.

The converse comes again from Lemma 2.2 and Remark 2.3. □

Remark 2.5. The condition (2.3) implies $\det J_{\mathfrak{p}} = 1$ and Trace $(J_{\mathfrak{p}}) = 0$, hence if $r = 1$, (2.4) and (2.5) follow from (2.3) and can be omitted.

Definition 2.6. A linear map $J : \mathfrak{g}_0 \to \mathfrak{g}_0$ is said to have zero torsion if it satisfies the condition

$$[JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0 \quad \forall X, Y \in \mathfrak{g}_0. \tag{2.6}$$

If J has zero torsion and satisfies in addition $J^2 = -1$, J is an (integrable) complex structure on $\mathfrak{g}_0$. That means that G_0 can be given the structure of a complex manifold with the same underlying real structure and such that the canonical complex structure on G_0 is the left invariant almost complex structure $\widehat{J}$ associated to J (for more details, see [8]).

When computing the matrices of the zero torsion maps in some fixed basis $(x_j)_{1\leqslant j\leqslant n}$ of $\mathfrak{g}_0$, we will denote by $ij \mid k$ $(1 \leqslant i, j, k \leqslant n)$ the torsion equation obtained by projecting on x_k the equation (2.6) with $X = x_i$, $Y = x_j$.

The automorphism group Aut $\mathfrak{g}_0$ of $\mathfrak{g}_0$ acts on the set of all zero torsion linear maps and on the set of all complex structures on $\mathfrak{g}_0$ by $J \mapsto \Phi \circ J \circ \Phi^{-1}$ for all $\Phi \in$ Aut $\mathfrak{g}_0$. It acts also on the set of CR-structures by $(\mathfrak{p}, J_{\mathfrak{p}}) \mapsto (\Phi\mathfrak{p}, \Phi \circ J_{\mathfrak{p}} \circ \Phi^{-1})$. Two J, J' (resp., $(\mathfrak{p}, J_{\mathfrak{p}})$, $(\mathfrak{p}', J_{\mathfrak{p}'})$) are said to be *equivalent* (notation: $J \equiv J'$ (resp., $J_{\mathfrak{p}} \equiv J_{\mathfrak{p}'}$)) if they are on the same Aut $\mathfrak{g}_0$ orbit. This means that the corresponding left invariant CR-structures on the connected simply connected real Lie group associated to $\mathfrak{g}_0$ are intertwined by some Lie group automorphism. It is a stronger notion than CR-diffeomorphy, where the intertwining is simply required to be a diffeomorphism.

Lemma 2.7. *Let $J : \mathfrak{g}_0 \to \mathfrak{g}_0$ be a linear map, $\mathfrak{m} = \{\widetilde{X} = X - iJX; X \in \mathfrak{g}_0\}$ and $\mathfrak{m}' = i\mathfrak{m} = \{JX + iX; X \in \mathfrak{g}_0\}$;*

(i) $\mathfrak{m} \cap \mathfrak{m}' = \{\widetilde{X}; X \in \ker(J^2 + 1)\}$,

(ii) *$\mathfrak{m}'$ is a real subalgebra of $\mathfrak{g}$ if and only if J has zero torsion,*

(iii) *if J has zero torsion, $\mathfrak{m} \cap \mathfrak{m}'$ is a complex subalgebra of $\mathfrak{g}$.*

Proof. (i) For any $Z \in \mathfrak{g}$ one has

$$\begin{aligned} Z \in \mathfrak{m} \cap \mathfrak{m}' &\iff \exists X, Y \in \mathfrak{g}_0, \quad Z = X - iJX = JY + iY \\ &\iff \exists X, Y \in \mathfrak{g}_0, \quad Z = X - iJX, X = JY, Y = -JX \\ &\iff \exists X \in \mathfrak{g}_0, \quad Z = X - iJX, X = -J^2X \\ &\iff \exists X \in \ker\left(J^2 + 1\right), \quad Z = \widetilde{X}. \end{aligned} \tag{2.7}$$

(ii) The result follows from Lemma 2.2(iii) since the first condition in (2.2) (with $\mathfrak{p} = \mathfrak{g}_0$ and $J_{\mathfrak{p}} = J$) is trivially satisfied and the second condition is the zero torsion condition.

(iii) From (ii), $[\mathfrak{m}', \mathfrak{m}'] \subset \mathfrak{m}'$, hence $[\mathfrak{m}, \mathfrak{m}'] \subset \mathfrak{m}$ and $[\mathfrak{m} \cap \mathfrak{m}', \mathfrak{m} \cap \mathfrak{m}'] \subset \mathfrak{m} \cap \mathfrak{m}'$. Clearly $\mathfrak{m} \cap \mathfrak{m}'$ is stable by multiplication by i. □

Definition 2.8. Let $J : \mathfrak{g}_0 \to \mathfrak{g}_0$ be a zero torsion linear map. We say that J is an extension of a CR-structure if there exists a vector subspace $\mathfrak{p} \neq \{0\}$ of $\mathfrak{g}_0$ such that $\mathfrak{p}$ equipped with the restriction $J_{\mathfrak{p}}$ of J to $\mathfrak{p}$ is a CR-structure on $\mathfrak{g}_0$.

Definition 2.9. A real form $\mathfrak{u}$ of $\mathfrak{g}$ is said to be of type I (with respect to $\mathfrak{g}_0$) if $\mathfrak{g}_0 \cap \mathfrak{u} \neq \{0\}$. $\mathfrak{g}_0$ is said to be type I if any real form $\mathfrak{u}$ of $\mathfrak{g}$ is of type I.

Remark 2.10. Introduce the real linear projections $\pi_1 : \mathfrak{g} \to \mathfrak{g}_0$, $\pi_2 : \mathfrak{g} \to \mathfrak{g}_0$ defined by $z = \pi_1(z) + i\pi_2(z)$ for $z \in \mathfrak{g}$. Then a real form $\mathfrak{u}$ of $\mathfrak{g}$ is of type I if and only if $\ker \pi_2 |_{\mathfrak{u}} \neq \{0\}$.

Proposition 2.11. *Let $J : \mathfrak{g}_0 \to \mathfrak{g}_0$ be a zero torsion linear map, $\mathfrak{m} = \{\widetilde{X} = X - iJX; X \in \mathfrak{g}_0\}$ and $\mathfrak{m}' = i\mathfrak{m}$. J is an extension of a CR-structure if and only if $\mathfrak{m} \cap \mathfrak{m}' \neq \{0\}$.*

Proof. From Lemma 2.7, $\mathfrak{m} \cap \mathfrak{m}'$ is a complex subalgebra of $\mathfrak{g}$ and $\mathfrak{m} \cap \mathfrak{m}' = \{\widetilde{X}; X \in \ker(J^2 + 1)\}$. If J is an extension of a CR-structure, one has $\{0\} \neq \mathfrak{p} \subset \ker(J^2 + 1)$ hence $\mathfrak{m} \cap \mathfrak{m}' \neq \{0\}$. Conversely, if $\mathfrak{m} \cap \mathfrak{m}' \neq \{0\}$, let $\mathfrak{p} = \ker(J^2 + 1)$. Then $\mathfrak{p}$ is stable under J, and if $J_{\mathfrak{p}}$ denotes the restriction of J to $\mathfrak{p}$, conditions (2.3), (2.5) are trivially satisfied. Condition (2.4) holds true since, from Lemma 2.7(iii), $\mathfrak{m} \cap \mathfrak{m}'$ is a subalgebra of $\mathfrak{g}$. Precisely, for $X, Y \in \mathfrak{p}$, $\widetilde{X}, \widetilde{Y} \in \mathfrak{m} \cap \mathfrak{m}'$ hence $[i\widetilde{X}, \widetilde{Y}] = [J_{\mathfrak{p}}X + iX, Y - iJ_{\mathfrak{p}}Y] \in \mathfrak{m} \cap \mathfrak{m}'$ and (2.4) follows. □

Corollary 2.12. *There is a one-to-one correspondence between nontype I real forms of $\mathfrak{g}$ and zero torsion linear maps $J : \mathfrak{g}_0 \to \mathfrak{g}_0$ which are no extension of a CR-structure.*

Proof. Let $J : \mathfrak{g}_0 \to \mathfrak{g}_0$ be a zero torsion linear map that is no extension of a CR-structure. Then $\mathfrak{m} \cap \mathfrak{m}' = \{0\}$, hence $\mathfrak{m}' = \{JX + iX; X \in \mathfrak{g}_0\}$ is a real form of $\mathfrak{g}$ which is nontype I. Conversely, if $\mathfrak{u}$ is a nontype I real form of $\mathfrak{g}$, then $\pi_2(\mathfrak{u}) = \mathfrak{g}_0$ implies $\mathfrak{u} = \{JX + iX; X \in \mathfrak{g}_0\}$ for some linear map $J : \mathfrak{g}_0 \to \mathfrak{g}_0$, that is, $\mathfrak{u} = \mathfrak{m}'$. As $\mathfrak{m}'$ is a real subalgebra, J has zero torsion from Lemma 2.7(ii). Now $\mathfrak{m} \cap \mathfrak{m}' = \{0\}$ since $\mathfrak{u}$ is a real form hence J is not an extension of a CR-structure. □

Corollary 2.13. *If $\mathfrak{g}_0$ is of type I, then any zero torsion linear map $J : \mathfrak{g}_0 \to \mathfrak{g}_0$ is an extension of a CR-structure.*

Proposition 2.14. *Let $\mathfrak{u}$ be any real form of $\mathfrak{g}$, τ, σ the conjugations with respect to $\mathfrak{u}$, $\mathfrak{g}_0$, and $N = \sigma\tau \in \operatorname{Aut} \mathfrak{g}$. If N has a nonzero fixed point, then $\mathfrak{u}$ is type I.*

Proof. Let Z be a fixed point of N. $NZ = Z$ reads $\sigma Z = \tau Z$. Consider $V = \sigma Z = \tau Z$. Then $\sigma V = \tau V = Z$. Hence $W = V + Z$ has $\tau W = Z + \tau Z = Z + V = W$ and similarly $\sigma W = W$. Hence $W \in \mathfrak{g}_0 \cap \mathfrak{u}$. Now, $W = 0$ if and only if $\sigma Z = \tau Z = -Z$, that is, $iZ \in \mathfrak{g}_0 \cap \mathfrak{u}$. □

Corollary 2.15. *If $\mathfrak{g}_0$ is nonsolvable, then it is type I.*

Proof. If $\mathfrak{g}_0$ is nonsolvable, so is $\mathfrak{g}$. Now, it is known that any automorphism of a nonsolvable Lie algebra over a characteristic 0 field has a nonzero fixed point ([9]). Hence any real form $\mathfrak{u}$ of $\mathfrak{g}$ is type I. □

3. Case of $\mathfrak{sl}(2,\mathbb{R})$

$G = SL(2,\mathbb{R})$ denotes the Lie group of real 2×2 matrices with determinant 1

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1. \tag{3.1}$$

Its Lie algebra $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R})$ consists of the zero trace real 2×2 matrices

$$X = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = xH + yX_+ + zX_-, \tag{3.2}$$

with basis $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $X_+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $X_- = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and commutation relations

$$[H, X_+] = 2X_+, \qquad [H, X_-] = -2X_-, \qquad [X_+, X_-] = H. \tag{3.3}$$

Beside the basis $(H,\ X_+,\ X_-)$, we will also make use of the basis (Y_1, Y_2, Y_3) where $Y_1 = (1/2)H$, $Y_2 = (1/2)(X_+ - X_-)$, $Y_3 = (1/2)(X_+ + X_-)$, with commutation relations

$$[Y_1, Y_2] = Y_3, \quad [Y_1, Y_3] = Y_2, \quad [Y_2, Y_3] = Y_1. \tag{3.4}$$

The adjoint representation of G on $\mathfrak{g}_0$ is given by $\mathrm{Ad}(\sigma)X = \sigma X \sigma^{-1}$. The matrix Φ of $\mathrm{Ad}(\sigma)$ (σ as in (3.1)) in the basis (H, X_+, X_-) is

$$\Phi = \begin{pmatrix} 1+2bc & -ac & bd \\ -2ab & a^2 & -b^2 \\ 2cd & -c^2 & d^2 \end{pmatrix}. \tag{3.5}$$

The adjoint group $\mathrm{Ad}(G)$ is the identity component of $\mathrm{Aut}\ \mathfrak{g}_0$ and one has

$$\mathrm{Aut}\,\mathfrak{g}_0 = \mathrm{Ad}(G) \cup \Psi_0 \mathrm{Ad}(G), \qquad \Psi_0 = \mathrm{diag}(1,-1,-1). \tag{3.6}$$

The adjoint action of G on $\mathfrak{g}_0$ preserves the form $x^2 + yz$. The orbits are as follows:

(i) the trivial orbit $\{0\}$;

(ii) the upper sheet $z > 0$ of the cone $x^2 + yz = 0$ (orbit of X_-);

(iii) the lower sheet $z < 0$ of the cone $x^2 + yz = 0$ (orbit of $-X_-$);

(iv) for all $s > 0$ the one-sheet hyperboloid $x^2 + yz = s^2$ (orbit of sH);

(v) for all $s > 0$ the upper sheet $z > 0$ of the hyperboloid $x^2 + yz = -s^2$ (orbit of $s(-X_+ + X_-)$);

(vi) for all $s > 0$ the lower sheet $z < 0$ of the hyperboloid $x^2 + yz = -s^2$ (orbit of $s(X_+ - X_-)$).

The orbits of $\mathfrak{g}_0$ under the whole $\operatorname{Aut}\mathfrak{g}_0$ are as follows, beside $\{0\}$:

(I) the cone $x^2 + yz = 0$ (orbit of X_-);

(II) the one-sheet hyperboloid $x^2 + yz = s^2$ (orbit of $s\mathrm{H}$) $(s > 0)$;

(III) the two-sheet hyperboloid $x^2 + yz = -s^2$ (orbit of $s(X_+ - X_-)$) $(s > 0)$.

Proposition 3.1. *Let $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R})$, and $J : \mathfrak{g}_0 \to \mathfrak{g}_0$ any linear map. J has zero torsion if and only if it is equivalent to the endomorphism defined in the basis (Y_1, Y_2, Y_3) (resp., (H, X_+, X_-)) by*

$$J_*(\lambda) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \lambda & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}, \tag{3.7}$$

$J_*(\lambda) \not\equiv J_*(\mu)$ for $\lambda \neq \mu$ (resp.,

$$J(\alpha) = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & \alpha & -\alpha \\ 1 & -\alpha & \alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}, \tag{3.8}$$

$J(\alpha) \not\equiv J(\beta)$ for $\alpha \neq \beta$).

Proof. Let $J = (\xi^i_j)_{1\leqslant i,j\leqslant 3}$ in the basis (H, X_+, X_-). The 9 torsion equations are in the basis (H, X_+, X_-):

$$\begin{aligned}
&12 \mid 1 \quad 2\left(\xi^2_2 + \xi^1_1\right)\xi^1_2 + \left(\xi^2_2 - \xi^1_1\right)\xi^3_1 - \left(\xi^2_1 + 2\xi^1_3\right)\xi^3_2 = 0,\\
&12 \mid 2 \quad 2\left(\xi^2_1\xi^1_2 + 1 + \left(\xi^2_2\right)^2\right) - \xi^3_1\xi^2_1 - 2\xi^3_2\xi^2_3 = 0,\\
&12 \mid 3 \quad \left(\xi^3_1 + 2\xi^1_2\right)\xi^3_1 - 2\left(\xi^2_2 + 2\xi^1_1\right)\xi^3_2 + 2\xi^3_3\xi^3_2 = 0,\\
&13 \mid 1 \quad \left(\xi^2_1 - 2\xi^1_3\right)\xi^1_1 + 2\xi^2_3\xi^1_2 + \xi^3_1\xi^2_3 - \left(\xi^2_1 + 2\xi^1_3\right)\xi^3_3 = 0,\\
&13 \mid 2 \quad 2\left(\xi^2_2 - 2\xi^1_1\right)\xi^2_3 + \left(\xi^2_1 + 2\xi^1_3\right)\xi^2_1 - 2\xi^3_3\xi^2_3 = 0,\\
&13 \mid 3 \quad \xi^3_1\xi^2_1 - 2\xi^3_1\xi^1_3 - 2 + 2\xi^3_2\xi^2_3 - 2\left(\xi^3_3\right)^2 = 0,\\
&23 \mid 1 \quad 4\xi^1_3\xi^1_2 - 1 - \xi^2_2\xi^1_1 - \xi^3_2\xi^2_3 + \left(\xi^2_2 - \xi^1_1\right)\xi^3_3 = 0,\\
&23 \mid 2 \quad 4\xi^2_3\xi^1_2 - \left(\xi^2_2 + \xi^3_3\right)\xi^2_1 = 0,\\
&23 \mid 3 \quad 4\xi^3_2\xi^1_3 - \left(\xi^2_2 + \xi^3_3\right)\xi^3_1 = 0.
\end{aligned} \tag{3.9}$$

J has at least one real eigenvalue λ. Let $v \in \mathfrak{g}_0$, $v \neq 0$, an eigenvector associated to λ. From the classification of the Aut $\mathfrak{g}_0$ orbits of $\mathfrak{g}_0$, we then get 3 cases according to whether v is on the orbit (I), (II), (III) (in the cases (II), (III) one may choose v so that $s = 1$).

Case 1. There exists $\varphi \in \operatorname{Aut} \mathfrak{g}_0$ such that $v = \varphi(X_-)$. Then, replacing J by $\varphi^{-1} J \varphi$, we may suppose $\xi_3^1 = \xi_3^2 = 0$. That case is impossible from $13 \mid 2$ and $13 \mid 3$.

Case 2. There exists $\varphi \in \operatorname{Aut} \mathfrak{g}_0$ such that $v = \varphi(H)$. Then we may suppose $\xi_1^2 = \xi_1^3 = 0$. Then from $12 \mid 2$, $\xi_3^2 \xi_2^3 \neq 0$, and $23 \mid 2$, $23 \mid 3$ yield $\xi_2^1 = \xi_3^1 = 0$. Then $12 \mid 3$ and $13 \mid 2$ successively give $\xi_3^3 = \xi_2^2 + 2\xi_1^1$ and $\xi_1^1 = 0$. Now $12 \mid 2$ and $23 \mid 1$ read, respectively, $-\xi_3^2 \xi_2^3 + (\xi_2^2)^2 + 1 = 0$, and $\xi_3^2 \xi_2^3 - (\xi_2^2)^2 + 1 = 0$. Hence that case is impossible.

Case 3. There exists $\varphi \in \operatorname{Aut} \mathfrak{g}_0$ such that $v = \varphi(X_+ - X_-)$. Then we may suppose that $v = X_+ - X_-$. Now instead of the basis (H, X_+, X_-), we consider the basis (Y_1, Y_2, Y_3). The matrix of J in the basis (Y_1, Y_2, Y_3) has the form

$$J_* = \begin{pmatrix} \eta_1^1 & 0 & \eta_3^1 \\ \eta_1^2 & \lambda & \eta_3^2 \\ \eta_1^3 & 0 & \eta_3^2 \end{pmatrix}. \tag{3.10}$$

Then the 9 torsion equations $*ij \mid k$ (the star is to underline that the new basis is in use) for J in that basis are as follows:

$$\begin{aligned}
&*12 \mid 1 \quad \left(\eta_1^3 + \eta_3^1\right)\lambda - \left(\eta_1^3 - \eta_3^1\right)\eta_1^1 = 0, \\
&*12 \mid 2 \quad \left(\eta_1^1 + \lambda\right)\eta_3^2 - \eta_1^2 \eta_1^3 = 0, \\
&*12 \mid 3 \quad \eta_1^1 \lambda - 1 + \left(\eta_1^3\right)^2 - \left(\eta_1^1 + \lambda\right)\eta_3^3 = 0, \\
&*13 \mid 1 \quad \eta_3^2 \eta_3^1 + \eta_1^2 \eta_1^1 + \eta_3^2 \eta_1^3 - \eta_1^2 \eta_3^3 = 0, \\
&*13 \mid 2 \quad \eta_1^1 \lambda + 1 + \left(\eta_1^2\right)^2 + \left(\eta_3^2\right)^2 + \eta_1^3 \eta_3^1 - \left(\eta_1^1 - \lambda\right)\eta_3^3 = 0, \\
&*13 \mid 3 \quad \eta_3^2 \eta_1^1 - \eta_1^2\left(\eta_3^1 + \eta_1^3\right) - \eta_3^2 \eta_3^3 = 0, \\
&*23 \mid 1 \quad \eta_1^1 \lambda + 1 - \left(\eta_3^1\right)^2 + \left(\eta_1^1 - \lambda\right)\eta_3^3 = 0, \\
&*23 \mid 2 \quad \eta_3^2 \eta_3^1 - \left(\eta_3^3 + \lambda\right)\eta_1^2 = 0, \\
&*23 \mid 3 \quad \left(\eta_1^3 + \eta_3^1\right)\lambda + \left(\eta_1^3 - \eta_3^1\right)\eta_3^3 = 0.
\end{aligned} \tag{3.11}$$

From $*12 \mid 1$ and $*23 \mid 3$,

$$\eta_1^1\left(\eta_1^3 - \eta_3^1\right) = -\eta_3^3\left(\eta_1^3 - \eta_3^1\right). \tag{3.12}$$

Case 1. Suppose first that $\eta_1^3 = \eta_3^1$. Then $\lambda\eta_1^3 = 0$.

Subcase 1. Consider the subcase $\eta_1^3 = 0$. $*13 \mid 1$ and $*13 \mid 3$ read, respectively, $(\eta_3^3 - \eta_1^1)\eta_1^2 = 0, (\eta_3^3 - \eta_1^1)\eta_3^2 = 0$. Suppose $\eta_3^3 \neq \eta_1^1$. Then $\eta_1^2 = \eta_3^2 = 0$, and $*13 \mid 2$ gives $\eta_1^1\lambda + 1 = (\eta_1^1 - \lambda)\eta_3^3$, which implies $\eta_3^3 = 0$ by $*23 \mid 1$. As $*12 \mid 3$ then reads $1 = 0$, this case $\eta_3^3 \neq \eta_1^1$ is not possible. Now, the case $\eta_3^3 = \eta_1^1$ is not possible either since then $*23 \mid 1$ would read $(\eta_1^1)^2 + 1 = 0$. We conclude that the Subcase 1 is not possible.

Subcase 2. Hence we are in the Subcase 2: $\eta_1^3 \neq 0$. Then $\lambda = 0$. From $*13 \mid 2$, $\eta_3^3\eta_1^1 \neq 0$. Then $*23 \mid 1$ yields $\eta_3^3 = (-1 + (\eta_1^3)^2)/\eta_1^1$ and $*13 \mid 2$ reads $(\eta_1^2)^2 + (\eta_3^2)^2 + 2 = 0$. This Subcase 2 is not possible either.

Case 2. Hence Case 1 is not possible, and we are necessarily in the Case 2: $\eta_1^3 \neq \eta_3^1$. From (3.12), $\eta_3^3 = -\eta_1^1$. Then $*13 \mid 2$ reads $(\eta_1^1)^2 + (\eta_1^2)^2 + (\eta_3^2)^2 + 1 + \eta_1^3\eta_3^1 = 0$ hence $\eta_1^3 \neq 0$ and $\eta_3^1 = -((\eta_1^1)^2 + (\eta_1^2)^2 + (\eta_3^2)^2 + 1)/\eta_1^3$. From $*12 \mid 2, \eta_1^2 = (\eta_3^2(\eta_1^1 + \lambda))/\eta_1^3$. Then $*23 \mid 2$ reads $\eta_3^2(((\eta_3^2)^2 + \lambda^2 + 1)(\eta_1^3)^2 + (\eta_1^1 + \lambda)^2(\eta_3^2)^2) = 0$, that is, $\eta_3^2 = 0$, which implies $\eta_1^2 = 0$. Now $*12 \mid 1$ reads $\lambda(1 + (\eta_1^1)^2 - (\eta_1^3)^2) = -\eta_1^1(1 + (\eta_1^1)^2 + (\eta_1^3)^2)$. The subcase $\eta_1^1 \neq 0$ is not possible since then $*12 \mid 3$ would yield $\lambda = -((\eta_1^1)^2 + (\eta_1^3)^2 - 1)/2\eta_1^1$ and $*12 \mid 1$ would read $((\eta_1^1)^2 + (\eta_1^3 + 1)^2)((\eta_1^1)^2 + (\eta_1^3 - 1)^2) = 0$. Hence $\eta_1^1 = 0$. Then $*12 \mid 3$ reads $(\xi_1^3)^2 = 1$. The condition $(\xi_1^3)^2 = 1$ now implies the vanishing of all the torsion equations. In that case

$$J_* = \begin{pmatrix} 0 & 0 & -\varepsilon \\ 0 & \lambda & 0 \\ \varepsilon & 0 & 0 \end{pmatrix}, \quad \varepsilon = \pm 1. \tag{3.13}$$

Then in the basis (H, X_+, X_-)

$$J = \begin{pmatrix} 0 & -\dfrac{\varepsilon}{2} & -\dfrac{\varepsilon}{2} \\ \varepsilon & \dfrac{\lambda}{2} & -\dfrac{\lambda}{2} \\ \varepsilon & -\dfrac{\lambda}{2} & \dfrac{\lambda}{2} \end{pmatrix}. \tag{3.14}$$

The cases $\varepsilon = \pm 1$ are equivalent under Ψ_0. □

Remark 3.2. $J_*(\lambda)$ is an extension of a CR-structure, in agreement with Corollary 2.15.

4. Complex Structures on $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$

We consider the basis $(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}, Y_1^{(2)}, Y_2^{(2)}, Y_3^{(2)})$ of $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$, with the upper index referring to the first or second factor. The automorphisms of $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ fall into 2 kinds: the first kind is comprised of the $\operatorname{diag}(\Phi_1, \Phi_2)$, $\Phi_1, \Phi_2 \in \operatorname{Aut} \mathfrak{sl}(2,\mathbb{R})$, and the second kind is comprised of the $\Gamma \circ \operatorname{diag}(\Phi_1, \Phi_2)$, with Γ the switch between the two factors of $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$.

Proposition 4.1. *Any integrable complex structure J on $\mathfrak{sl}(2,\mathbb{R})\times\mathfrak{sl}(2,\mathbb{R})$ is equivalent under some first kind automorphism to the endomorphism given in the basis $(Y_1^{(1)},Y_2^{(1)},Y_3^{(1)},Y_1^{(2)},Y_2^{(2)},Y_3^{(2)})$ by the matrix*

$$\widetilde{J}_*\left(\xi_2^2,\xi_5^2\right)=\begin{pmatrix}0 & 0 & -1 & 0 & 0 & 0\\ 0 & \xi_2^2 & 0 & 0 & \xi_5^2 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1\\ 0 & -\dfrac{\left(\xi_2^2\right)^2+1}{\xi_5^2} & 0 & 0 & -\xi_2^2 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\end{pmatrix},\quad \xi_2^2,\xi_5^2\in\mathbb{R},\ \xi_5^2\neq 0. \tag{4.1}$$

$\widetilde{J}_*(\xi_2^2,\xi_5^2)$ is equivalent to $\widetilde{J}_*(\xi'^2_2,\xi'^2_5)$ under some first (resp., second) kind automorphism if and only if $\xi'^2_2=\xi_2^2$, $\xi'^2_5=\xi_5^2$ (resp., $\xi'^2_2=-\xi_2^2$, $\xi'^2_5=-((\xi_2^2)^2+1)/\xi_5^2$).

Proof. Let $J=(\xi_j^i)_{1\leqslant i,j\leqslant 6}=\begin{pmatrix}J_1 & J_2\\ J_3 & J_4\end{pmatrix}$, ($J_1,J_2,J_3,J_4$ 3×3 blocks), an integrable complex structure in the basis $(Y_\ell^{(k)})$. From Proposition 3.1, with some first kind automorphism, one may suppose $J_1=\begin{pmatrix}0&0&-1\\0&\xi_2^2&0\\1&0&0\end{pmatrix}$, $J_4=\begin{pmatrix}0&0&-1\\0&\xi_5^5&0\\1&0&0\end{pmatrix}$. As $\mathrm{Tr}(J)=0$, $\xi_5^5=-\xi_2^2$. Then one is led to (4.1), and the result follows. □

Remark 4.2. The complex subalgebra $\mathfrak{m}$ associated to $\widetilde{J}_*(\xi_2^2,\xi_5^2)$ has basis $\widetilde{Y}_1^{(1)}=Y_1^{(1)}-iY_3^{(1)}$, $\widetilde{Y}_1^{(2)}=Y_1^{(2)}-iY_3^{(2)}$, $\widetilde{Y}_2^{(2)}=-i\xi_5^2Y_2^{(1)}+(1+i\xi_2^2)Y_2^{(2)}$. The complexification $\mathfrak{sl}(2)\times\mathfrak{sl}(2)$ of $\mathfrak{sl}(2,\mathbb{R})\times\mathfrak{sl}(2,\mathbb{R})$ has weight spaces decomposition with respect to the Cartan subalgebra $\mathfrak{h}=\mathbb{C}Y_2^{(1)}\oplus\mathbb{C}Y_2^{(2)}$:

$$\mathfrak{h}\oplus\mathbb{C}\left(Y_1^{(1)}+iY_3^{(1)}\right)\oplus\mathbb{C}\left(Y_1^{(2)}+iY_3^{(2)}\right)\oplus\mathbb{C}\widetilde{Y}_1^{(1)}\oplus\mathbb{C}\widetilde{Y}_1^{(2)}. \tag{4.2}$$

Then $\mathfrak{m}=(\mathfrak{h}\cap\mathfrak{m})\oplus\mathbb{C}\widetilde{Y}_1^{(1)}\oplus\mathbb{C}\widetilde{Y}_1^{(2)}$ with $\mathfrak{h}\cap\mathfrak{m}=\mathbb{C}\widetilde{Y}_2^{(2)}$, which is a special case of the general fact proved in [10] that any complex (integrable) structure on a reductive Lie group of class I is regular.

5. Case of $\mathfrak{n}$

Let $\mathfrak{n}$ be the real 3-dimensional Heisenberg Lie algebra with basis (x_1,x_2,x_3) and commutation relations $[x_1,x_2]=x_3$.

Proposition 5.1. *Let $J:\mathfrak{n}\to\mathfrak{n}$ any linear map. J has zero torsion if and only if it is equivalent to one of the endomorphisms defined in the basis (x_1,x_2,x_3) by the following:*

(i)

$$S\left(\xi_3^3\right) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \xi_3^3 \end{pmatrix}, \quad \xi_3^3 \in \mathbb{R}, \tag{5.1}$$

(ii)

$$D\left(\xi_1^1\right) = \begin{pmatrix} \xi_1^1 & 0 & 0 \\ 0 & \xi_1^1 & 0 \\ 0 & 0 & \dfrac{\left(\xi_1^1\right)^2 - 1}{2\xi_1^1} \end{pmatrix}, \quad \xi_1^1 \in \mathbb{R},\ \xi_1^1 \neq 0, \tag{5.2}$$

(iii)

$$T(a,b) = \begin{pmatrix} 0 & -ab & 0 \\ 1 & b & 0 \\ 0 & 0 & \dfrac{ab-1}{b} \end{pmatrix}, \quad a,b \in \mathbb{R},\ b \neq 0. \tag{5.3}$$

Any two distinct endomorphisms in the preceding list are nonequivalent. $T(a,b)$ is equivalent to

$$T'(a,b) = \begin{pmatrix} b & -b & 0 \\ a & 0 & 0 \\ 0 & 0 & \dfrac{ab-1}{b} \end{pmatrix}. \tag{5.4}$$

Proof. Let $J = (\xi_j^i)_{1\leqslant i,j\leqslant 3}$ in the basis (x_1, x_2, x_3). The 9 torsion equations reduce to $\xi_3^1 = \xi_3^2 = 0$ and equation $12 \mid 3$ (with the general notation introduced after Definition 2.6) which reads

$$\xi_3^3 \operatorname{Tr}(A) = \det(A) - 1, \tag{5.5}$$

where $A = \left(\begin{smallmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{smallmatrix}\right)$. Suppose first $\operatorname{Tr}(A) = 0$. Then $A^2 = -I$, so that A is similar over $\mathbb{C}$, hence over $\mathbb{R}$, to $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Hence $J \equiv \left(\begin{smallmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ * & * & \xi_3^3 \end{smallmatrix}\right)$. Now, since ξ_3^3 does not belong to the spectrum of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, taking the automorphism $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & 1 \end{smallmatrix}\right)$ of $\mathfrak{n}$ for suitable $\alpha, \beta \in \mathbb{R}$, one gets $J \equiv S(\xi_3^3)$. Suppose now $\operatorname{Tr}(A) \neq 0$. Then $\xi_3^3 = (\det(A) - 1)/\operatorname{Tr}(A)$. If A is a scalar matrix, that is, $A = \xi_1^1 I$, then $J = \left(\begin{smallmatrix} \xi_1^1 & 0 & 0 \\ 0 & \xi_1^1 & 0 \\ * & * & ((\xi_1^1)^2 - 1)/2\xi_1^1 \end{smallmatrix}\right) \equiv D(\xi_1^1)$. If A is not a scalar matrix, then A is similar to $\left(\begin{smallmatrix} 0 & -ab \\ 1 & b \end{smallmatrix}\right)$

for some $a, b \in \mathbb{R}$, and $b \neq 0$ from the trace. Then $J \equiv T(a,b)$. Finally, $T'(a,b) \equiv T(a,b)$ since the matrices $\left(\begin{smallmatrix} 0 & -ab \\ 1 & b \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} b & -b \\ a & 0 \end{smallmatrix}\right)$ are similar for they have the same spectrum and are no scalar matrices. □

Remark 5.2. $S(\xi_3^3)$ is an extension of a rank 1 CR-structure; however, $D(\xi_1^1), T(a,b)$ are not.

6. CR-Structures on $\mathfrak{n}$

Proposition 6.1. (i) *Any linear map $J : \mathfrak{n} \to \mathfrak{n}$ which has zero torsion and is an extension of a rank 1 CR-structure on $\mathfrak{n}$ such that $\mathfrak{p}$ is nonabelian is equivalent to a unique*

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \xi_3^3 \end{pmatrix}, \quad \xi_3^3 \in \mathbb{R}. \tag{6.1}$$

(ii) *Any linear map $J : \mathfrak{n} \to \mathfrak{n}$ which is an extension of a rank 1 CR-structure on $\mathfrak{n}$ such that $\mathfrak{p}$ is abelian is equivalent to a unique*

$$\begin{pmatrix} \xi_1^1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \xi_1^1 \in \mathbb{R}. \tag{6.2}$$

J has nonzero torsion.

Proof. For any nonzero $X \in \mathfrak{p}$, $(X, J_{\mathfrak{p}}X)$ is a basis of $\mathfrak{p}$. In case (i), $[X, J_{\mathfrak{p}}X] \neq 0$, since $\mathfrak{p}$ is nonabelian. Then $[X, J_{\mathfrak{p}}X] = \mu x_3$, $\mu \neq 0$, and $x_3 \notin \mathfrak{p}$ since otherwise $\mathfrak{p}$ would be abelian. One may extend $J_{\mathfrak{p}}$ to $\mathfrak{n}$ in the basis $(X, J_{\mathfrak{p}}X, \mu x_3)$ as

$$J = \begin{pmatrix} 0 & -1 & \xi_3^1 \\ 1 & 0 & \xi_3^2 \\ 0 & 0 & \xi_3^3 \end{pmatrix}, \tag{6.3}$$

and J has zero torsion only if $\xi_3^1 = \xi_3^2 = 0$. In case (ii), necessarily $x_3 \in \mathfrak{p}$ since $\mathfrak{p}$ is abelian. Hence $(x_3, J_{\mathfrak{p}}x_3)$ is a basis for $\mathfrak{p}$. Take any linear extension J of $J_{\mathfrak{p}}$ to $\mathfrak{n}$. There exists some eigenvector $y_1 \neq 0$ of J associated to some eigenvalue $\xi_1^1 \in \mathbb{R}$. Then $y_1 \notin \mathfrak{p}$, which implies $[y_1, Jx_3] \neq 0$, for otherwise y_1 would commute to the whole of $\mathfrak{n}$ and then be some multiple of $x_3 \in \mathfrak{p}$. Hence $[y_1, Jx_3] = \lambda x_3$, $\lambda \neq 0$, and dividing y_1 by λ one may suppose $\lambda = 1$. In the basis $y_1, y_2 = Jx_3, y_3 = x_3$ one has

$$J = \begin{pmatrix} \xi_1^1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{6.4}$$

and (ii) follows. □

Remark 6.2. Let $N = \{[x, y, z]; x, y, z \in \mathbb{R}\}$ denote the Heisenberg group, where $[x, y, z] = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$. N can also be realized ([11, 12]) as the boundary $M_2 = \{(\zeta, w) \in \mathbb{C}^2; \Im\zeta = w\overline{w}\}$ of the Siegel half-space equipped with the multiplication $(\zeta_1, w_1)(\zeta_2, w_2) = (\zeta_1 + \zeta_2 + 2i\overline{w_1}w_2, w_1 + w_2)$. The map $\Psi : N \to M_2$ defined by $\Psi([x, y, z]) = (z-(1/2)xy+i(x^2+y^2)/4, (1/2)(x-iy))$ is an isomorphism. If P, Q denote the left invariant vector fields associated, respectively, to x_1, x_2, then $(d\Psi)(P + iQ) = 2i\overline{w}(\partial/\partial\zeta) + (\partial/\partial w)$, hence the left invariant CR-structure on N associated to the CR-structure on $\mathfrak{n}$ introduced in (i) is the CR-structure on M_2 induced from $\mathbb{C}^2$. The left invariant CR-structure on N associated to the CR-structure on $\mathfrak{n}$ introduced in (ii) is not CR-diffeomorphic to the CR-structure on M_2 induced from $\mathbb{C}^2$, since the former has zero Levi form.

7. Complex Structures on $\mathfrak{n} \times \mathfrak{n}$

We will use for commutation relations $[x_1, x_2] = x_5$, $[x_3, x_4] = x_6$. The automorphisms of $\mathfrak{n} \times \mathfrak{n}$ fall into 2 kinds. The first kind is comprised of the matrices

$$\Phi = \left(\begin{array}{cccccc} b_1^1 & b_2^1 & 0 & 0 & 0 & 0 \\ b_1^2 & b_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3^3 & b_4^3 & 0 & 0 \\ 0 & 0 & b_3^4 & b_4^4 & 0 & 0 \\ \hline b_1^5 & b_2^5 & b_3^5 & b_4^5 & b_1^1b_2^2 - b_2^1b_1^2 & 0 \\ b_1^6 & b_2^6 & b_3^6 & b_4^6 & 0 & b_3^3b_4^4 - b_4^3b_3^4 \end{array}\right), \quad \left(b_1^1b_2^2 - b_2^1b_1^2\right)\left(b_3^3b_4^4 - b_4^3b_3^4\right) \neq 0. \tag{7.1}$$

The second kind ones are $\Psi = \Theta\Phi$ where Φ is first kind and

$$\Theta = \left(\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right). \tag{7.2}$$

Proposition 7.1. *Any integrable complex structure J on $\mathfrak{n} \times \mathfrak{n}$ is equivalent under some first kind automorphism to one of the following:*

(i)

$$\widetilde{S}_{\varepsilon}\left(\xi_5^5\right) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_5^5 & -\varepsilon\left(\left(\xi_5^5\right)^2+1\right) \\ 0 & 0 & 0 & 0 & \varepsilon & -\xi_5^5 \end{pmatrix}, \quad \varepsilon = \pm 1,\ \xi_5^5 \in \mathbb{R}, \tag{7.3}$$

$\widetilde{S}_{\varepsilon'}(\xi'^5_5)$ is equivalent to $\widetilde{S}_{\varepsilon}(\xi_5^5)$ ($\varepsilon, \varepsilon' = \pm 1; \xi'^5_5, \xi_5^5 \in \mathbb{R}$) under some first (resp., second) kind automorphism if and only if $\varepsilon' = \varepsilon,\ \xi'^5_5 = \xi_5^5$ (resp., $\varepsilon' = -\varepsilon,\ \xi'^5_5 = -\xi_5^5$).

(ii)

$$\widetilde{D}\left(\xi_1^1\right) = \begin{pmatrix} \xi_1^1 & 0 & -\left(\left(\xi_1^1\right)^2+1\right) & 0 & 0 & 0 \\ 0 & \xi_1^1 & 0 & -\left(\left(\xi_1^1\right)^2+1\right) & 0 & 0 \\ 1 & 0 & -\xi_1^1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\xi_1^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{\left(\xi_1^1\right)^2-1}{2\xi_1^1} & -\dfrac{\left(\left(\xi_1^1\right)^2+1\right)^2}{2\xi_1^1} \\ 0 & 0 & 0 & 0 & \dfrac{1}{2\xi_1^1} & \dfrac{1-\left(\xi_1^1\right)^2}{2\xi_1^1} \end{pmatrix}, \quad \xi_1^1 \in \mathbb{R} \setminus \{0\}, \tag{7.4}$$

$\widetilde{D}(\xi'^1_1)$ is equivalent to $\widetilde{D}(\xi_1^1)$ ($\xi'^1_1, \xi_1^1 \in \mathbb{R}$) under some first (resp., second) kind automorphism if and only if $\xi'^1_1 = \xi_1^1$ (resp., $\xi'^1_1 = -\xi_1^1$),

(iii)

$$\widetilde{T}\left(\xi_3^3,\xi_3^4\right)=\begin{pmatrix} 0 & -\xi_3^4\xi_3^3 & -\xi_3^4\xi_3^3 & \xi_3^4\xi_3^3-1 & 0 & 0 \\ 1 & -\xi_3^3 & -\dfrac{\left(\xi_3^3\right)^2+1-\xi_3^4\xi_3^3}{\xi_3^3} & \xi_3^3 & 0 & 0 \\ 0 & \xi_3^3 & \xi_3^3 & -\xi_3^3 & 0 & 0 \\ 1 & 0 & \xi_3^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{\xi_3^4\xi_3^3-1}{\xi_3^3} & -\dfrac{\left(\xi_3^4\xi_3^3-2\right)\xi_3^4\xi_3^3+\left(\xi_3^3\right)^2+1}{\left(\xi_3^3\right)^2} \\ 0 & 0 & 0 & 0 & 1 & \dfrac{\xi_3^4\xi_3^3-1}{\xi_3^3} \end{pmatrix},$$

$$\xi_3^3\in\mathbb{R}\setminus\{0\},\ \xi_3^4\in\mathbb{R}, \tag{7.5}$$

$\widetilde{T}(\xi'^3_3,\xi'^4_3)$ is equivalent to $\widetilde{T}(\xi_3^3,\xi_3^4)$ ($\xi'^3_3,\xi_3^3\in\mathbb{R}\setminus\{0\}$, $\xi'^4_3,\xi_3^4\in\mathbb{R}$) under some first (resp., second) kind automorphism if and only if $\xi'^3_3=\xi_3^3$, $\xi'^4_3=\xi_3^4$ (resp. $\xi'^3_3=-\xi_3^3$, $\xi'^4_3=-\xi_3^4$).

Finally, the cases (i), (ii), (iii) are mutually nonequivalent, either under first or second kind automorphism.

Proof. Let $J=(\xi_j^i)_{1\leqslant i,j\leqslant 6}$ an integrable complex structure in the basis $(x_k)_{1\leqslant k\leqslant 6}$. Denote $J_1=\begin{pmatrix}\xi_1^1 & \xi_2^1\\ \xi_1^2 & \xi_2^2\end{pmatrix}$, $J_2=\begin{pmatrix}\xi_3^1 & \xi_4^1\\ \xi_3^2 & \xi_4^2\end{pmatrix}$, $J_3=\begin{pmatrix}\xi_1^3 & \xi_2^3\\ \xi_1^4 & \xi_2^4\end{pmatrix}$, $J_4=\begin{pmatrix}\xi_3^3 & \xi_4^3\\ \xi_3^4 & \xi_4^4\end{pmatrix}$. Then $J_1^*=\begin{pmatrix}\xi_1^1 & \xi_2^1 & \xi_5^1\\ \xi_1^2 & \xi_2^2 & \xi_5^2\\ \xi_1^5 & \xi_2^5 & \xi_5^5\end{pmatrix}$ and $J_3^*=\begin{pmatrix}\xi_3^3 & \xi_4^3 & \xi_6^3\\ \xi_3^4 & \xi_4^4 & \xi_6^4\\ \xi_3^6 & \xi_4^6 & \xi_6^6\end{pmatrix}$ are zero torsion linear maps from $\mathfrak{n}$ to $\mathfrak{n}$, hence equivalent to type (5.1), (5.2), or (5.3) in Proposition 5.1. It can be checked that their being different types would contradict with $J^2=-1$. Hence, modulo equivalence under some first kind automorphism, we get 3 cases:

Case 1. $J_1^*=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & \xi_5^5\end{pmatrix}$, $J_3^*=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & \xi_6^6\end{pmatrix}$,

Case 2. $J_1^*=D(\xi_1^1)$, $J_3^*=D(\xi_3^3)$, (ξ_1^1, $\xi_3^3\neq 0$), and

Case 3. $J_1^*=\begin{pmatrix}0 & \xi_2^1 & 0\\ 1 & \xi_2^2 & 0\\ 0 & 0 & \xi_5^5\end{pmatrix}$, $J_3^*=\begin{pmatrix}\xi_3^3 & -\xi_3^3 & 0\\ \xi_3^4 & 0 & 0\\ 0 & 0 & \xi_6^6\end{pmatrix}$, ($\xi_2^2$, $\xi_3^3\neq 0$).

Case 1 (resp., 2, 3) leads to (7.3) (resp., (7.4), (7.5)). The assertion about equivalence in Cases 1 and 2 are readily proved, as is equivalence under some first kind automorphism in Case 3 and the nonequivalence of the 3 types. Consider now $\Theta\widetilde{T}(\xi_3^3,\xi_3^4)\Theta^{-1}$. It is equivalent under some first kind automorphism to some $\widetilde{T}(\eta_3^3,\eta_3^4)$. That implies that the matrices $\begin{pmatrix}\xi_3^3 & -\xi_3^3\\ \xi_3^4 & 0\end{pmatrix}$, $\begin{pmatrix}0 & -\eta_3^4\eta_3^3\\ 1 & -\eta_3^3\end{pmatrix}$ are similar, which amounts to their having same trace and same determinant, that is, $\eta_3^3=-\xi_3^3,\eta_3^4=-\xi_3^4$. As $\widetilde{T}(\xi'^3_3,\xi'^4_3)$ is equivalent to $\widetilde{T}(\xi_3^3,\xi_3^4)$ under some second kind automorphism if and only if it is equivalent to $\Theta\widetilde{T}(\xi_3^3,\xi_3^4)\Theta^{-1}$ under some first kind automorphism, the assertion about second kind equivalence in Case 3 follows. □

Remark 7.2. In Case 3, had we used $J_3^* = \begin{pmatrix} 0 & \xi_4^3 & 0 \\ 1 & \xi_4^4 & 0 \\ 0 & 0 & \xi_6^6 \end{pmatrix}$, then we would have to separate further into 2 subcases: subcase $\xi_2^1 \neq 0$,

$$\widetilde{T}\left(\xi_2^1, \xi_2^2\right) = \begin{pmatrix} 0 & \xi_2^1 & -\dfrac{\xi_2^2}{\xi_2^1} & -\left(\xi_2^1+1\right) & 0 & 0 \\ 1 & \xi_2^2 & \dfrac{\xi_2^1+1}{\xi_2^1} & -\xi_2^2 & 0 & 0 \\ 0 & -\xi_2^1 & 0 & \xi_2^1 & 0 & 0 \\ 1 & \xi_2^2 & 1 & -\xi_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{\xi_2^1+1}{\xi_2^2} & -\dfrac{\left(\xi_2^2\right)^2+\left(\xi_2^1+1\right)^2}{\xi_2^2\xi_2^1} \\ 0 & 0 & 0 & 0 & \dfrac{\xi_2^1}{\xi_2^2} & \dfrac{\xi_2^1+1}{\xi_2^2} \end{pmatrix}, \quad \xi_2^1\xi_2^2 \neq 0, \tag{7.6}$$

subcase $\xi_2^1 = 0$,

$$\widetilde{T}\left(\xi_2^2\right) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & \xi_2^2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -\xi_2^2 & -\left(\left(\xi_2^2\right)^2+1\right) & 1 & -\xi_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{1}{\xi_2^2} & \dfrac{1}{\xi_2^2} \\ 0 & 0 & 0 & 0 & -\dfrac{\left(\xi_2^2\right)^2+1}{\xi_2^2} & \dfrac{1}{\xi_2^2} \end{pmatrix}, \quad \xi_2^2 \neq 0. \tag{7.7}$$

Remark 7.3. $\widetilde{S}_\varepsilon(\xi_5^5)$ is abelian (i.e., the corresponding complex subalgebra $\mathfrak{m}$ is abelian).

Remark 7.4. If one looks for zero torsion linear maps instead of complex structures, then J_1^* and J_3^* may be of different types.

References

[1] L. Magnin, "Left invariant complex structures on $U(2)$ and $SU(2) \times SU(2)$ revisited," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 55, no. 4, pp. 269–296, 2010, (see also errata to appear in vol. 56).

[2] J.-Y. Charbonnel and H. O. Khalgui, "Classification des structures CR invariantes pour les groupes de Lie compacts," *Journal of Lie Theory*, vol. 14, no. 1, pp. 165–198, 2004.

[3] A. V. Kelarev, "Directed graphs and Lie superalgebras of matrices," *Journal of Algebra*, vol. 285, no. 1, pp. 1–10, 2005.

[4] A. V. Kelarev, *Ring Constructions and Applications*, World Scientific, River Edge, NJ, USA, 2002.

[5] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, *Real Submanifold in Complex Space and Their Mappings*, Princeton University Press, Princeton, NJ, USA, 1999.

[6] G. Gigante and G. Tomassini, "*CR*-structures on a real Lie algebra," *Advances in Mathematics*, vol. 94, no. 1, pp. 67–81, 1992.
[7] S. Dragomir and G. Tomassini, *Differential Geometry and Analysis on CR Manifolds*, vol. 246 of *Progress in Mathematics*, Birkhauser, Boston, Mass, USA, 2006.
[8] L. Magnin, "Complex structures on indecomposable 6-dimensional nilpotent real Lie algebras," *International Journal of Algebra and Computation*, vol. 17, no. 1, pp. 77–113, 2007.
[9] N. Jacobson, "A note on automorphisms of Lie algebras," *Pacific Journal of Mathematics*, vol. 12, pp. 303–315, 1962.
[10] D. M. Snow, "Invariant complex structures on reductive Lie groups," *Journal für die Reine und Angewandte Mathematik*, vol. 371, pp. 191–215, 1986.
[11] A. Boggess, *CR Manifolds and the Tangential Cauchy Riemann Complex*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 1991.
[12] G. B. Folland and E. M. Stein, "Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group," *Communications on Pure and Applied Mathematics*, vol. 27, pp. 429–522, 1974.